Three-body approach to the multiple scattering direct reactions: Statistical theory of the continuum spectrum

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The physical idea behind the present approach is that the (a,ab) reaction continuum consists mainly of those particles a,b, following a quasifree knockout collision, which undergo an energy degradation due to rescattering on their way out from the target nucleus. The Feshbach, Kerman, and Koonin statistical two-body theory of the (a,b) reaction continuum is then extended to also describe this exclusive continuum process. A computable expression for the direct (a,ab) continuum cross section is deduced as a convolution integral over a doorway quasifree cross section and a number of factors describing the probability of multiple rescattering of the quasifree particles on the residual nucleus. The resemblance between the two-body and the three-body expressions is remarkable and reflects their common physical support.

I. INTRODUCTION

To account for the continuum spectrum in nuclear reactions would appear at first sight to be a formidable, if not impossible, formal as well as numerical task. However, the pioneering work of Griffin and many other investigations[1] which followed demonstrated that the job can be approached using mostly the physics learned from the earlier studies of reactions leading to the excitation of discrete nuclear states. The basic idea which underlies any of these theories is the multi-step (MS) character of the microscopic nuclear collisions which determines a hierarchy of nuclear excitations over the duration of the macroscopic interaction.

For instance, in the exciton model of Griffin, the projectile, say a nucleon, strikes a nucleon in the target nucleus thereby causing a two-particle—one-hole (2p-1h) excitation, one of the particles being the projectile itself; as the initial energy brought in by the projectile continues to dissipate inside the target nucleus, subsequent nucleon-nucleon (NN) collisions can lead to 3p-2h, 4p-3h, etc., excitations. At each stage, one or more nucleons may escape into the final channel thus producing the continuum spectrum of the reaction. Eventually, the projectile initial energy spreads completely over a large number of degrees of freedom of the nuclear system such that a statistical equilibrium of the energy distribution is achieved. This is the extreme compound nuclear state which evaporates particles isotropically in the low-energy part of the continuum spectrum. Toward higher energies, the compound particle emission becomes anisotropic, but still symmetric about 90° c.m., and is eventually overwhelmed by the direct particle emission peaking in the forward direction. Thus the entire continuum can be thought to be composed of an equilibrium (compound-evaporation) spectrum plus a preequilibrium (direct + compound) MS spectrum, obtained by collecting the contributions from all the intermediate np-$(n-1)h$ interaction stages. The difference between various theories lies ultimately in how these stages are modulated and summed together into a cross section.

All of the activity outlined above has been devoted to the study of the continuum spectrum in the inclusive (a,b) reactions, such as (p,p′) or (p,n). In contrast, still very little has been done to understand the continuum coincidence spectrum in the exclusive (a,ab) quasifree (QF) knockout reactions. Until now it has been customary in the analyses of knockout reactions, such as (e,ep) or (p,2p), leading to deep hole states lying in the continuum, to draw smooth background lines in an arbitrary manner. However, there are reasons to expect that the continuum background can be structured[2] and thus mock up some direct knockout characteristics. Such occurrences are obviously unwanted, but could perhaps be accounted for if one had a formal description of the continuum background. Having such a formalism appears even more desirable if one realizes that, in general, the continuum represents a large fraction of all measured events and in many cases could exceed the number of QF single-step (SS) events corresponding to discrete nuclear states. For example, a simulated Monte Carlo calculation[2] for the quasifree (e,ep) scattering at 700 MeV on ^{12}C indicates that 35% (57%) of the protons knocked out from the $0p_{3/2}$ ($0s_{1/2}$) state eventually end up in the continuum.

The present paper reports, apparently for the first time, on an attempt to formulate a three-body theory of the preequilibrium direct continuum spectrum. The aim will be to obtain a simple framework capable of providing not only theoretical developments but also a computable basis for comparison with the experiment. It will be shown that this task can be achieved by extending the Feshbach, Kerman, and Koonin (FKK)[1] statistical two-body theory of the inclusive (a,b) spectra to also describe the continuum direct (a,ab) processes. It will be pointed out that the formalism presented here cannot be further extended, in the general case, to the (a,abc) reaction continuum.

II. MULTI-STEP THREE-BODY MODEL FOR THE DIRECT (a,ab) CONTINUUM SPECTRUM

The continuum direct reaction formalism which will be developed here bases its premise on the multi-scattering model which was considered recently[3] in order to explore the role of inelastic scattering in the QF processes. Before we start to explain the details of the model we wish to introduce a general framework for the description of such reactions. In the simplest $A(a,ab)S$ QF process, the projectile particle with atomic number a and mass m_0 approaching with energy E_0 and momentum $\vec{k}_0$, after being elastically scattered from the target nucleus with atomic number A, knocks out a particle with atomic number b and mass m_b residing with momentum $\vec{k}_\nu$ and binding energy E_ν in a certain quantum state ν inside the target nucleus. Then, the QF particles, i.e., the projectile a and the knocked out nuclear particle b, scatter elastically from the residual nucleus with atomic numer $S=A-b$ into the exit channel with final energies E_1 and E_2 and momenta $\vec{k}_1$ and $\vec{k}_2$.

We make here the usual assumption that the core S, to which the particle b is initially bound in the target nucleus, is identical to the residual nucleus, i.e., it remains merely a spectator with momentum $\vec{k}_S=-\vec{k}_\nu$ which just

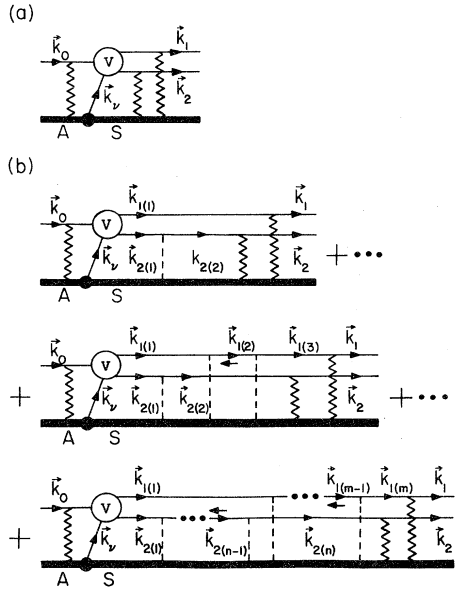

FIG. 1. Diagrammatic representation of the $A(a,ab)S$ knockout scattering. (a) The single-step three-body scattering. The zig-zag line indicates the particle interaction with the optical potential. The heavy line indicates the nucleus, the black dot the struck particle-spectator relative motion wave function ϕ_ν, and the open circle a two-body type $v=\sum_{i\in a,j\in A}v(i,j)$ nuclear interaction. (b) The multi-step, three-body scattering. The dashed lines indicate the interaction $v=\sum_{i\in a,b;j\in S}v(i,j)$ of the QF unbound particles with the S nucleus which produces certain modes, e.g., p-h, of nuclear excitations. The arrows pointing backwards indicate possible retrograde collisions of particles taking up energy from the nucleus, although these are discarded in the text on statistical likelihood grounds. The numbers in parentheses denote the scattering stages of the particles a and b on the nucleus S.

