# Generalized eikonal approximation

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A new approximation method for potential scattering is proposed. The usual idea of linearizing propagators in eikonal approximation is incorporated with the second Born approximation in this method such that the scattering amplitude retains more accurate low order Born terms. Consequently, the linearized propagator is dependent on the specifics of the potential. The new formula thus obtained gives spectacular improvements to usual eikonal formulas. Interestingly it works also for scatterings at not so high energy.

## I. INTRODUCTION

For scattering of a particle of energy $E$ by a potential of strength $V_0$ and range $a$, eikonal approximation has been shown to be valid as long as $V_0/E \ll 1$ and $ka \gg 1$ ($k=\sqrt{2mE}/\hbar$). Basically eikonal approximation suggests that the particle propagates linearly inside the potential and results in a useful closed formula for scattering amplitudes. Among them, the Glauber amplitude[1] has been used most successfully in the studies of particle collisions in atomic and nuclear physics.[2]

Eikonal approximation, however, suffers from the limitations that it is good only for high energy scattering at small angles. In pion-nucleus scattering at medium energy of 100—300 MeV, $ka$ is actually not much greater than one and the validity of the eikonal approximation in this case is therefore questionable. Not surprisingly, numerical calculations[2–5] show fairly large errors in the Glauber approximations when the energy is not high enough to ensure that $ka \gg 1$ and $|V_0|/E \ll 1$. In fact it has been pointed out[2,4] that even the second Born approximation is more accurate than the Glauber approximation for the weak coupling case in which

$$|V_0|a/k=(|V_0|/k^2)(ka) \ll 1 \ .$$

This paper reports a new approximation scheme in which the usual eikonal approximation method is generalized to incorporate with the second Born approximation. Consequently the resulting formula for scattering amplitude contains the dominant real part as well as the dominant imaginary part of the second Born term and yields spectacular improvements to the Glauber formula. The importance of including the real part, $\mathrm{Re} f_{B_2}$, in an eikonal formula has been pointed out before by Byron *et al.*[4] They noted that a vast improvement for Yukawa potentials is achieved by simply adding $\mathrm{Re} f_{B_2}$ to the Glauber amplitude.

In the present approximation method, an eikonal-type amplitude is formulated from a generalized eikonal propagator which consists of some arbitrary parameters. These parameters are then determined in such a way that both the dominant real part and the dominant imaginary part of $f_{B_2}$ are retained. The determination of parameters, interestingly enough, amounts to adjustment of the propagation momentum inside the potential and, therefore, is dependent on the specifics of the potential.

Closely following the work of Sugar and Blankenbecler,[6] I begin with a brief introduction of $T$ matrices in Sec. II. With the parametrized linearized propagator, an eikonal-type $T$-matrix element is obtained. The determination of parameters involved is then discussed in Sec. III and carried out in Secs. IV and V for Gaussian potentials. The resulting new formula for scattering amplitude is then numerically tested in Sec. VI. The results from the present method, the Glauber method, and the second Born approximation are closely compared with the exact values. It is shown that the new method works excellently for a much wider region of scattering angle even when the $ka$ value is not much greater than one and the $V_0/k^2$ value is not much smaller than one. Equally impressive results are obtained for exponential potentials which are briefly reported in Sec. VII. The paper ends with a conclusion in Sec. VIII.

## II. FORMULATION

### A. Lippmann-Schwinger equation

Our purpose is to solve for the $T$ matrix satisfying the well-known equation

$$T=V+VGT=V+TGV \tag{1}$$

in an approximate, but simple and closed form. In (1), $G$ is the free Green's function given by

$$G(p)=(k^2-p^2+i\epsilon)^{-1} \ . \tag{2}$$

The units in which $\hbar=\frac{1}{2}m=c=1$ are used here and throughout the paper. $k$ is the magnitude of the momentum $\vec{k}$ (or $\vec{k}'$) of incoming (or outgoing) projectile in the c.m. system, which is scattered by the potential $V$.

In order to solve $T$, I shall approximate the free Green's function by a linearized propagator to be discussed later. Here we note, following Sugar and Blankenbecler,[6] that if the Green's function is separated into two parts

$$G=G_l+G_lNG=G_l+GN_lG_l \ , \tag{3}$$

 

where

$$N_l = G_l^{-1} - G^{-1} , \tag{4}$$

Eq. (1) becomes

$$T = T_l + TGN_lG_lT_l , \tag{5}$$

where

$$T_l = V + T_lG_lV = V + Vg_lV \tag{6}$$

and

$$g_l = G_l + G_lVg_l . \tag{7}$$

### B. Linearized propagator

A general linearized propagator which restricts the propagation in some direction $\hat{z}$ can be written as

$$G_l^{-1} = 2\alpha(\beta - p_z) + i\epsilon , \tag{8}$$

where $\alpha$ and $\beta$ are parameters with the dimensions of a momentum. Note that the direction of the propagation $\hat{z}$, $\alpha$, and $\beta$ are all arbitrary at this point. Our purpose is to choose them in such a way that the error involved in the present approximation scheme is small. I shall come to this point later. Here we see that (8) gives the familiar propagators if the parameters are chosen as follows: For the projectile of initial and final momenta $\vec{k}$ and $\vec{k}\,'$, $|\vec{k}| = |\vec{k}\,'| = k$ and $\vec{\Delta} = \frac{1}{2}(\vec{k} + \vec{k}\,')$, (i) if $\alpha = \beta = k$ and $\hat{z} || \vec{k}$, $G_l$ is nothing but the usual eikonal propagator, (ii) if $\alpha = k$, $\beta = \Delta$, and $\hat{z} || \vec{\Delta}$, $G_l$ is precisely the propagator in the Glauber approximation,[1] (iii) if $\alpha = \beta = \Delta$, and $\hat{z} || \Delta$, $G_l$ is then the propagator resulting in the Abarbanel-Itzykson amplitude,[7] and (iv) if $\alpha = \beta = k$, and $\hat{z} || \vec{\Delta}$, $G_l$ is the propagator used in the approximation method reported earlier by the present author.[5]

It should be noted that the propagator (8) in the spacial space is

$$G_l(\vec{\rho}) = \frac{1}{2\alpha i}\delta^2(\rho_\perp)\theta(\rho_z)e^{i\beta\rho_z} . \tag{9}$$

This clearly indicates that the propagation is in the $z$ direction with the momentum $\beta$. It should also be noted that the difference between $G_l^{-1}$ and the exact $G^{-1}$ is

$$N_l = (\vec{p} - \alpha\hat{z})^2 - (\alpha - \beta)^2 - (k^2 - \beta^2) . \tag{10}$$

### C. Scattering amplitude

The approximation scheme proposed here uses the linearized propagator (8) as the lowest order term in place of the exact Green's function in (1). The free parameters in the linearized propagator are then to be determined by demanding that the neglected contribution from higher order terms is small.

