Semiclassical and quantal inversion of nuclear scattering at fixed energy

H. Fiedeldey* and R. Lipperheide
Hahn-Meitner-Institut für Kernforschung Berlin and Freie Universität Berlin, D-1000 Berlin 39, Federal Republic of Germany

K. Naidoo
Physics Department, University of Durban-Westville, Durban, South Africa

S. A. Sofianos
Department of Physics, University of South Africa, Pretoria, South Africa
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The semiclassical and quantal inversion problems at fixed energy are solved for a number of nuclear scattering systems at various energies. The scattering function is represented as a particular rational or nonrational function of angular momentum. At low energies the semiclassical inversion generally breaks down; it works, however, surprisingly well in many cases.

I. INTRODUCTION

The first attempts to solve the inverse scattering problem at fixed energy were made by Hoyt,[1] Firsov,[2] and Wheeler[3] in the semiclassical Wentzel-Kramers-Brillouin (WKB) approximation. Since then semiclassical inversion methods for real potentials[4–8] have been further developed and widely applied in molecular physics (cf. the review in Ref. 9). Applications to nuclear physics, involving complex potentials, were made by Kujawski.[10]

A fully quantal inversion scheme was devised by Newton[11] and extended by Newton and Sabatier.[12,13] It has been applied to inversion problems in nuclear physics by Coudray,[14] and more recently, by Münchow and Scheid,[15] Baldock *et al.*,[16] and Barrett *et al.*[17]

In the last few years a method has been proposed to solve the quantal inverse scattering problem at fixed energy for a class of scattering functions which are rational functions of the angular momentum (of the "Bargmann type")[18] or nonrational modifications thereof.[19,20] A given scattering function can then be inverted after representing it by a function of either class, fitted at the physical (integer) values of the angular momentum. The method has been successfully applied to inversion problems in nuclear and atomic physics, involving complex potentials and charged particles.[20,21] (Inversion approaches based on particular forms of the scattering function have been reviewed by Zakhar'ev *et al.*[22])

The approach[18–20] makes use of certain simple analytic forms of the scattering function. On the real axis, these can be written as rational functions of the angular momentum. Owing to this, the WKB inversion can be carried out analytically; this is particularly useful for the case of complex potentials, when the usually real WKB inversion formalism has to be continued into the complex plane. In the present paper we take advantage of this circumstance to reconstruct various real and complex nuclear potentials at different energies by quantal and WKB inversion of the same scattering function fitted to the physical angular momenta. In this way the quality of the two types of inversion can be compared. It will be seen that the quantal and WKB methods are generally equally effective at high energies (reproducing the input potential almost perfectly), while at low energies, the WKB method will sometimes break down.

Rational representations of the scattering function were introduced earlier on heuristic grounds by Remler.[23] These were used by Rich *et al.*[8] to obtain molecular scattering potentials by WKB inversion. However, since these authors included a nonrational repulsive-core contribution in the scattering function, the inversion had to be done numerically. No analytic continuation was required, as the potentials were real.

Kujawski[10] was the first to apply the WKB inversion method to complex, nuclear potentials. He did not use the rational representation of the scattering function which renders the analytic continuation of the formalism trivial; instead the inversion was carried out by numerical integration, and the analytic continuation was done via Padé approximants. A point-Coulomb potential was introduced as reference potential for charged particles; in this way Kujawski found, at all energies, classically forbidden regions near the origin in which no WKB inversion could be carried out.

In Sec. II we present the formalism of our semiclassical inversion method and compare it with that employed by Kujawski.[10] We also briefly recapitulate our quantal inversion schemes. Our results are presented in Sec. III. First the semiclassical and quantal inversions are compared in detail for α-α scattering. This will be done over a wide range of energies, via the reconstruction of an energy-dependent α-α potential derived by Kukulin *et al.*[23] from microscopic considerations. Subsequently, we reconstruct realistic p-^{58}Ni optical potentials using the WKB and quantal methods, and compare the results for various energies. Then we consider a few schematic cases of n-α scattering which have been previously investigated by Coudray,[14] using the Newton-Sabatier method. Finally, we reconsider the case investigated by Kujawski,[10] α-^{12}C scattering at 104 MeV. In the last section we discuss the results and present our conclusions.