recoils after the knockout collision. Thus, the particles a and b collide as if they were free, except for distortion effects and three-body kinematics. In this single-step reaction, which is represented diagrammatically in Fig. 1(a), the QF particles will be found along a kinematic locus resulting from the energy conservation condition

$$E_1+E_2=E_0+E_\nu^\alpha-E_S\ , \tag{1a}$$

where E_S is the recoil energy of the residual nucleus S and the binding energy E_ν^α is the sum of the ground state (g.s.) binding energy ϵ_ν^0 of particle b and the excitation energy ϵ_α of the residual nucleus found in a quantum state α, i.e.,

$$E_\nu^\alpha=\epsilon_\nu^0+\epsilon_\alpha\ . \tag{1b}$$

However, not all the QF particles will lie on the kinematic locus (1a) when they leave the nucleus. A certain fraction of them which emerge from the first QF collision with the energies $E_{(1)}\equiv E_{1(1)},E_{2(1)}$ related by Eq. (1a), and the momenta $\vec{k}_{(1)}\equiv\vec{k}_{1(1)},\vec{k}_{2(1)}$ related by momentum conservation, will experience a series of subsequent collisions with the nucleus S. (Note the convention which will be used throughout the paper of labeling various quantities with the appropriate scattering stage number in parentheses.) These collisions can degrade the energies and momenta of the particles which, by the time they are finally scattered elastically into the exit channel, become $E_{(f)}\equiv E_1,E_2$ and $\vec{k}_{(f)}\equiv\vec{k}_1,\vec{k}_2$, and thereby no longer lie on the original kinematic locus (1a). We take the point of view that the continuum spectrum in the knockout scattering results mainly from these multi-step nuclear interactions initiated by a QF three-body collision, acting here as a doorway phenomenon, as represented diagrammatically in Fig. 1(b). Another source for the continuum events could also be the multi-nucleon decay of the deep-hole states formed by the nuclear QF doorway scattering, but this has been shown[4] to be of a relatively minor importance and therefore will not be considered here.

The intention now is to implement these ideas within a theoretical framework which should be as close to being computable as possible. The exact transition amplitude from the initial channel i to the final channel f of the direct knockout scattering can be conveniently divided into the single-step $T_{fi}^{(SS)}$ and multi-step $T_{fi}^{(MS)}$ amplitudes

$$T_{fi}=T_{fi}^{(SS)}+T_{fi}^{(MS)}\ . \tag{2}$$

In the standard two-potential scattering theory,[5] this is written as

$$T_{fi}=\langle f\,|\,v+vGv\,|\,i\rangle\ , \tag{3}$$

where v represents the interaction giving rise to the direct excitation of the target nucleus, and

$$G=(E_0+i\eta-H)^{-1}\ , \tag{4}$$

with the implicit limit $\eta\to 0^+$, is the scattering Green's function corresponding to the energy-shell E_0 of the nuclear system described by the Hamiltonian operator

$$H=H_1+H_2+H_S+W_1+W_2+\sum_{i<j\in a,b,S}v(i,j)\ . \tag{5}$$

Consistent with the general practice of the two-potential scattering formalism, the above division consists of terms describing the kinetic and internal motion of the QF particles and the residual nucleus (H_1, H_2, and H_S), the elastic scattering of the QF particles (W_1 and W_2), and the residual particle interaction of the two-body type (v). Since the final-state interaction between the QF particles after the doorway stage will be neglected [in which case the residual interaction will be taken as

$$\sum_{i\in a,b;j\in S} v(i,j) ,$$

an approximation probably very good for lighter QF particles] the Hamiltonian (5) is separable and therefore the Green's function (4) is solvable in terms of known two-body states[6] for the rescattering stages.

It is the first term of Eq. (3), represented by the diagram in Fig. 1(a), that is usually considered in the standard distorted-wave impulse approximation (DWIA) calculations to account for the single-step QF events from the kinematic loci (1a). Overlooking detailed antisymmetry requirements and suppressing the explicit reference to the spin and isospin quantum numbers, this transition amplitude essentially has the form[7]

$$T_{fi}^{(SS)}=\sum_{\nu}\langle\chi_1^{(-)}\chi_2^{(-)}\Phi_1\Phi_2\Phi_3\,|\,v\,|\,\Phi_0\Phi_A\chi_0^{(+)}\rangle \times(2\pi)^3\delta(\vec{k}_1+\vec{k}_2-\vec{k}_0-\vec{k}_\nu) , \quad (6)$$

where $\chi_0^{(+)}$, $\chi_1^{(-)}$, and $\chi_2^{(-)}$ are the distorted-wave functions describing the elastic scattering of the projectile and the outgoing QF particles on the target A and on the nucleus S, respectively; the $(+)$ and $(-)$ superscripts indicate incoming and outgoing boundary conditions, respectively; and Φ_0, Φ_A, Φ_1, Φ_2, and Φ_S are the internal motion wave functions of the projectile a, the target A, the QF particles a and b after the doorway knockout collision, and the residual nucleus S, respectively. The momentum conserving $(2\pi)^3\delta$-function results from the integration over the plane wave describing the c.m. motion of the whole a-A nuclear system.

We now wish to project out the virtual cluster b wave function, before the knockout, from the target wave function Φ_A. The usual procedure is to perform a cluster fractional parentage expansion[8] as

$$\Phi_A(\{\vec{r}_A\})=\sum_{\nu,\lambda,\gamma} A_{\nu\lambda\gamma}\phi_\nu(\vec{R}) \times\Phi_{b\lambda}(\{\vec{r}_b\})\Phi_{S\gamma}(\{\vec{r}_{A\text{-}b}\}) , \quad (7)$$

where $A_{\nu\lambda\gamma}$ are the expansion coefficients, ϕ_ν is the b-S relative motion wave function, and $\Phi_{b\lambda}$ and $\Phi_{S\gamma}$ are the internal motion wave functions of the virtual cluster and the spectator nuclear fragment, respectively. Next, the overlap of the spectator wave functions before and after the knockout renders the quantum numbers γ identical to the observed quantum numbers α describing the residual state; also, the quantum numbers λ will be taken to coincide with those of the g.s. of particle b ($\Phi_{b\lambda}\equiv\Phi_{b(\text{g.s.})}$), assumed to remain unchanged during the reaction.

Thus, Eq. (6) may now be written as

$$T_{fi}^{(SS)}=\left\langle\chi_1^{(-)}\chi_2^{(-)}\left|\bar{v}\right|\sum_\nu f_\nu\chi_0^{(+)}\right\rangle , \quad (8)$$

where we introduced the new quantity

$$f_\nu(\vec{R})=A_{\nu(\lambda\gamma)}\phi_\nu(\vec{R})(2\pi)^3\delta(\vec{k}_1+\vec{k}_2-\vec{k}_0-\vec{k}_\nu) \quad (9)$$

with the quantum numbers $\lambda\gamma$ in the parentheses fixed as specified above. The integral over the internal nucleon coordinates $\{\ \}$ produces the two-body pseudopotential

$$\bar{v}(\vec{r})\equiv\langle\Phi_1\Phi_2\,|\,v\,|\,\Phi_0\Phi_b\rangle , \quad (10)$$

where $\Phi_1\equiv\Phi_0$ and $\Phi_2\equiv\Phi_b$, and r is the distance between the c.m. of the QF particles. In practice, instead of using this double-folding interaction which would require some technical manipulations,[9] one usually prefers[7] an impulse zero-range approximation of (10), such as $t(\vec{q})\delta(\vec{r})$, where $t(\vec{q})$ is the t matrix for the free $a+b$ scattering evaluated at the asymptotic final relative momentum $\vec{q}=\frac{1}{2}(\vec{k}_1-\vec{k}_2)$ of the QF particles. This prescription, together with the factorization of the free $a+b$ collision cross section, although raising questions which are still being investigated,[7] reduces the problem of obtaining the single-step QF cross section to a quite tractable integration.