With $G \simeq G_l$, the lowest order $T$ matrix is given by (6). The lowest order full Green's function $g_l(\vec{r},\vec{r}\,')$ is the solution of the differential equation

$$2\alpha\left[\beta + i\frac{\partial}{\partial z}\right]g_l(\vec{r},\vec{r}\,') = \delta^3(\vec{r} - \vec{r}\,') + V(\vec{r})g_l(\vec{r},\vec{r}\,') .$$

Thus

$$g_l(\vec{r},\vec{r}\,') = G_l(\vec{r} - \vec{r}\,')e^{-i\chi(\vec{r},\vec{r}\,')} , \tag{11}$$

where

$$\chi(\vec{r},\vec{r}\,') = \chi(\vec{r}) - \chi(\vec{r}\,') , \tag{12}$$

and

$$\chi(\vec{r}) = \frac{1}{2\alpha}\int_{-\infty}^{z} V(\vec{r}\,')dz' .$$

The lowest order $T$-matrix element is then

$$\begin{aligned}\langle \vec{k}\,' | T_l | \vec{k} \rangle &= \langle \vec{k}\,' | V + Vg_lV | \vec{k} \rangle \\ &= \langle T_B \rangle + \frac{1}{2\alpha i}\int d^3r\, e^{-i\Lambda'(\vec{r})}V(\vec{r}) \\ &\quad \times \int_{-\infty}^{z} V(\vec{r}\,')e^{i\Lambda(\vec{r}\,')}dz' ,\end{aligned} \tag{13}$$

where $\vec{r} = (x,y,z)$ and $\vec{r}\,' = (x,y,z')$, $\vec{q} = \vec{k} - \vec{k}\,'$,

$$\Lambda(\vec{r}) = \vec{k}\cdot\vec{r} - \beta z + \chi(\vec{r}) ,$$

$$\Lambda'(\vec{r}) = \vec{k}\,'\cdot\vec{r} - \beta z + \chi(\vec{r}) ,$$

and $\langle T_B \rangle$ is the first Born given by

$$\langle T_B \rangle = \int d^3r\, e^{i\vec{q}\cdot\vec{r}}V(\vec{r}) .$$

When the propagation of the particle is not all the way along the incident direction, there are in general two hard scatterings in our approximation scheme where the changes of direction occur. In (13), one such hard scattering takes place at $\vec{r}\,'$ with a sudden phase change (or momentum transfer) of $\vec{k} - \beta\hat{z}$. The particle then continues in the $\hat{z}$ direction, accumulating a phase $\chi(\vec{r},\vec{r}\,')$. It is then scattered into the final direction at $\vec{r}$ with another sudden phase change of $\beta\hat{z} - \vec{k}\,'$. Obviously, when $\beta = k_z$, (13) reduces to the familiar eikonal-type form:

$$\langle \vec{k}\,' | T_E | \vec{k} \rangle = \int d^3r\, e^{i\vec{q}\cdot\vec{r}}V(\vec{r})e^{-i\chi(\vec{r})} . \tag{14}$$

With the above-mentioned specific choices of $\alpha$, $\beta$, and $\hat{z}$, this gives the usual eikonal amplitude, the Glauber amplitude,[1] or the Abarbanel-Itzykson amplitude.[7] They are known to work for small-angle scattering at high energy.

Equation (13) at this point is merely a solution to the $T$-matrix equation with the linearized propagator. The crucial question is whether it has anything to do with the solution of the exact $T$-matrix equation. My purpose next is to choose $\alpha$, $\beta$, and the $\hat{z}$ direction such that Eq. (13) gives a good approximation to the exact solution.

## III. DETERMINATION OF PARAMETERS

The exact $T$ matrix is related to $T_l$ through (5) and, therefore, the $T$-matrix element is related to $\langle \vec{k}\,' | T_l | \vec{k} \rangle$ through

$$\langle \vec{k}\,' | T | \vec{k} \rangle = \langle \vec{k}\,' | T_l | \vec{k} \rangle + \langle \vec{k}\,' | TGN_lG_lT_l | \vec{k} \rangle . \tag{15}$$

It is then obvious that $\langle \vec{k}\,' | T_l | \vec{k} \rangle$ is a good approximation of $\langle \vec{k}' | T | \vec{k} \rangle$ if the second term above, i.e.,

$$\Delta T \equiv \langle \vec{k}' | TGN_l G_l T_l | \vec{k} \rangle , \tag{16}$$

is small compared with the first term. Thus the smallness of $\Delta T$ is the natural validity condition of our approximation scheme. The parameters $\alpha$, $\beta$, and the $z$ direction, therefore, have to be determined such that $\Delta T$ is small.