II. THE SEMICLASSICAL AND QUANTAL INVERSION METHODS

A. Semiclassical inversion

Our semiclassical inversion scheme has already been described in Ref. 18 for the case of real potentials and applied to a few schematic examples. Here we present the general case of complex potentials, including a reference potential with a Coulomb tail.

We start with real potentials. The real classical deflection function $\theta(\lambda)$ is written as

$$\theta(\lambda)=2\frac{d\delta(\lambda)}{d\lambda} \tag{1}$$

in terms of the phase shifts

$$\delta(\lambda)=(1/2i)\ln S(\lambda) , \tag{2}$$

where $S(\lambda)$ is the scattering function and $\lambda=l+\frac{1}{2}$. The inversion problem with one turning point then becomes entirely classical. Introducing the "quasi-potential"

$$Q(\sigma)=\frac{2E}{\pi}\int_{\sigma}^{\infty}\frac{\theta(\lambda)d\lambda}{(\lambda^2-\sigma^2)^{1/2}} , \tag{3}$$

we define

$$\rho=\rho(\sigma)=\sigma\exp[Q(\sigma)/2E] . \tag{4}$$

The potential is then given by

$$\begin{aligned} V(\rho)&=E(1-\sigma^2/\rho^2)\\ &=E\{1-\exp[-Q(\sigma)/E]\} , \end{aligned} \tag{5}$$

where $\rho=kr$.

The semiclassical inversion method produces a unique potential if the relation (4), or equivalently, the relation

$$\sigma=\sigma(\rho)=\rho[1-V(\rho)/E]^{1/2} \tag{6}$$

defines a one-to-one correspondence between ρ and σ. This requires that $\sigma(\rho)$ and $\rho(\sigma)$ are monotonic functions of their arguments, i.e.,

$$d\sigma/d\rho>0 \text{ for all } \rho , \tag{7}$$

or equivalently,

$$d\rho/d\sigma>0 \text{ for all } \sigma . \tag{8}$$

The relations Eqs. (6) and (7) together imply the "nonorbiting" condition

$$E>V(\rho)+\tfrac{1}{2}\rho\, dV/d\rho , \quad E>V(\rho) , \tag{9}$$

while Eqs. (4) and (8) yield

$$dQ/d\sigma>-2E/\sigma . \tag{10}$$

Since $Q(\sigma)$ is given in terms of $\theta(\lambda)$, this is essentially a condition on the S function to yield a unique semiclassical potential. As such it is more useful for our purposes.

The WKB inversion for complex phase shifts (potentials) was discussed by Kujawski.[10] In this case the deflection function $\theta(\lambda)$ is complex for real λ, and so is the quasi-potential $Q(\sigma)$ for real σ. The function $Q(\sigma)$ in Eq. (4) must be continued in the complex σ plane so as to make ρ real, since the latter must be identified with the radial coordinate, $\rho=kr$. From the conditions

$$\begin{aligned} 0&=\mathrm{Im}\rho=\mathrm{Im}\{\sigma\exp[Q(\sigma)/2E]\} ,\\ \rho&=\mathrm{Re}\rho=\mathrm{Re}\{\sigma\exp[Q(\sigma)/2E]\} , \end{aligned} \tag{4$'$}$$

a complex function $\sigma(\rho)$ of the real variable ρ is obtained, which after substitution in Eq. (5) yields the complex potential $V(\rho)$. The relation between ρ and σ must again be one to one. Otherwise the complex "path" $\sigma=\sigma(\rho)$ is in the "critical domain,"[10] where the WKB inversion breaks down.