The object of our interest here is, however, the multi-step amplitude $T_{fi}^{(MS)}$, also represented by the diagrams of Fig. 1(b), which we expect to describe the three-body continuum events, with source contributions from a certain number of possible orbitals of particle b. Presumably, we are to associate these events with the second term of Eq. (3), which has the three-body form

$$T_{fi}^{(MS)}=\left\langle\chi_1^{(-)}\chi_2^{(-)}\left|vGv\right|\sum_\nu f_\nu\chi_0^{(+)}\right\rangle , \quad (11)$$

in which the iteration of $G=G_\alpha+G_\alpha v_\alpha G$ leads to the chain of scattering processes

$$G=G_\alpha+G_\alpha v_\alpha G_\beta+G_\alpha v_\alpha G_\beta v_\beta G_\gamma+\cdots$$

involving an undefined number of intermediate nuclear states $\alpha,\beta,\gamma,\ldots$. Since the QF particles undergo a sequence of scatterings from one target particle to another before emerging from the nucleus, it would be desirable to describe process (11) in terms of such successive scatterings occurring, most probably, through NN collisions. We are thus led, almost inevitably, to the FKK (Ref. 1) statistical theory of the multi-step direct implicit reactions, which will be extended next to the (a,ab) knockout reactions.

III. FUNDAMENTAL MODEL ASSUMPTIONS

Consistent with the FKK theory, we define the projection operators P and Q, such that

$$P^2=P,\ \ Q^2=Q,\ \ P+Q=1,\ \ PQ=QP=0 , \quad (12)$$

to divide the Hilbert space into two orthogonal subspaces, $\mathscr{P}$ and $\mathscr{Q}$. $\mathscr{P}$ contains the open channels on the energy shell E_0 with at least two particles in the continuum and the initial channel with just the projectile particle in the continuum, and $\mathscr{Q}$ contains the closed channels with all the particles of the nuclear system bound. Each of these

subspaces are then partitioned again into sets of mutually orthogonal subspaces $\mathcal{P}_n$ and $\mathcal{Q}_n$ such that, loosely speaking, increasing n corresponds to increasing their complexity. What this really means depends on the specific representation chosen for the nuclear states, e.g., the shell model, the vibrational model, etc. Thus, in the shell-model framework we will find it convenient to use the p-h language such that $\mathcal{P}_n$ and $\mathcal{Q}_n$ will be the subspaces of the $(n+1)p$-nh states.

To make the formalism computable by reasonable means, the two fundamental FKK assumptions are also adopted here:

(i) *The chaining hypothesis.* It is assumed that, in both $\mathcal{P}$ and $\mathcal{Q}$ subspaces, the residual interaction v can induce transitions from the nth stage only to the $(n\pm1)$st stages, or on to the final state through elastic scattering; it is not possible to miss a stage and effect the $n\rightarrow n\pm2$ transition, for example. This condition will be expressed for the open channel subspace $\mathcal{P}$ as

$$v_{nm}=0 \text{ if } |n-m|\geq 2 , \tag{13}$$

with

$$v_{nm}\equiv P_n v P_m , \tag{14}$$

where P_m and P_n are the projection operators on the $\mathcal{P}_m$ and $\mathcal{P}_n$ subspaces, respectively.

Using this assumption we see immediately that Eq. (3) can be written like its FKK two-body equivalent as

$$T_{fi}=v_{fi}+\sum_n T_{fi}^{(n)} , \tag{15}$$

where

$$v_{fi}\equiv P_f v P_i \tag{16}$$

uses the projectors P_i and P_f on the initial and final channel wave functions, respectively, which operate in the $\mathcal{P}_i$ and $\mathcal{P}_f$ subspaces belonging to $\mathcal{P}$. Furthermore, the nth scattering stage amplitude

$$T_{fi}^{(n)}=v_{fn}G_n v_{n,n-1}G_{n-1}\cdots v_{21}G_1 v_{1i} \tag{17}$$

uses the continuum FKK propagators

$$G_n=P_n\frac{1}{E_0+i\eta-H_{nn}^{(d)}-v_{n,n+1}G_{n+1}v_{n+1,n}}P_n \tag{18}$$

with

$$H_{nn}^{(d)}\equiv P_n H^{(d)} P_n , \tag{19}$$

where

$$H^{(d)}=H_1+H_2+H_S+W_1+W_2 \tag{20}$$

is the part of Hamiltonian (5) diagonal with respect to the disorted channel wave functions.

The chained transition amplitude (17) can also be used for the multi-step compound reaction if the projectors P_n are replaced by Q_n. But for either reaction, direct or compound, one could make in practice the "never-come-back" assumption that retrograde transitions $|n\rangle \rightarrow |n-1\rangle$, depicted by backward arrows in Fig. 1(b), can be neglected. The physical grounds for this assumed irreversible scattering through the sequence of stages n is the rapid increase of the nuclear level density with increasing complexity (or equivalently, n) of the configurations from the subspaces $\mathcal{P}_n$ or $\mathcal{Q}_n$. Therefore, up to a point, it is more likely that the particle scattering will lead to exciting, rather than deexciting, the residual nucleus.

(ii) *The random phase hypothesis.* This is the statistical assumption that the relative phases of matrix elements are random, such that the only interference direct terms surviving upon an averaging over the excitation energy U of the residual nucleus S will be those involving the same change in the momentum of the particles in the continuum. In contrast, no interference compound terms will remain upon such averaging. To be more specific, we should stress the point that, aiming at a theory of the continuum, we are not concerned here with cross sections for reactions leading to individual final states, which are not experimentally measurable, but rather with energy averaged cross sections corresponding to groups of final states characterized by the same quantum numbers such as the angular momentum parity or the channel spin. These states $\Psi_{ne}^{(-)}$ involving only those channels with particular momenta $\vec{k}_1$ and $\vec{k}_2$ of the emergent QF particles, which represent the exit modes e from a stage n, are eigenfunctions of the $H_{nn}\equiv P_n H P_n$ Hamiltonian. They form an expansion basis for the final state eigenfunction $\Psi_f^{(-)}$ of the $H_{PP}\equiv PHP$ Hamiltonian of the continuum as

$$\Psi_f^{(-)}(\vec{k}_1,\vec{k}_2)=\sum_n\sum_{e=n-1}^{n+1} C_{ne}^{(f)}\Psi_{ne}^{(-)}(\vec{k}_1,\vec{k}_2) , \tag{21}$$

where the expansion coefficients $C_{ne}^{(f)}$ are assumed to have random signs. Here, introducing the $e=n,n-1$ exit modes we mimic the FKK paper although, to be consistent with our never-come-back assumption, only the $e=n+1$ exit mode should actually be considered, even though this may be less justified for larger n's.