$\Delta T$, however, is a complicated function of $\alpha$, $\beta$, and $\hat{z}$. Fortunately, there is some understanding of $\Delta T$ at high energy from past work which can be made use of here for the purpose of determining the parameters. It is known that an eikonal-type approximation, especially Glauber's approximation, works at small angle at large $k$.[1,6] This means that at large $k$ and small $\theta$, $\Delta T$ is small if $\hat{z} \| \vec{\Delta}$, $\alpha = k$, and $\beta = \Delta$. In fact it has been proven that the Glauber amplitude with this choice selects out in each term of the Born series the dominant contribution (to order $k^{-1}$) which is alternatively real and imaginary.[2,4] The Glauber amplitude, consequently, has a rather simple form as given in (14). However, the fact that the Glauber terms do not have both the dominant real term and the dominant imaginary term seems to limit its accuracy quite severely. For a weak coupling case, for example, the Glauber approximation does not work as well as the second Born approximation. This was studied by Byron *et al.*[4] They pointed out that this is because of the missing real term in the second Glauber term. In fact Byron *et al.* showed that the mere inclusion of the real part of the second Born term to the Glauber amplitude results in spectacular improvements for Yukawa-type potentials.

It is thus clear that the Glauber choice is a good one for small-angle scattering at high energy, but the second Born term needs to be considered more carefully. Accordingly, I choose the parameters $\hat{z}$, $\alpha$, and $\beta$ in our formulation to be consistent with the Glauber approximation by assuming $\hat{z}$ to be parallel to $\vec{\Delta}$ and

$$\alpha = k + \Delta\alpha, \quad \beta = k + \Delta\beta , \tag{17}$$

where $\Delta\alpha$ and $\Delta\beta$ are correction terms which are much smaller than $k$ for forward scattering at large $k$.[8] I shall then determine $\Delta\alpha$ and $\Delta\beta$ such that both the dominant real term and the dominant imaginary term of the second Born term are kept in our approximation formula. To this end I need to consider the second Born term of $\Delta T$ in (16)

$$\Delta T_2 = \langle \vec{k}' | VGN_l G_l V | \vec{k} \rangle ,$$

which, from (3), can be rewritten as

$$\Delta T_2 = \langle \vec{k}' | VG_l N_l G_l V + VG_l N_l G_l N_l G_l V + \cdots | \vec{k} \rangle . \tag{18}$$

Note that $G_l$ has a pole at $\vec{p} = \beta\hat{z}$ and $N_l$ is small at the pole if $\alpha \simeq \beta \simeq k$. The first term in (18) is therefore the dominant term for small-angle scattering at large $k$.[6] Thus, to the lowest order in $N_l$, I have

$$\Delta T_2 = \langle \vec{k}' | VG_l N_l G_l V | \vec{k} \rangle . \tag{19}$$

From (9), this becomes

$$\Delta T_2 = -\frac{1}{4\alpha^2} \int d^3r \left[ \int_z^{\infty} e^{-i\lambda'(\vec{r}')} V(\vec{r}') dz' \right] e^{-i\beta z} \times N_1 e^{i\beta z} \left[ \int_{-\infty}^{z} e^{i\lambda(\vec{r}')} V(\vec{r}') dz' \right] , \tag{20}$$

where $N_l$ in the coordinate space is given by

$$N_l = -\nabla^2 + 2\alpha i \nabla_z - k^2 + 2\alpha\beta ,$$

$$\lambda(\vec{r}) = \vec{k}\cdot\vec{r} - \beta z ,$$

and

$$\lambda'(\vec{r}) = \vec{k}'\cdot\vec{r} - \beta z .$$

After integrations by parts, (20) can be reduced to the simpler form:

$$\Delta T_2 = -\frac{1}{4\alpha^2} \int d^3r\, V(\vec{r}) e^{-i\lambda(\vec{r}')} \times N_1' \int_{-\infty}^{z} dz'(z-z') V(\vec{r}') e^{i\lambda'(\vec{r}')} , \tag{21}$$

where

$$N_l' = -\nabla^2 + 2i(\alpha - \beta)\nabla_z - (k^2 - \beta^2) . \tag{22}$$

Thus our objective is to find $\alpha$ and $\beta$ such that $\Delta T_2$ of (21) is small. In general, this depends on the details of the potential $V(\vec{r})$ and requires a more quantitative analysis. In the following, I shall consider spherical potentials, especially Gaussian potentials and Yukawa-type potentials.

## IV. SPHERICAL POTENTIAL

For a spherical potential $V(\vec{r}) = V(r)$, Eq. (21) can be rewritten in a more convenient form. First, $\nabla_z^2$ and $\nabla_z$ operations can be carried out easily. I obtain

$$\Delta T_2 = T_A + T_B + T_C + T_D , \tag{23}$$

where

$$T_A = \frac{1}{4\alpha^2} \int d^3r\, e^{i\vec{q}\cdot\vec{r}} V^2 , \tag{24}$$

$$T_B = -i\frac{\alpha-\beta}{2\alpha^2} \int d^3r\, e^{i\vec{q}\cdot\vec{r}} V(r) e^{-i\kappa z} \times \int_{-\infty}^{z} dz' V(r') e^{i\kappa z'} , \tag{25}$$

$$T_C = \frac{(k^2-\beta^2)}{2\alpha^2} \int d^3r\, e^{i\vec{q}\cdot\vec{r}} z V(r) e^{-i\kappa z} \times \int_{-\infty}^{z} dz' V(r') e^{i\kappa z'} , \tag{26}$$

and

$$T_D = -\frac{1}{4\alpha^2} \int d^3r\, \vec{\nabla}_\perp [e^{(i/2)\vec{q}\cdot\vec{r}} V(r)] e^{-i\kappa z} \times \int_{-\infty}^{z} dz'(z-z') \vec{\nabla}_\perp [e^{(i/2)\vec{q}\cdot\vec{r}} V(r')] \times e^{i\kappa z'} , \tag{27}$$

where $\kappa = \Delta - \beta$.