We now apply this formalism to our particular rational form of scattering function, λ real,

$$S_{\mathrm{rat}}(\lambda)=S^{(0)}(\lambda)\prod_{n=1}^{N}\frac{\lambda^2-\beta_n^2}{\chi^2-\alpha_n^2} . \tag{11}$$

The background function $S^{(0)}(\lambda)$ is introduced to take account of the long-range Coulomb tail (in the scattering function as well as in the potential) for charged-particle scattering. It has the form[20]

$$S^{(0)}(\lambda)=\exp[i\eta\ln(\lambda^2+\lambda_c^2)] , \tag{12}$$

where η is the Sommerfeld parameter and λ_c is a real parameter, $\lambda_c=x\eta$, $x=4$ to 10. Applying the WKB inversion scheme to this unitary scattering function, we have

$$\theta^{(0)}(\lambda)=2\eta\lambda/(\lambda^2+\lambda_c^2) , \tag{13}$$

$$Q^{(0)}(\sigma)=2\eta E/(\sigma^2+\lambda_c^2)^{1/2} , \tag{14}$$

$$V^{(0)}(r)\underset{r\to\infty}{\longrightarrow}2\eta E/kr , \tag{15a}$$

$$\underset{r\to 0}{\longrightarrow}E(1-e^{2\eta/\lambda_c})\approx 2\eta E/\lambda_c . \tag{15b}$$

The background scattering function $S^{(0)}(\lambda)$ thus corresponds to a "quasi-Coulomb" potential which asymptotically behaves like a Coulomb potential, but does not contain a $1/r$ singularity at the origin.

The scattering function (11) leads to

$$\begin{aligned} \theta(\lambda)&=2\eta\lambda(\lambda^2+\lambda_c^2)^{-1}\\ &\quad+2i\lambda\sum_n[(\lambda^2-\alpha_n^2)^{-1}-(\lambda^2-\beta_n^2)^{-1}] \end{aligned} \tag{16}$$

and

$$\begin{aligned} Q(\sigma)&=2E\eta(\sigma^2+\lambda_c^2)^{-1/2}\\ &\quad+2iE\sum_n[(\sigma^2-\alpha_n^2)^{-1/2}-(\sigma^2-\beta_n^2)^{-1/2}] . \end{aligned} \tag{17}$$

If $\beta_n=\alpha_n^*$, $Q(\sigma)$ is real for real σ, and the real potential $V(\sigma)$ is directly read off from Eqs. (4) and (5), choosing a value for σ and calculating $V(\rho)$ at the corresponding value of ρ. In general, $Q(\sigma)$ is complex for real σ, and Eqs. (4$'$) have first to be solved for $\sigma=\sigma(\rho)$, ρ real, before going to Eq. (5).

The approach of Kujawski[10] differs from this in two respects: (i) In Ref. 10 the background function $S^{(0)}(\lambda)$ is taken as the classical point-Coulomb scattering function, corresponding to the Rutherford deflection function

$$\theta^{(R)}(\lambda)=2\arctan(\eta/\lambda)$$

and the Rutherford potential

$$V^{(R)}(r)=2\eta E/kr\ .$$

Kujawski then finds that his inversion method is restricted to the domain $r>2\eta/k$. (ii) Instead of expression (17), Kujawski obtains $Q(\sigma)$ in numerical form, which is then continued analytically with the help of Padé approximants.

Whenever the semiclassical inversion works well, the path $\sigma=\sigma(\rho)$ does not approach any of the poles or zeros $\{\alpha_n,\beta_n\}$, and the square roots in Eq. (17) can be determined unambiguously. Problems only crop up when the semiclassical inversion actually starts to break down and multivalued solutions of Eqs. (4′) become possible (in the "critical domain").

B. Quantal inversion

We briefly recapitulate the quantal inversion and refer to Refs. 18–20 for a full explanation of the methods employed. The potential $V(r)$ is determined iteratively,

$$V(r)=V_N(r)\ , \tag{18}$$

where

$$V_n(r)=V_{n-1}(r)+V^{(n)}(r)\ ,\quad n=1,\ldots,N \tag{19}$$

and

$$V^{(n)}(r)=\frac{2}{r}(\beta_n^2-\alpha_n^2)\frac{d}{dr}\left[\frac{1}{r}\left[\frac{1}{L_{\beta_n}^{(n-1)-}(r)-L_{\alpha_n}^{(n-1)+}(r)}\right]\right]. \tag{20}$$