Invoking the above-mentioned statistical assumptions and using the FKK chaining identity, the sum over the final states of the modulus squared of the multi-step reaction amplitude from Eq. (15) then becomes

$$\sum_{n,n',f} T_{fi}^{(n)*}T_{fi}^{(n')}=\sum_n\sum_{e=n-1}^{n+1}|v_{ne}G_n v_{n,n-1}G_{n-1}\cdots v_{21}G_1 v_{1i}|^2 . \tag{22}$$

In order to obtain the projection operators for the above equation we will use the eigenstates $\Psi_{\alpha_n}^{(-)}$ of the effective Hamiltonian associated with each Green's operator (18) through the Schrödinger equation

$$[H_{nn}^{(d)}+v_{n,n+1}G_{n+1}v_{n+1,n}]\Psi_{\alpha_n}^{(-)}=[k_{1(n)}^2\hbar^2/(2m_1)+k_{2(n)}^2\hbar^2/(2m_2)+\epsilon_{\alpha_n}]\Psi_{\alpha_n}^{(-)} , \tag{23}$$

where, overlooking the Pauli principle, we take

$$\Psi_{\alpha_n}^{(-)}=\chi_{1(n)}^{(-)}(\vec{k}_{1(n)},\vec{r}_1)\chi_{2(n)}^{(-)}(\vec{k}_{2(n)},\vec{r}_2)\Phi_1(\{\xi_a\})\Phi_2(\{\xi_b\})\Phi_{S\alpha_n}(\{\xi_A\text{-}b\}) , \tag{24}$$

with the distorted wave functions $\chi_{1(n)}^{(-)}$ and $\chi_{2(n)}^{(-)}$ describing the elastic scattering of the QF particles, Φ_1 and Φ_2 describing the internal motion of the QF particles which are assumed to remain in their ground states throughout the scattering process, and $\Phi_{S\alpha_n}$ describing the intrinsic state α_n with energy ϵ_{α_n} of the residual nucleus after the nth scattering stage.

The distorted wave functions obey the following normalization conditions:

$$\int d\vec{r}\,\tilde{\chi}^{(\pm)*}(\vec{k}',\vec{r})\chi^{(\pm)}(\vec{k},\vec{r})=(2\pi)^3\delta(\vec{k}-\vec{k}') , \tag{25a}$$

$$\int d\vec{r}\,\tilde{\chi}^{(\pm)*}(\vec{k}',\vec{r})\tilde{\chi}^{(-)}(\vec{k},\vec{r})=(2\pi)^3k^{-2}\delta(k-k')\sum_l S_l^{-1}(Y^l(\Omega_{\vec{k}'})\cdot Y^l(\Omega_{\vec{k}})) , \tag{25b}$$

where S_l is the elastic scattering S-matrix element corresponding to a particular partial wave l, and the tilde denotes a time reversal operation; also, the normalization of the intrinsic wave functions is given by the equations

$$\sum_{\{\sigma_a\},\{\tau_a\}}\int\prod_{i=1}^{a}d\vec{r}_i\,\delta\left[\frac{1}{a}\sum_{i=1}^{a}\vec{r}_i\right]\Phi_1^*(\{\xi_a\})\Phi_1(\{\xi_a\})=1 , \tag{26a}$$

$$\sum_{\{\sigma_b\},\{\tau_b\}}\int\prod_{j=1}^{b}d\vec{r}_j\,\delta\left[\frac{1}{b}\sum_{j=1}^{b}\vec{r}_j\right]\Phi_2^*(\{\xi_b\})\Phi_2(\{\xi_b\})=1 , \tag{26b}$$

$$\sum_{\{\sigma_{A-b}\},\{\tau_{A-b}\})}\int\prod_{m=1}^{A-b}d\vec{r}_m\,\delta\left[\frac{1}{A-b}\sum_{m=1}^{A-b}\vec{r}_m\right]\Phi_{S\alpha_n'}^*(\{\xi_{A\text{-}b}\})\Phi_{S\alpha_n}(\{\xi_{A\text{-}b}\})=\delta_{\alpha_n,\alpha_n'} , \tag{26c}$$

where $\{\xi\}\equiv\{\sigma,\tau,\vec{r}\}$ represents the aggregate of spin, isospin, and position coordinates of the nucleons constituting the nuclei. The δ-function constraint ensuring the c.m. translational invariance in Eqs. (26) is usually discarded in the shell model framework which should normally be employed for heavier nuclei.

Due to the fact that the relative motion distorted wave functions $\chi^{(\pm)}$ are not orthogonal, it is necessary to use a biorthogonal set of states (24) to build the projection operators P_n, needed in Eq. (22), as

$$P_n=(2\pi)^{-6}\,|\,\tilde{\Psi}_{\alpha_n}^{(-)}\rangle\langle\Psi_{\alpha_n}^{(-)}\,|\ . \tag{27}$$

Unfortunately, the insertion of these operators in Eq. (22) does not give an expression composed of transition (T-type) matrix elements. One possible way out would be to use the modified projector form[10]

$$P_n=(2\pi)^{-6}S^{-1}\,|\,\Psi_{\alpha_n}^{(+)}\rangle\langle\Psi_{\alpha_n}^{(-)}\,|\ , \tag{28}$$

based on the known relations of the disorted waves

$$\chi^{(+)}(\vec{k},\vec{r})=4\pi/(kr)\sum_l i^l\chi_l(k,r)(Y^l(\Omega_{\vec{k}})\cdot Y^l(\Omega_{\vec{r}})) , \tag{29a}$$

$$\tilde{\chi}^{(-)}(\vec{k},\vec{r})=4\pi/(kr)\sum_l i^lS_l^{-1}\chi_l(k,r)\times(Y^l(\Omega_{\vec{k}})\cdot Y^l(\Omega_{\vec{r}})) . \tag{29b}$$

The above technique does not seem to reveal any divergence problems;[11] however, it is not appealing from a numerical point of view. While this question is not yet resolved, we will continue to use in the interim the FKK implicit assumption noted by Tamura *et al.*;[1] namely, we will take $S_l=1$ for all partial waves l, even though this approximation tends to underestimate the reaction cross section. Hopefully, in practice, this effect might be diluted in the present treatment by the other approximations with opposing effects. In any event, the result of a recent investigation[12] points out that the above approximation is basically legitimate.

IV. STATISTICAL TREATMENT OF THE MULTI-STEP DIRECT (a,ab) REACTION

To illustrate the calculation of Eq. (22) we will first consider the following factor of the summand

$$\begin{aligned}v_{1i}^*G_1^*v_{21}^*v_{21}G_1v_{1i}=&\int\int\frac{d\vec{k}_{1(1)}}{(2\pi)^3}\frac{d\vec{k}_{2(1)}}{(2\pi)^3}\int\int\frac{d\vec{k}'_{1(1)}}{(2\pi)^3}\frac{d\vec{k}'_{2(1)}}{(2\pi)^3}\sum_{\nu,\nu'}\sum_{\alpha_1,\alpha_1'}v_{\alpha_1'i}^*(\vec{k}'_{1(1)},\vec{k}'_{2(1)};\vec{k}_0,\vec{k}_{\nu'})v_{\alpha_2'\alpha_1'}^*(\vec{k}'_{1(2)},\vec{k}'_{2(2)};\vec{k}'_{1(1)},\vec{k}'_{2(1)})\\&\times v_{\alpha_2\alpha_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})v_{\alpha_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)\\&\times[E_0-i\eta-\vec{k}'^2_{1(1)}\hbar^2/(2m_1)-\vec{k}'^2_{2(1)}\hbar^2/(2m_2)-\epsilon_{\alpha_1'}]^{-1}\\&\times[E_0+i\eta-k_{1(1)}^2\hbar^2/(2m_1)-k_{2(1)}^2\hbar^2/(2m_2)-\epsilon_{\alpha_1}]^{-1} ,\end{aligned} \tag{30}$$

related to the first stage, namely the QF doorway scattering. The above matrix elements, which will be written explicitly later, are regular T-type matrices of the two-body interaction v obtained using the projectors (28) with $S=1$. Further, following the FKK prescription, we again invoke the random phase hypothesis (ii) to take $\nu=\nu'$ and $\alpha_1=\alpha_1'$, and then average the integrand (30) over the energies ϵ_{α_1} of the residual nucleus states α_1 as

$$I_{\alpha_1}=\int d\epsilon_{\alpha_1}\rho_S(\epsilon_{\alpha_1})M_{(1)}(\epsilon_{\alpha_1})G_1^*G_1 \ . \tag{31}$$