Noting that

$$z \vec{\nabla}_\perp V(r) = r_\perp \frac{\partial V(r)}{\partial z} ,$$

I can further reduce $T_D$ to the following, after a few algebraical steps:

$$T_D=-\frac{1}{2\alpha^2}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\left\{\left[1+\frac{i\vec{q}\cdot\vec{r}}{2}\right]V^2-\left[i(\Delta-\beta)\left[1+\frac{i\vec{q}\cdot\vec{r}}{2}\right]-\frac{q^2}{4}z\right]Ve^{-i\kappa z}\int_{-\infty}^{z}dz'Ve^{i\kappa z'}\right\}. \tag{28}$$

Combining $T_A$, $T_B$, $T_C$, and $T_D$, I have

$$\Delta T_2=\frac{1}{4\alpha^2}(J_1+J_2+J_3+J_4)\,, \tag{29}$$

where

$$J_1=-\int d^3r\, e^{i\vec{q}\cdot\vec{r}}(1+i\vec{q}\cdot\vec{r})V(r)^2\,,$$

$$J_2=i2(\Delta-\alpha)\int d^3r\, e^{i\vec{q}\cdot\vec{r}}V(r)e^{-i\kappa z}\times\int_{-\infty}^{z}dz'V(r')e^{i\kappa z'}\,,$$

$$J_3=2(\Delta^2-\beta^2)\int d^3r\, e^{i\vec{q}\cdot\vec{r}}zV(r)e^{-i\kappa z}\times\int_{-\infty}^{z}dz'\,V(r')e^{i\kappa z'}\,,$$

and

$$J_4=-\kappa\vec{q}\cdot\int d^3r\, e^{i\vec{q}\cdot\vec{r}}\vec{r}_\perp V(r)e^{-i\kappa z}\times\int_{-\infty}^{z}dz'V(r')e^{i\kappa z'}\,.$$

For a given potential, $\alpha$ and $\beta$ can now be determined by minimizing $\Delta T_2$ in (29). To examine how the present formulation actually works, I have carried out the numerical computation for Gaussian potentials and Yukawa-type potentials. In the following the results for Gaussian and exponential potentials are reported.

## V. GAUSSIAN POTENTIAL

For a potential

$$V(r)=V_0e^{-r^2/a^2}\,, \tag{30}$$

$\vec{r}V(r)=-a^2/2\vec{\nabla}V(r)$ and the integrals $J_1$, $J_2$, $J_3$, and $J_4$ in (29) can easily be seen to become

$$J_1=-\left[1-\frac{q^2a^2}{4}\right]I\,,$$

$$J_2=i2(\Delta-\alpha)I'\,,$$

$$J_3=(\Delta^2-\beta^2)a^2(I-i\kappa I')\,,$$

and

$$J_4=-i\kappa\frac{q^2a^2}{4}I'\,,$$

where

$$I=\int d^3r\, e^{i\vec{q}\cdot\vec{r}}V^2 \tag{31}$$

and

$$I'=\int d^3r\, e^{i\vec{q}\cdot\vec{r}}Ve^{-i\kappa z}\int_{-\infty}^{z}dz'\,Ve^{i\kappa z'}\,. \tag{32}$$

Thus the error $\Delta T_2$ can be rewritten in terms of two integrals $I$ and $I'$ of (31) and (32):

$$\Delta T_2=\frac{1}{4\alpha^2}\{[-1+(k^2-\beta^2)a^2]I+i[2(\Delta-\alpha)-\kappa(k^2-\beta^2)a^2]I'\}\,. \tag{33}$$

With proper choices for $\alpha$ and $\beta$, $\Delta T_2$ can actually be made to vanish in this case. The choices are obviously

$$\beta=k(1-1/k^2a^2)^{1/2} \tag{34}$$

and

$$\alpha=\tfrac{1}{2}[\Delta+k(1-1/k^2a^2)^{1/2}]\,. \tag{35}$$

It is interesting to note that, for Glauber's amplitude ($\alpha=k$ and $\beta=\Delta$), the error $\Delta T_2$ is

$$(\Delta T_2)_{\text{Glauber}}=\frac{1}{4k^2}[(\tfrac{1}{4}q^2a^2-1)I+2i(\Delta-k)I']\,.$$

Noting that the Glauber amplitude is made up of terms like $I/V_0$ and $I'/2k$, one sees that the error involved in the Glauber amplitude is large unless the scattering is nearly forward such that $q^2a^2\ll1$ and $(\Delta-k)/k\ll1$. Even then the error is of the order $V_0/k^2$.

## VI. NUMERICAL RESULTS

With $\alpha$ and $\beta$ determined, the approximate formula (13) can now be evaluated. Equation (13) yields the scattering amplitude from potential $V(r)$:

$$f=f_B-\frac{1}{8\pi\alpha i}\int d^3r\, e^{i\vec{q}\cdot\vec{r}}V(r)e^{-i\kappa z-i\chi(\vec{r})}\times\int_{-\infty}^{z}dz'V(r')e^{i\kappa z'+i\chi(\vec{r}\,')}\,, \tag{36}$$

where $\vec{r}=(x,y,z)$, $\vec{r}\,'=(x,y,z')$, $\vec{q}=\vec{k}-\vec{k}\,'$, $\kappa=\Delta-\beta$,

$$\chi(\vec{r})=\frac{1}{2\alpha}\int_{-\infty}^{z}V(\vec{r}\,')dz'\,,$$

$$f_B=-\frac{1}{4\pi}\int e^{i\vec{q}\cdot\vec{r}}V(\vec{r})d^3r\,,$$

and the $z$ axis is in the direction of $\vec{\Delta}=\frac{1}{2}(\vec{k}+\vec{k}\,')$.