The $L_\lambda^{(n)\pm}(r)$ are the logarithmic derivatives of the Jost solutions $f_\lambda^{(n)\pm}(r)$ to the potential $V_n(r)$ in the rational scheme ($\mathrm{Im}\alpha_n^2>0$, $\mathrm{Im}\beta_n^2<0$) and of the regular functions $\phi_\lambda^{(n)}(r)$ in the nonrational scheme ($\mathrm{Im}\alpha_n^2<0$, $\mathrm{Im}\beta_n^2>0$). (Note that these functions differ from the ones defined in Ref. 18 by a factor $\pm i$.) The corresponding rational scattering function is given by Eq. (11), whereas the nonrational scattering function has the form

$$S_{\mathrm{nonrat}}(\lambda)=S^{(0)}(\lambda)\frac{\left|\left|\dfrac{\sigma_{\beta_n}^{(0)}-\sigma_{\alpha_m}^{(0)}}{\beta_n^2-\alpha_m^2}-\dfrac{\sigma_\lambda^{(0)}-\sigma_{\alpha_m}^{(0)}}{\lambda^2-\alpha_m^2}\dfrac{\sigma_{\beta_n}^{(0)}}{\sigma_\lambda^{(0)}}\right|\right|}{\left|\left|\dfrac{\sigma_{\beta_n}^{(0)}-\sigma_{\alpha_m}^{(0)}}{\beta_n^2-\alpha_m^2}-\dfrac{\sigma_\lambda^{(0)}-\sigma_{\alpha_m}^{(0)}}{\lambda^2-\alpha_m^2}\right|\right|} \tag{21}$$

with

$$\sigma_\lambda^{(0)}=\exp[-i\pi(\lambda-\tfrac{1}{2})]S^{(0)}(\lambda)\ . \tag{22}$$

If $|\mathrm{Im}\alpha_n|$, $|\mathrm{Im}\beta_n|$ are sufficiently large (≥ 2),

$$S_{\mathrm{nonrat}}(\lambda)=S_{\mathrm{rat}}(\lambda)\ \text{for real}\ \lambda\ .$$

The two schemes can be combined by writing $S(\lambda)$ as a product of a rational and a nonrational scattering function (mixed scattering function).[20]

As explained previously the semiclassical inversion employs the rational scattering function to which our quantal S function in general only reduces on the real λ axis. When comparing our quantal and semiclassical inversions, we therefore do not, strictly speaking, employ analytically the same scattering function. However, as we will show in the next section, the fact that the two S functions coincide numerically near the real λ axis is sufficient to guarantee practically identical quantal and semiclassi-