By applying Cauchy's theorem of residues we immediately obtain

$$I_{\alpha_1}=-\rho_S(U_{(1)})M_{(1)}(U_{(1)})2\pi i/[(k_{1(1)}^2-\vec{k}\,'^2_{1(1)})\hbar^2/(2m_1)+(k_{2(1)}^2-\vec{k}\,'^2_{2(1)})\hbar^2/(2m_2)+2i\eta] \ , \tag{32}$$

where $\rho_S(\epsilon_{\alpha_1})$ is the density of residual nuclear states at the excitation energy ϵ_{α_1} after the QF first stage, $M_{(1)}$ is the product of matrices $v^*_{\alpha_1' i}v^*_{\alpha_2'\alpha_1'}v_{\alpha_2\alpha_1}v_{\alpha_1 i}$ assumed to vary slowly in the neighborhood of the propagator singularities, and

$$U_{(1)}\equiv E_0-k_{1(1)}^2\hbar^2/(2m_1)-k_{2(1)}^2\hbar^2/(2m_2) \tag{33}$$

is the fraction of the available energy dissipated over the degrees of freedom of the residual nucleus. According to Eqs. (1), in the case of the doorway QF stage this is just the residual excitation energy ϵ_{α_1} rescaled by the ground state binding energy ϵ_ν^0 of the knocked out particle and the recoil energy E_S of the residual nucleus S.

In order to proceed further, following the FKK method we assume a slow variation of $\rho_S M_{(1)}$ with $k_{1(1)}$ and $k_{2(1)}$ so that a pole approximation can be made for I_{α_1} to obtain

$$I_{\alpha_1}=2\pi^2\rho_S(U_{(1)})M_{(1)}(U_{(1)})\delta[(k_{1(1)}^2-k_{1(1)}'^2)\hbar^2/(2m_1)+(k_{2(1)}^2-k_{2(1)}'^2)\hbar^2/(2m_2)] \ . \tag{34}$$

For the moment, it is convenient to switch representation to the total momentum

$$\vec{Q}_{(1)}\equiv\vec{k}_{1(1)}+\vec{k}_{2(1)}, \quad \vec{Q}\,'_{(1)}\equiv\vec{k}\,'_{1(1)}+\vec{k}\,'_{2(1)} \tag{35a}$$

and the relative momentum

$$\vec{q}_{(1)}\equiv(m_2\vec{k}_{1(1)}-m_1\vec{k}_{2(1)})/m, \quad \vec{q}\,'_{(1)}\equiv(m_2\vec{k}\,'_{1(1)}-m_1\vec{k}\,'_{2(1)})/m \tag{35b}$$

of the QF particles, where $m=m_1+m_2$, although only the primed quantities will actually be needed for the integration in Eq. (30).

Using this notation and the reduced mass μ of the QF particles

$$\mu=m_1m_2/m \ , \tag{36}$$

the quantity (31) can be written as

$$I_{\alpha_1}=2\pi^2\rho_S(U_{(1)})M_{(1)}(U_{(1)})\delta[(q_{(1)}^2-q_{(1)}'^2)\hbar^2/(2\mu)+(Q_{(1)}^2-Q_{(1)}'^2)\hbar^2/(2m)] \ . \tag{37}$$

Recalling definition (9) we separate the momentum conservation conditions from the matrix elements as

$$v_{\alpha_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)\equiv v^{(\nu)}_{\alpha_1 i}(\vec{Q}_{(1)},\vec{q}_{(1)};\vec{k}_0,\vec{k}_\nu)(2\pi)^3\delta(\vec{Q}_{(1)}-\vec{k}_0-\vec{k}_\nu) \ , \tag{38a}$$

$$v_{\alpha_1 i}(\vec{k}\,'_{1(1)},\vec{k}'_{2(1)};\vec{k}_0,\vec{k}_\nu)\equiv v^{(\nu)}_{\alpha_1 i}(\vec{Q}\,'_{(1)},\vec{q}\,'_{(1)};\vec{k}_0,\vec{k}_\nu)(2\pi)^3\delta(\vec{Q}\,'_{(1)}-\vec{k}_0-\vec{k}_\nu) \ . \tag{38b}$$

Having made these rearrangements, the integral over $\vec{Q}\,'_{(1)}$ can easily be performed to obtain the result

$$\begin{aligned} v^*_{1i}G_1^*v^*_{21}v_{21}G_1v_{1i} = \int\int \frac{d\vec{k}_{1(1)}}{(2\pi)^3}\frac{d\vec{k}_{2(1)}}{(2\pi)^3}\int\frac{d\vec{q}\,'_{(1)}}{(2\pi)^3}2\pi^2\rho_S(U_{(1)})\sum_\nu\sum_{\alpha_1}{}' v^{(\nu)*}_{\alpha_1 i}(\vec{Q}_{(1)},\vec{q}\,'_{(1)};\vec{k}_0,\vec{k}_\nu)v^*_{\alpha_2'\alpha_1}(\vec{k}\,'_{1(2)},\vec{k}\,'_{2(2)};\vec{Q}_{(1)},\vec{q}\,'_{(1)}) \\ \times v_{\alpha_2\alpha_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{Q}_{(1)},\vec{q}_{(1)})v^{(\nu)}_{\alpha_1 i}(\vec{Q}_{(1)},\vec{q}_{(1)};\vec{k}_0,\vec{k}_\nu) \\ \times(2\pi)^3\delta(\vec{Q}_{(1)}-\vec{k}_0-\vec{k}_\nu)\delta[(q_{(1)}^2-q_{(1)}'^2)\hbar^2/(2\mu)] \ , \end{aligned} \tag{39}$$

which will further become

$$v_{1i}^{*}G_1^{*}v_{21}^{*}v_{21}G_1v_{1i}$$

$$=\int\int\frac{d\vec{k}_{1(1)}}{(2\pi)^3}\frac{d\vec{k}_{2(1)}}{(2\pi)^3}\int d\Omega'_{(1)}\frac{\mu q_{(1)}}{4\pi\hbar^2}\rho_S(U_{(1)})\sum_{\nu}\sum_{\alpha_1}{}'v_{\alpha_1 i}^{(\nu)*}(\vec{Q}_{(1)},q_{(1)}\Omega'_{(1)};\vec{k}_0,\vec{k}_\nu)v^{*}_{\alpha'_2\alpha_1}(\vec{k}\,'_{1(2)},\vec{k}\,'_{2(2)};\vec{Q}_{(1)},q_{(1)}\Omega'_{(1)})$$
$$\times v_{\alpha_2\alpha_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{Q}_{(1)},\vec{q}_{(1)})v^{(\nu)}_{\alpha_1 i}(\vec{Q}_{(1)},\vec{q}_{(1)};\vec{k}_0,\vec{k}_\nu)$$
$$\times(2\pi)^3\delta(\vec{Q}_{(1)}-\vec{k}_0-\vec{k}_\nu)\,, \tag{40}$$

where $\sum'_{\alpha_1}$ includes only those configurations in $\mathscr{P}_1$ whose energies equal $U_{(1)}$. Continuing to follow the FKK procedure, we now invoke the random phase hypothesis (ii) to define a new averaged expression as

$$\sum_{\alpha_1}{}'v_{\alpha_1 i}^{(\nu)*}(\vec{Q}_{(1)},q_{(1)}\Omega'_{(1)};\vec{k}_0,\vec{k}_\nu)v^{*}_{\alpha'_2\alpha_1}(\vec{k}\,'_{1(2)},\vec{k}\,'_{2(2)};\vec{Q}_{(1)},q_{(1)}\Omega'_{(1)})v_{\alpha_2\alpha_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{Q}_{(1)},\vec{q}_{(1)})v^{(\nu)}_{\alpha_1 i}(\vec{Q}_{(1)},\vec{q}_{(1)};\vec{k}_0,\vec{k}_\nu)$$
$$=\delta(\Omega'_{(1)}-\Omega_{(1)})\,|\,v^{(\nu)}_{\bar{\alpha}_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)\,|^2 v^{*}_{\alpha'_2\bar{\alpha}_1}(\vec{k}\,'_{1(2)},\vec{k}\,'_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})v_{\alpha_2\bar{\alpha}_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{k}_{1(11)},\vec{k}_{2(1)})\,, \tag{41}$$

where the bar over α_1 indicates an averaging over the states corresponding to the quantum numbers α_1.