For a Gaussian potential of (33), I find

$$\chi(\vec{r})=\frac{\sqrt{\pi}V_0a}{4\alpha}e^{-(x^2+y^2)/a^2}\Phi(z/a)\,, \tag{37}$$

where $\Phi(\rho)$ is the error function of argument $\rho$, and the integrations over $x$ and $y$ in (36) can be carried out if $\exp(\pm i\chi)$ is expanded in series. The scattering amplitude (36) then becomes

$$f=-\frac{V_0a^3\sqrt{\pi}}{4}e^{-q^2a^2/4}-\frac{V_0^2a^4}{8i\alpha}\sum_{n,n'=0}^{\infty}\left[-\frac{V_0a\sqrt{\pi}}{4\alpha}i\right]^{n+n'}\frac{e^{-q^2a^2/4(2+n+n')}}{n!n'!(2+n+n')}I_{nn'}\,, \tag{38}$$

where

TABLE I. Cross sections for Gaussian potential at $(V_0,k,a)=(-2,4,0.75)$.

| | $\theta$ 0° | 30° | 60° | 90° | 120° |
|---|---|---|---|---|---|
| Exact | 1.422(−1) | 4.215(−2) | 1.429(−3) | 1.06(−5) | 3.59(−7) |
| Eq. (38) | 1.423(−1) | 4.214(−2) | 1.417(−3) | 1.00(−5) | 2.76(−7) |
| Second Born | 1.439(−1) | 4.299(−2) | 1.528(−3) | 1.56(−5) | 6.68(−7) |
| Glauber | 1.390(−1) | 4.162(−2) | 1.569(−3) | 2.37(−5) | 1.11(−6) |

TABLE II. Cross sections for Gaussian potential at $(V_0,k,a)=(-2,4,0.5)$.

| | $\theta$ 0° | 30° | 60° | 90° | 120° | 150° |
|---|---|---|---|---|---|---|
| Exact | 1.253(−2) | 7.312(−3) | 1.663(−3) | 2.10(−4) | 2.3(−6) | 3.9(−6) |
| Eq. (38) | 1.254(−2) | 7.316(−3) | 1.656(−3) | 2.05(−4) | 2.1(−6) | 3.0(−6) |
| Second Born | 1.260(−2) | 7.368(−3) | 1.687(−3) | 2.17(−4) | 2.6(−6) | 4.8(−6) |
| Glauber | 1.224(−2) | 7.163(−3) | 1.657(−3) | 2.25(−4) | 3.1(−5) | 7.5(−6) |

TABLE III. Cross sections for Gaussian potential for $k=2, a=1$ at $\theta=0°$ and 60°.

| | $-V_0/k^2$ <br> $-V_0a/4k$ | 1/4 <br> 1/8 | 1/2 <br> 1/4 | 1 <br> 1/2 | 3/2 <br> 3/4 | 2 <br> 1 | 4 <br> 2 |
|---|---|---|---|---|---|---|---|
| $\theta=0°$ | Exact | 2.03(−1) | 8.24(−1) | 3.21(+0) | 6.59(+0) | 1.02(+1) | 1.7(+1) |
| | Eq. (38) | 2.04(−1) | 8.30(−1) | 3.22(+0) | 6.43(+0) | 0.94(+1) | 1.2(+1) |
| $\theta=60°$ | Exact | 2.64(−2) | 1.03(−1) | 3.66(−1) | 7.03(−1) | 1.02(+0) | 1.3(+0) |
| | Eq. (38) | 2.63(−2) | 1.01(−1) | 3.49(−1) | 6.09(−1) | 0.75(+0) | 0.5(+0) |

$$I_{nn'}=\int_{-\infty}^{\infty} dz\, e^{-i\kappa a z} e^{-z^2}\Phi^n(z) \times \int_{-\infty}^{z} dz' e^{i\kappa a z'} e^{-z'^2}\Phi^{n'}(-z') \; . \tag{39}$$

In (39), the parameters $\alpha$ and $\kappa$ are determined from (34) and (35) to be

$$\alpha=\tfrac{1}{2}[\Delta+k(1-1/k^2a^2)^{1/2}]$$

and

$$\kappa=\Delta-k(1-1/k^2a^2)^{1/2} \; .$$

For various values of $(V_0,k,a)$ (38) is computed and compared with exact values from phase shift analysis in Tables I—III and Figs. 1—7. It is evident that the present formula works excellently for wide ranges of $(V_0,k,a)$ values and scattering angles. More specifically, I have the following comments in order.

(A) In general the present approximate formula works excellently for small angle scatterings at high energy. The resulting cross sections agree with the exact ones well within 10% for all angles ranging from 0° to 60° if $ka>1$ and $|V_0|/k^2<1$. For the wide ranges of $V_0$, $k$, and $a$ values I tested, $0\lesssim ka\lesssim 8$ and $0<|V_0|\leq 20$, the present formula is consistently far superior to either the Glauber formula or the second Born approximation.

(B) For the weak coupling case for which $V_0a/k \ll 1$, the accuracy of the present formula is truly outstanding. It is interesting to note that the second Born approximation may actually be more accurate than the Glauber formula for the weak coupling case, especially when $ka$ is not much greater than one. The present formula with the more accurate second order term works remarkably well in this case even for not so high energy. This is illustrated in Figs. 1 and 2 for $(V_0,k,a)=(-2,4,1)$ and Tables I and II for $(-2,4,0.75)$ and $(-2,4,0.5)$. It is evident that the results from Eq. (38) agree with exact ones within 10% for a wide range of scattering angle (up to over 100°). In fact, the resulting scattering amplitudes in Figs. 1 and 2 reproduce excellently the exact ones for all angles from 0° to 140°. The accuracy in the resulting cross sections for small angles is even more impressive as can be seen in Tables I and II.

(C) For a stronger coupling case, the present formula continues to give good results for small-angle scattering as long as $ka>1$ and $|V_0|/k^2\lesssim 1$. The amplitude resulting from Eq. (38) continues to reproduce overall structure of exact amplitude for all scattering angles and gives excellent agreement (within 10%) with exact results for angles ranging from 0° to about 60°. This is illustrated in Figs. 3 and 4 for $(V_0,k,a)=(-12,4,1)$ where the results from the Glauber amplitude are also included. The resulting cross sections relative to the exact ones are depicted in Fig. 5. It should be noted that the present formula behaves like a good small-angle approximate formula should: It always