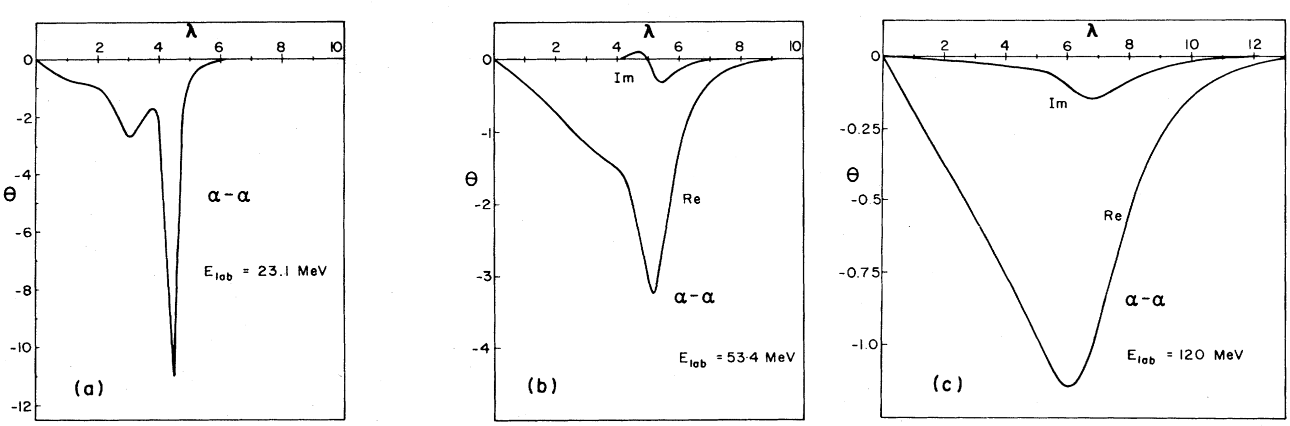

FIG. 1. Deflection functions $\theta(\lambda)$ for α-α scattering (Ref. 24) at $E_{\mathrm{lab}}=23.1$ (a), 53.4 (b), and 120 (c) MeV.

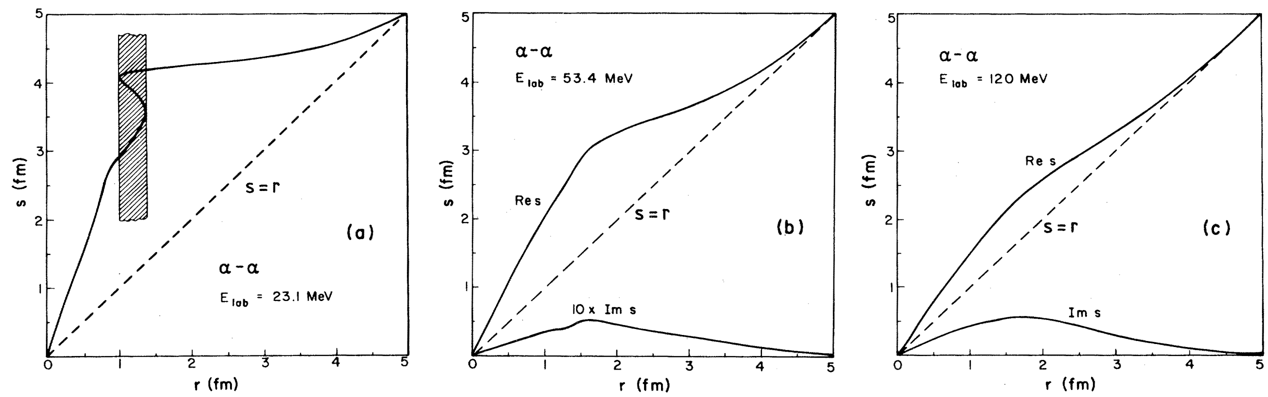

FIG. 2. The function $s = s(r)$ for α-α scattering (Ref. 24) at $E_{\text{lab}} = 23.1$ (a), 53.4 (b), and 120 (c) MeV.

cal inverted potentials, in those cases where the semiclassical inversion works well. This confirms that the differences in $S(\lambda)$ far away from the real axis have little significance for the potential obtained in the semiclassical inversion scheme. In particular, it is here irrelevant whether the scattering function contains unphysical poles in the complex λ plane, in contrast to the situation in the quantal scheme.[19]

III. RESULTS

In this section we present a representative selection of our comparisons of the semiclassical and quantal inversions. To be able to check our results, we reconstruct given input potentials from their scattering functions.

For the quantal inversion we employ in general a "mixed" scattering function, while for the semiclassical inversion the rational approximation is used. The parameters $\{\alpha_n, \beta_n\}$ are obtained by fitting the rational S function of Eq. (11) to the input scattering function at the physical angular momenta $l = 0, 1, 2, \ldots$. The fit using the corresponding mixed S function is nearly identical, in all these cases, to the one obtained with the rational S function.

A. Inversion of α-α scattering

Here we consider the α-α potential of Kukulin *et al.*,[24] which has a Woods-Saxon shape, with parameters $V_0(E) = -125 + 0.33E_{\text{c.m.}}$ (MeV); $W_0 = 0$, -5, and -10 MeV at $E_{\text{lab}} = 23.1$, 53.4, and 120 MeV, respectively; and $R_v = R_w = 1.78$ fm and $a_v = a_w = 0.66$ fm. These deep local α-α potentials[25] are essentially phase equivalent local potentials for the nonlocal exchange potentials resulting from a microscopic resonating group model[26] calculation.