As a result of these manipulations, Eq. (40) finally becomes

$$v_{1i}^{*}G_1^{*}v_{21}^{*}v_{21}G_1v_{1i}=\int\int\frac{d\vec{k}_{1(1)}}{(2\pi)^3}\frac{d\vec{k}_{2(1)}}{(2\pi)^3}2\pi^2\rho(\vec{q}_{(1)})\rho_S(U_{(1)})$$
$$\times\sum_{\nu}\,|\,v^{(\nu)}_{\bar{\alpha}_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)\,|^2(2\pi)^3\delta(\vec{Q}_{(1)}-\vec{k}_0-\vec{k}_\nu)$$
$$\times v^{*}_{\alpha'_2\bar{\alpha}_1}(\vec{k}\,'_{1(2)},\vec{k}\,'_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})v_{\alpha_2\bar{\alpha}_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})\,, \tag{42}$$

where

$$\rho(\vec{q}_{(1)})\equiv\mu q_{(1)}/[(2\pi)^3\hbar^2] \tag{43}$$

is the density of continuum states of the QF particles with the relative momentum $\vec{q}_{(1)}$, and the averaged matrix element

$$v^{(\nu)}_{\bar{\alpha}_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)\equiv\langle\chi^{(-)}_{1(1)}\chi^{(-)}_{2(1)}\,|\,\bar{v}\,|\,A_{\nu(\lambda\gamma)}\phi_\nu\chi^{(+)}_0\rangle_{\text{av}} \tag{44}$$

is reminiscent of the SS transition amplitude (8) with the momentum conserving δ-function removed.

We will illustrate the calculation of the rest of the rescattering chain steps by adding to Eq. (42) the next factors from Eq. (22) related to the second stage, namely,

$$v_{1i}^{*}G_1^{*}v_{21}^{*}G_2^{*}v_{32}^{*}v_{32}G_2v_{21}G_1v_{1i}$$

$$=\int\int\frac{d\vec{k}_{1(1)}}{(2\pi)^3}\frac{d\vec{k}_{2(1)}}{(2\pi)^3}\int\int\frac{d\vec{k}_{1(2)}}{(2\pi)^3}\frac{d\vec{k}_{2(2)}}{(2\pi)^3}\int\int\frac{d\vec{k}\,'_{1(2)}}{(2\pi)^3}\frac{d\vec{k}\,'_{2(2)}}{(2\pi)^3}$$
$$\times\sum_{\alpha_2,\alpha'_2}\sum_{\nu}2\pi^2\rho(\vec{q}_{(1)})\rho_S(U_{(1)})\,|\,v^{(\nu)}_{\bar{\alpha}_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)\,|^2(2\pi)^3\delta(\vec{Q}_{(1)}-\vec{k}_0-\vec{k}_\nu)v^{*}_{\alpha'_2\bar{\alpha}_1}(\vec{k}\,'_{1(2)},\vec{k}\,'_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})$$
$$\times v^{*}_{\alpha'_3\alpha'_2}(\vec{k}\,'_{1(3)},\vec{k}\,'_{2(3)};\vec{k}\,'_{1(2)},\vec{k}\,'_{2(2)})v_{\alpha_3\alpha_2}(\vec{k}_{1(3)},\vec{k}_{2(3)};\vec{k}_{1(2)},\vec{k}_{2(2)})v_{\alpha_2\bar{\alpha}_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})$$
$$\times[E_0-i\eta-k'^2_{1(2)}\hbar^2/(2m_1)-k'^2_{2(2)}\hbar^2/(2m_2)-\epsilon_{\alpha'_2}]^{-1}$$
$$\times[E_0+i\eta-k^2_{1(2)}\hbar^2/(2m_1)-k^2_{2(2)}\hbar^2/(2m_2)-\epsilon_{\alpha_2}]^{-1}\,. \tag{45}$$

As before, we make use of assumption (ii) to take $\alpha_2=\alpha'_2$, average the integrand over the residual state energies ϵ_{α_2}, and then make the pole approximation to obtain the result

$$I_{\alpha_2}=2\pi^2\rho_S(U_{(2)})M_{(2)}(U_{(2)})\delta[(k^2_{1(2)}-k'^2_{1(2)})\hbar^2/(2m_1)+(k^2_{2(2)}-k'^2_{2(2)})\hbar^2/(2m_2)]\,, \tag{46}$$

in which the notations made should be obvious by analogy with Eqs. (31)—(34). The quantity

$$U_{(2)}=E_0-k^2_{1(2)}\hbar^2/(2m_1)-k^2_{2(2)}\hbar^2/(2m_2) \tag{47}$$

is the energy dissipated over the residual nucleus after the second scattering stage. Since the energy must be conserved at each step, we notice that $U_{(2)}$ is, in fact,

$$U_{(2)}=U_{(1)}+(k^2_{1(1)}-k^2_{1(2)})\hbar^2/(2m_1)+(k^2_{2(1)}-k^2_{2(2)})\hbar^2/(2m_2)\ , \tag{48}$$

when Eq. (33) is taken into account.

We observe now that, due to the special form of the two-body interaction employed in the Hamiltonian (5) which no longer couples the QF particles to each other after the first stage, and by the normalization properties (26), the matrix elements from Eq. (45) can be decomposed as

$$v_{\alpha_2\bar{\alpha}_1}(\vec{k}'_{1(2)},\vec{k}'_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})=(2\pi)^3[v_{\alpha_2\bar{\alpha}_1}(\vec{k}'_{1(2)},\vec{k}_{1(1)})\delta(\vec{k}'_{2(2)}-\vec{k}_{2(1)})+v_{\alpha_2\bar{\alpha}_1}(\vec{k}'_{2(2)},\vec{k}_{2(1)})\delta(\vec{k}'_{1(2)}-\vec{k}_{1(1)})]\ , \tag{49a}$$

$$v_{\alpha_2\bar{\alpha}_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})=(2\pi)^3[v_{\alpha_2\bar{\alpha}_1}(\vec{k}_{1(2)},\vec{k}_{1(1)})\delta(\vec{k}_{2(2)},-\vec{k}_{2(1)})+v_{\alpha_2\bar{\alpha}_1}(\vec{k}_{2(2)},\vec{k}_{2(1)})\delta(\vec{k}_{1(2)}-\vec{k}_{1(1)})]\ , \tag{49b}$$

and similarly for $v_{\alpha'_3\alpha_2}$ and $v_{\alpha_3\alpha_2}$. Inserting these forms together with the relation (46) in Eq. (45) and taking the integral $\int' I_{\alpha_2}$ over the primed momenta relevant to the second stage, we obtain