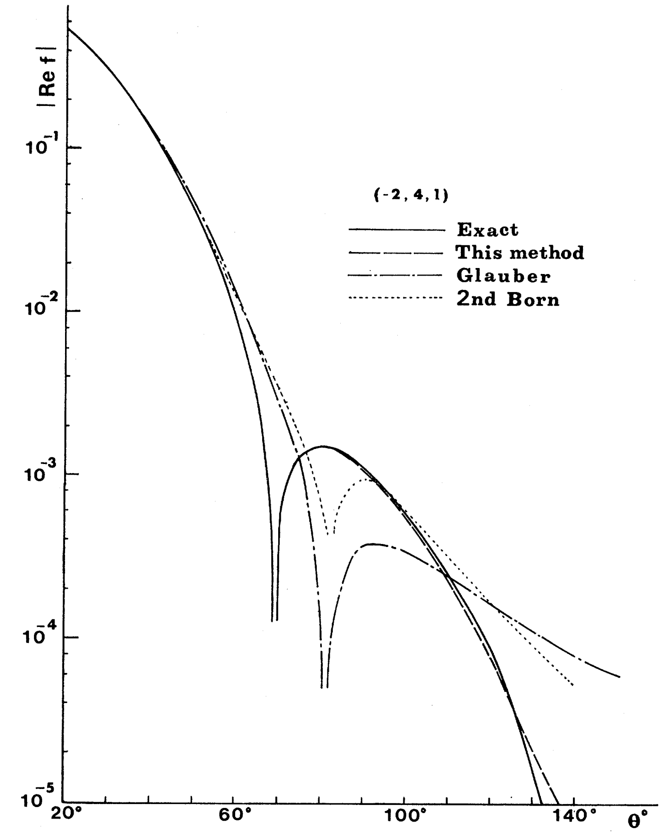


FIG. 1. The real part of scattering amplitude from Gaussian potential at $(V_0,k,a)=(-2,4,1)$. Note that the result from the present method is indistinguishable from the exact one from 0° up to about 90°.

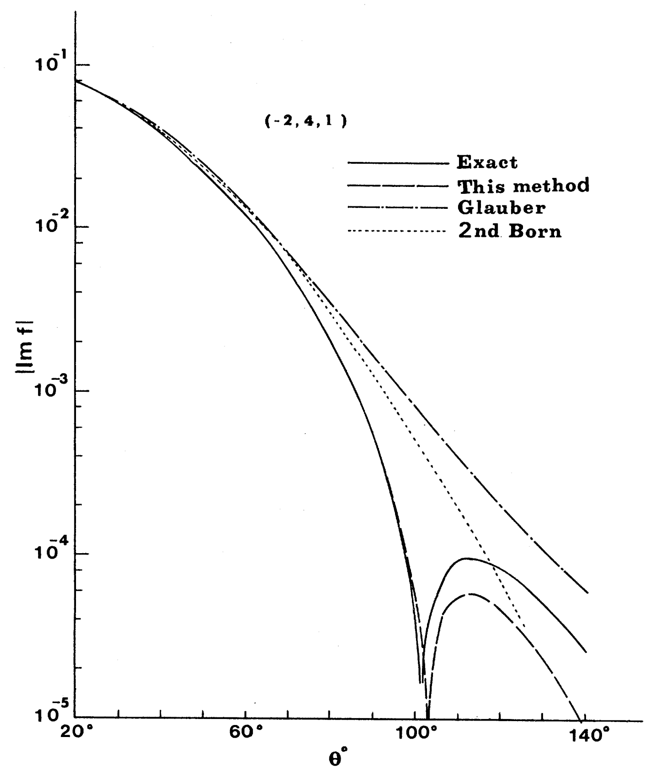


FIG. 2. The imaginary part of the amplitude of Fig. 1. Again the result from the present method agrees with the exact one indistinguishably for angles up to about 90°.

gives the best agreement at $\theta=0°$ and then the gradual deviation at larger angles. It is also interesting to note that the present formula works remarkably well even for the case in which $ka$ is not much greater than one. In fact it works surprisingly well even for the case in which $|V_0|/k^2$ is not smaller than one. This is best illustrated by the result for $(|V_0|a/k,ka)=(3,2)$ in Fig. 5 and leads to the next observation.

(D) For a fixed value of $ka$ or $|V_0|/k^2$, the Glauber formula works better if $|V_0|a/k$ is smaller. This is more evidently so for the present formula as can be seen in Figs. 6 and 7. Apparently this is because of low order Born terms which become more dominant as $|V_0|a/k$ is smaller. Since it retains the low order Born terms more accurately, the present formula actually works better for smaller $|V_0|a/k$: For scattering angles between 0° and 40° its accuracy is consistently within 10% if $ka \gtrsim 2$ as long as $|V_0|a/4k \lesssim 1$. This is more evident in Table III where numerical values of cross sections are listed for $V_0=-1$ to $-16$ at $k=2$ and $a=1$.

## VII. EXPONENTIAL POTENTIAL

I have further tested the present approximation scheme with Yukawa-type potentials, especially, the exponential potential of the type

$$V=V_0e^{-r/a}\,. \tag{40}$$

The minimization of $\Delta T_2$ for the exponential potential yields

$$\alpha=k \tag{41}$$

TABLE IV. Scattering amplitudes for exponential potential at $(V_0,k,a)=(-6,5,1/1.45)$.

| | 0° | | 30° | | 60° | | 90° | |
|---|---|---|---|---|---|---|---|---|
| | Real | Imaginary | Real | Imaginary | Real | Imaginary | Real | Imaginary |
| Exact | 3.8851(+0) | 5.376(−1) | 1.829(−1) | 1.267(−1) | 1.27(−2) | 1.67(−2) | 3.0(−3) | 4.1(−3) |
| Eq. (36) | 3.8849(+0) | 5.382(−1) | 1.833(−1) | 1.256(−1) | 1.21(−2) | 1.66(−2) | 2.2(−3) | 3.8(−3) |
| Glauber | 3.8563(+0) | 5.321(−1) | 1.900(−1) | 1.286(−1) | 1.55(−2) | 1.94(−2) | 3.9(−3) | 5.4(−3) |
| Second Born | 3.94(+0) | 5.43(−1) | 1.89(−1) | 1.36(−1) | 2.11(−2) | 2.15(−2) | 4.9(−3) | 6.0(−3) |