$$\begin{aligned}\int' I_{\alpha_2}\equiv&\int\int\frac{d\vec{k}'_{1(2)}}{(2\pi)^3}\frac{d\vec{k}'_{2(2)}}{(2\pi)^3}I_{\alpha_2}\\
=&2\pi^2\Big[\rho(\vec{k}_{1(2)})\rho_S(U_{1(2)})\int d\Omega'_{1(2)}v^*_{\alpha'_3\alpha_2}(\vec{k}'_{1(3)},\vec{k}'_{2(3)};k_{1(2)}\Omega'_{1(2)},\vec{k}_{2(2)})\\
&\times v^*_{\alpha_2\bar{\alpha}_1}(k_{1(2)}\Omega'_{1(2)},\vec{k}_{1(1)})v_{\alpha_2\bar{\alpha}_1}(\vec{k}_{1(2)},\vec{k}_{1(1)})(2\pi)^3\delta(\vec{k}_{2(2)}-\vec{k}_{2(1)})\\
&\times v_{\alpha_3\alpha_2}(\vec{k}_{1(3)},\vec{k}_{2(3)};\vec{k}_{1(2)},\vec{k}_{2(2)})\\
&+\rho(\vec{k}_{2(2)})\rho_S(U_{2(2)})\int d\Omega'_{2(2)}v^*_{\alpha'_3\alpha_2}(\vec{k}'_{1(3)},\vec{k}'_{2(3)};\vec{k}_{1(2)},k_{2(2)}\Omega'_{2(2)})\\
&\times v^*_{\alpha_2\bar{\alpha}_1}(k_{2(2)}\Omega'_{2(2)},\vec{k}_{2(1)})v_{\alpha_2\bar{\alpha}_1}(\vec{k}_{2(2)},\vec{k}_{2(1)})(2\pi)^3\delta(\vec{k}_{1(2)}-\vec{k}_{1(1)})\\
&\times v_{\alpha_3\alpha_2}(\vec{k}_{1(3)},\vec{k}_{2(3)};\vec{k}_{1(2)},\vec{k}_{2(2)})\Big]\ .\end{aligned} \tag{50}$$

Cross terms corresponding to residual excitations without momentum change of either QF particle which are incompatible with our model have been eliminated, and

$$U_{j(2)}=U_{(1)}+(k^2_{j(1)}-k^2_{j(2)})\hbar^2/(2m_j),\quad \text{for } j=1,2\ , \tag{51}$$

are the residual excitation energies due to the interaction of each of the QF particles separately. Similarly Eq. (41), upon summing Eq. (50) over the configurations from $\mathscr{P}_2$ whose energies equal $U_{(2)}$, we apply the random phase hypothesis (ii) to replace the matrix elements by their average over the quantum states α_2, operationally labeled by $\bar{\alpha}_2$. After performing the angular integrations which are possible due to the assumed occurrence of the $\delta(\Omega'_{1(2)}-\Omega_{1(2)})$ and $\delta(\Omega'_{2(2)}-\Omega_{2(2)})$ factors resulting from the above averaging, Eq. (45) becomes

$$\begin{aligned}v^*_{1i}G^*_1v^*_{21}G^*_2v^*_{32}v_{32}G_2v_{21}G_1v_{1i}=&\int\int\frac{d\vec{k}_{1(1)}}{(2\pi)^3}\frac{d\vec{k}_{2(1)}}{(2\pi)^3}\int\int\frac{d\vec{k}_{1(2)}}{(2\pi)^3}\frac{dk_{2(2)}}{(2\pi)^3}\\
&\times\sum_\nu\rho(\vec{q}_{(1)})\rho_S(U_{(1)})\,|\,v^{(\nu)}_{\bar{\alpha}_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)\,|^2(2\pi)^3\delta(\vec{Q}_{(1)}-\vec{k}_0-\vec{k}_\nu)\\
&\times[2\pi^2\rho(\vec{k}_{1(2)})\rho_S(U_{1(2)})\,|\,v_{\bar{\alpha}_2\bar{\alpha}_1}(\vec{k}_{1(2)},\vec{k}_{1(1)})\,|^2(2\pi)^3\delta(\vec{k}_{2(2)}-\vec{k}_{2(1)})\\
&+2\pi^2\rho(\vec{k}_{2(2)})\rho_S(U_{2(2)})\,|\,v_{\bar{\alpha}_2\bar{\alpha}_1}(\vec{k}_{2(2)},\vec{k}_{2(1)})\,|^2(2\pi)^3\delta(\vec{k}_{1(2)}-\vec{k}_{1(1)})]\\
&\times2\pi^2v^*_{\alpha'_3\bar{\alpha}_2}(\vec{k}'_{1(3)},\vec{k}'_{2(3)};\vec{k}_{1(2)},\vec{k}_{2(2)})\\
&\times v_{\alpha_3\bar{\alpha}_2}(\vec{k}_{1(3)},\vec{k}_{2(3)};\vec{k}_{1(2)},\vec{k}_{2(2)})\ .\end{aligned} \tag{52}$$

V. MULTI-STEP CROSS SECTION FOR THE DIRECT (a,ab) REACTION CONTINUUM SPECTRUM

We are now finally in a position to write the average cross section for the (a,ab) scattering continuum. After integrating the reaction amplitude over the momentum of the undetected recoiling nucleus S using the momentum conserving δ-function from Eq. (45), the usual procedure for obtaining reaction cross sections leads to the convolution integral

$$\left\langle \frac{d^2\sigma}{d\vec{k}_{(f)}} \right\rangle = \sum_n \sum_{e=n-1}^{n+1} \int\int \frac{d\vec{k}_{1(1)}}{(2\pi)^3}\frac{d\vec{k}_{2(1)}}{(2\pi)^3} \int\int \frac{d\vec{k}_{1(2)}}{(2\pi)^3}\frac{d\vec{k}_{2(2)}}{(2\pi)^3} \cdots \int\int \frac{d\vec{k}_{1(n)}}{(2\pi)^3}\frac{d\vec{k}_{2(n)}}{(2\pi)^3}$$
$$\times \sum_\nu \frac{d^2\sigma^{\rm QF}_{\bar{\alpha}_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)}{d\vec{k}_{(1)}} \frac{d^2W_{\bar{\alpha}_2\bar{\alpha}_1}(\vec{k}_{1(2)},\vec{k}_{2(2)};\vec{k}_{1(1)},\vec{k}_{2(1)})}{d\vec{k}_{(2)}} \cdots$$
$$\times \frac{d^2W_{\bar{\alpha}_n\bar{\alpha}_{n-1}}(\vec{k}_{1(n)},\vec{k}_{2(n)};\vec{k}_{1(n-1)},\vec{k}_{2(n-1)})}{d\vec{k}_{(n)}} \frac{d^2W_{\alpha_e\bar{\alpha}_n}(\vec{k}_1,\vec{k}_2;\vec{k}_{1(n)},\vec{k}_{2(n)})}{d\vec{k}_{(f)}} , \tag{53}$$

where $d\vec{k}_{(m)} \equiv d\vec{k}_{1(m)} d\vec{k}_{2(m)}$ and $\vec{k}_\nu = \vec{k}_{1(1)} + \vec{k}_{2(1)} - \vec{k}_0$. In this equation

$$\frac{d^2\sigma^{\rm QF}_{\bar{\alpha}_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu)}{d\vec{k}_{(1)}} \equiv \frac{2\pi m_0}{\hbar^2 k_0}\rho(\vec{q}_{(1)})\rho_S(U_{(1)}) \mid v^{(\nu)}_{\bar{\alpha}_1 i}(\vec{k}_{1(1)},\vec{k}_{2(1)};\vec{k}_0,\vec{k}_\nu) \mid^2 \delta(E_{1(1)}+E_{2(1)}-E_0-E^{\alpha_1}_\nu+E_S) \tag{54}$$