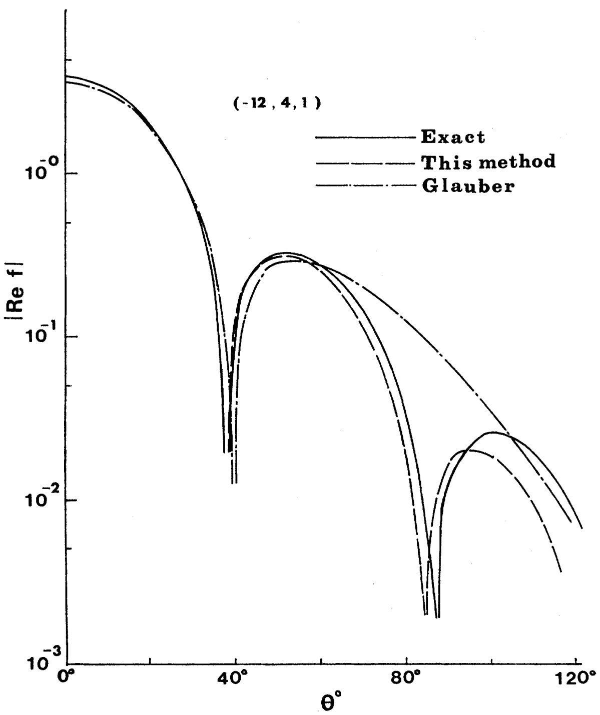


FIG. 3. The real part of scattering amplitude from Gaussian potential at $(V_0,k,a)=(-12,4,1)$. Note that the difference between the present method and the exact result is distinguishable only after 35°.

and

$$\kappa=\frac{1}{2ka^2}\left[\frac{4-3q^2a^2}{20+q^2a^2}\right]. \tag{42}$$

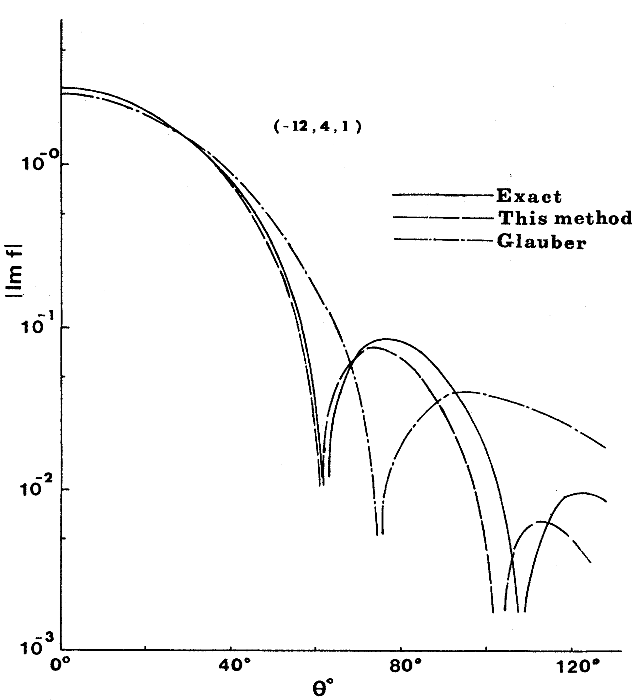


FIG. 4. The imaginary part of the amplitude of Fig. 3.

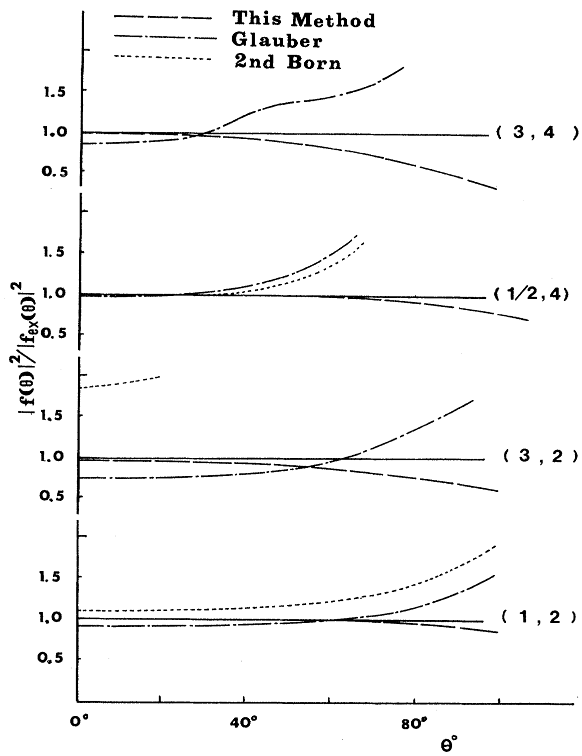


FIG. 5. The relative cross section, the ratio of the cross sections resulting from various approximation methods to the exact cross section, for scattering from Gaussian potentials. The numbers in the parentheses are the values for $(|V_0|a/k,ka)$. Note that the ratio of these two numbers gives the value for $|V_0|/k^2$.

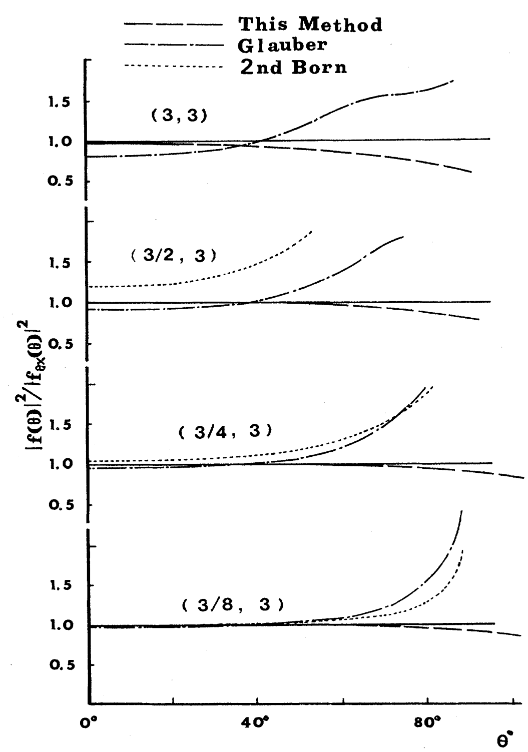


FIG. 6. The relative cross section for scattering from a Gaussian potential at fixed $ka$. The numbers in the parentheses are the values for $(|V_0|a/k,ka)$.