identifies the average cross section for the incident particle to undergo a doorway knockout collision with the emission of two QF particles with energies $E_{1(1)}$ and $E_{2(1)}$ and momenta $\vec{k}_{1(1)}$ and $\vec{k}_{2(1)}$. The methods for dealing with this quantity are those of the knockout SS process, briefly outlined in Sec. II. Also,

$$\frac{d^2W_{\bar{\alpha}_m\bar{\alpha}_{m-1}}(\vec{k}_{1(m)},\vec{k}_{2(m)};\vec{k}_{1(m-1)},\vec{k}_{2(m-1)})}{d\vec{k}_{(m)}}$$
$$\equiv (2\pi)^3 \left[\frac{dW_{\bar{\alpha}_m\bar{\alpha}_{m-1}}(\vec{k}_{1(m)},\vec{k}_{1(m-1)})}{d\vec{k}_{1(m)}} \delta(\vec{k}_{2(m)}-\vec{k}_{2(m-1)}) + \delta(\vec{k}_{1(m)}-\vec{k}_{1(m-1)}) \frac{dW_{\bar{\alpha}_m\bar{\alpha}_{m-1}}(\vec{k}_{2(m)},\vec{k}_{2(m-1)})}{d\vec{k}_{2(m)}} \right] \tag{55}$$

represents the averaged reduced probability per unit time that either one of the QF particles undergoes a scattering on the spectator nucleus S from the $(m-1)$st to the mth stage of the multi-scattering chain. The individual scattering probabilities corresponding to each QF particle are given by the familiar FKK formalism expression

$$\frac{dW_{\bar{\alpha}_m\bar{\alpha}_{m-1}}(\vec{k}_{j(m)},\vec{k}_{j(m-1)})}{d\vec{k}_{j(m)}} = 2\pi^2\rho(\vec{k}_{j(m)})\rho_S(U_{j(m)}) \mid v_{\bar{\alpha}_m\bar{\alpha}_{m-1}}(\vec{k}_{j(m)},\vec{k}_{j(m-1)}) \mid^2 , \tag{56}$$

with the residual excitation energies due to each particle

$$U_{j(m)} = U_{j(m-1)} + (k^2_{j(m-1)} - k^2_{j(m)})\hbar^2/(2m_j), \quad \text{for } j=1,2 , \tag{57}$$

and the average transition matrix elements of folding type[9]

$$v_{\bar{\alpha}_m\bar{\alpha}_{m-1}}(\vec{k}_{j(m)},\vec{k}_{j(m-1)}) = \langle \chi^{(-)}_{j(m)}\Phi_j\Phi_{S\bar{\alpha}_m} \mid v \mid \Phi_j\Phi_{S\bar{\alpha}_{m-1}}\chi^{(+)}_{j(m-1)}\rangle, \quad \text{for } j=1,2 . \tag{58}$$

The averaging procedure labeled with a bar in the above equations depends, of course, on the specific model chosen to describe the residual excitations. Perhaps the most attractive framework in this respect is the simple exciton model of Griffin[1] in which each subspace $\mathcal{P}_n$ is identified with shell model configurations containing $(n+1)p$-nh single particle states, i.e., $N=2n+1$ excitons. The level density of an N-exciton state at a certain excitation energy $U_{(n)}$ is given by the Ericson expression[14]

$$\rho_S(U_{(n)}) = \frac{g(gU_{(n)})^{N-1}}{(n+1)!n!(N-1)!} , \tag{59}$$

where g is the total density of single particle states in the equal-spacing model. As was commented earlier, since the level density increases rapidly with the number of excitons, neglecting the scattering terms with retrograde steps $n \to n-1$ appears to be an acceptable approximation, at least at the beginning of the excitation chain. Toward the end of longer excitation chains the level density increases more slowly and the above-mentioned never-come-back approximation becomes questionable, but still desirable for computational reasons. In this framework, Feshbach *et al.*[1] and Bonetti *et al.*[13] performed the bar averages using the energy-indepenent spin density $R_S(J_{(n)})$ of the exciton levels with total angular momen-

tum $J_{(n)}$ given by[14]

$$R_S(J_{(n)})=\frac{2J_{(n)}+1}{\pi^{1/2}N^{3/2}\sigma^3}\exp\left[-\frac{(J_{(n)}+\frac{1}{2})^2}{N\sigma^2}\right], \quad (60)$$

where σ is a spin cutoff parameter with the approximate normalization

$$\sum_{J_{(n)}}(2J_{(n)}+1)R_S(J_{(n)})\approx 1 .$$

VI. CONCLUSIONS

In this work the QF continuum was studied with the intention of formulating a theoretical framework such as those already existing for the two-body reactions. In order to focus on the basic physics of the problem a mechanism was proposed which assumes that the source of the (a,ab) continuum is the QF particles emerging from primary knockout collisions and degraded in energy by subsequent scattering. The multiple-scattering equations are in general impractical, if not impossible, to be solved exactly for most nuclear systems. On the other hand, one must realize that the measurable properties of the continuum spectrum are, in fact, average quantities. For this reason it was found to be quite natural to employ a statistical method to treat the problem of the (a,ab) continuum spectrum. The task was greatly simplified by the existence of such a type of treatment as formulated by Feshbach *et al.*[1] for the (a,b) continuum.

Somewhat surprisingly, no major difficulties were encountered in extending the FKK statistical ideas to the three-body case, although, to simplify the problem, one had to further assume that the particles following the doorway QF stage do not interact simultaneously with the residual nucleus and also do not interact with each other. Moreover, encountering the standard difficulty[10] of the multi-step Born expansion caused by the biorthogonality of the distorted wave base of the Green's function, in order to keep the formalism practicable one had to take the elastic scattering matrix S equal to unity. The effect of this approximation remains to be studied, but it should be mentioned that the two-body applications[13] of the FKK formalism did not seem to show a major drawback because of it. A recent assessment[15] of the FKK approximations showed that, at least in the light ion reactions, the net error remains fairly small.

As one could have perhaps anticipated, the cross section for the (a,ab) continuum differs only in specific detail but not in substance from its (a,b) equivalent. From this point of view one can state that, even though approximate, there now exists a unitary statistical description of both the two- and three-body continuum spectra of nuclear reactions. After the successful extension of the FKK formalism to the three-body reactions, one might find tempting an outright generalization of this result to the n-body reactions. It turns out, unfortunately, that this would generally be unwarranted. Our attempts to obtain an expression similar to Eq. (53) for the four-body QF reactions (a,abc), such as (e,epp) or (p,3p), failed, unless the relative momentum of the knocked out (bc) nucleon pair was assumed to be conserved throughout the reaction. In practice, this should only happen in those particular cases when the knocked out pair remains correlated and continues to behave as a pseudoparticle in the final channel.

From a practical point of view, the expression which was derived for the three-body cross section does not pose too much more computational complication than its FKK two-body equivalent, except for an inherently increased amount of computer resources required to calculate the doorway QF cross section and also the rescattering probabilities for each QF particle. In those cases, such as the (e,ep) reaction, where the rescattering of one of the particles can be neglected, the amount of computation is reduced drastically and becomes comparable with the two-body case. A three-body situation of this type in the (p,2p) reaction was examined in detail in two recent papers.[3] In conclusion, the author believes that the present work indicates promising prospects for studying the continuum spectrum in the (a,ab) knockout, and perhaps also the $A(a,bc)A$ breakup, reactions.

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