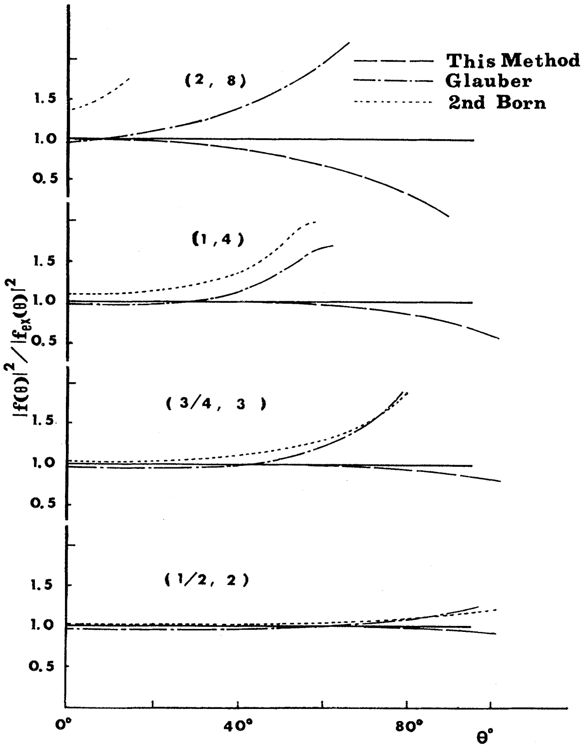


FIG. 7. The relative cross sections for Gaussian potentials at $|V_0|/k^2=\frac{1}{4}$. The numbers in the parentheses represent $(|V_0|a/k,ka)$.

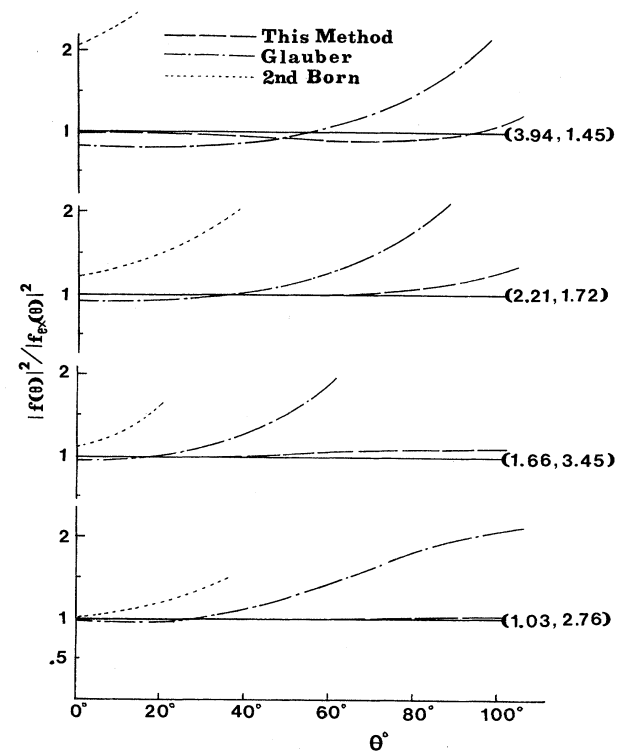


FIG. 8. The relative cross section from exponential potentials. The numbers in the parentheses are the values for $(|V_0|a/k,ka)$.

With the $\alpha$ and $\kappa$ so determined the approximate formula (36) is then computed for various values of $V_0$, $k$, and $a$. The results are again excellent. The improvement to the Glauber result is spectacular. Typical results are given in Table IV and Fig. 8. Scattering amplitudes resulting from (36) and the Glauber formula are compared with the exact ones in Table IV for $(V_0,k,a)=(-6,5,1/1.45)$. Both the real and imaginary parts from (36) agree excellently with the exact ones for angles up to about 70° in this case. In general, I found that the present method works for an exponential potential just as well as for a Gaussian potential. It works not only when $|V_0|/k^2 \ll 1$ and $ka \gg 1$, but also when $|V_0|a/4k \lesssim 1$ and $ka \gtrsim 1$. This is illustrated in Fig. 8, where the resulting cross sections relative to the exact ones are depicted for various values of $(|V_0|a/k,ka)$.

## VIII. CONCLUSION

Starting with the generalized eikonal propagator with adjustable parameters, I have obtained a new eikonal-type scattering amplitude which is shown to work excellently for elastic potential scattering when $|V_0|a/4k \lesssim 1$ and $ka \gtrsim 1$. The essential ingredient in the present approximation method is the determination of parameters such that more accurate low order Born terms are retained. Consequently, the present formula results in spectacular improvements of the usual eikonal formula in the high energy region where $|V_0|/k^2 \ll 1$ and $ka \gg 1$. More interestingly, the present formula continues to work well in the medium energy region where $|V_0|a/k \lesssim 4$ and $ka$ is slightly greater than one and promises to be useful in the studies of pion-nucleus scattering in the 100—300 MeV region.

*Note added in proof.* Instead of minimizing the error as is done in this paper, the parameters $\alpha$ and $\beta$ can also be determined by direct comparison with the second Born term in the case it is known. For Gaussian potentials, this yields slightly different but consistent results as discussed in a recent paper [T. W. Chen, Phys. Rev. D **29**, 1839 (1984)].

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[8]The choice $\Delta\alpha=\Delta\beta=0$ gives rise to the formula in Ref. 5 which is shown to work well for Gaussian potentials.