The scattering function calculated from this potential is fitted by the rational expression (11), which yields the parameter values given in Table I. In Figs. 1(a)–(c) we display the classical deflection function $\theta(\lambda)$ calculated from these parameters at the three energies considered. At 23.1 MeV the potential is real and so is the deflection function. It has a sharp structure at $\lambda \sim 4.5$, corresponding to a twofold revolution about the scattering center ($\theta \approx -4\pi$, near-orbiting). The complex deflection func-

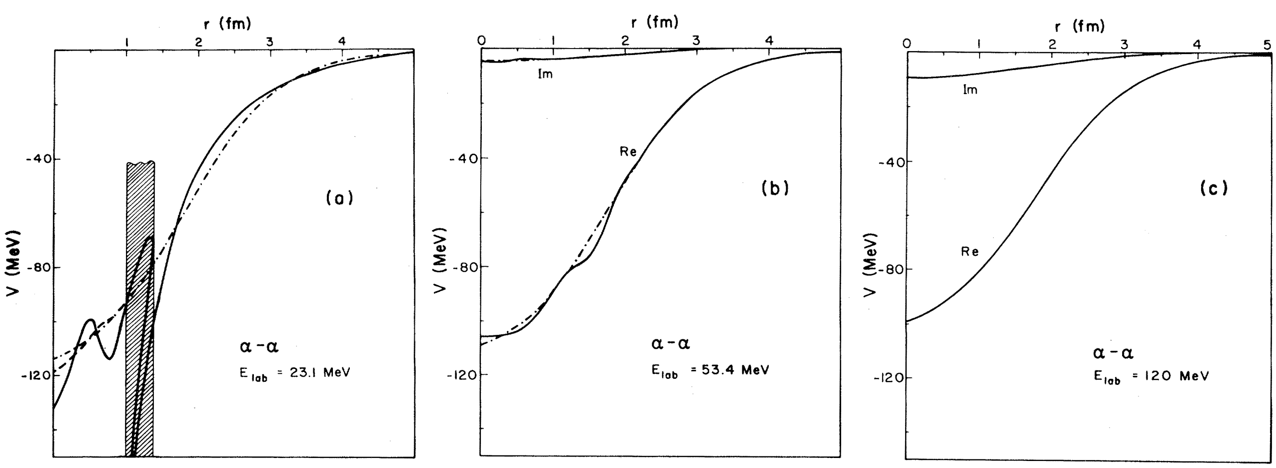

FIG. 3. Input potential (—·—·—·) and its reconstructions by quantal (— — —) and WKB (——) inversion for α-α scattering (Ref. 24) at $E_{\text{lab}} = 23.1$ (a), 53.4 (b), and 120 (c) MeV.

TABLE I. Parameters $\{\alpha_n, \beta_n\}$ for α-α inversion.

E_{lab} (MeV)	α_n	β_n
	$2.5959-2.8456i$	
	$5.6253-4.6466i$	
23.1	$0.0793+6.6226i$	
($\lambda_c=5.0$)	$1.0645+1.6272i$	$\beta_n=\alpha_n^*$
	$3.0455+0.7207i$	
	$4.4098+1.4774i$	
	$4.5018-3.9224i$	$4.7056+3.9971i$
	$8.7888-6.1692i$	$9.1269+6.1930i$
53.4	$2.4522+3.4613i$	$1.8382-3.2987i$
($\lambda_c=3.0$)	$3.5002+20214i$	$3.4303-1.6944i$
	$5.2170+0.7619i$	$5.1506-0.6137i$
	$9.1391+6.2427i$	$8.8059-6.2356i$
	$9.0292-9.7351i$	$-9.2837+1.0245i$
	$0.5369-3.1721i$	$5.0577+6.5669i$
120.0	$5.2060-6.3115i$	$8.8203+9.8012i$
($\lambda_c=5.0$)	$4.1442+3.3524i$	$8.1312-2.7502i$
	$6.4317+2.0951i$	$3.8679-2.8224i$
	$9.0313+1.0389i$	$6.1937-1.7654i$

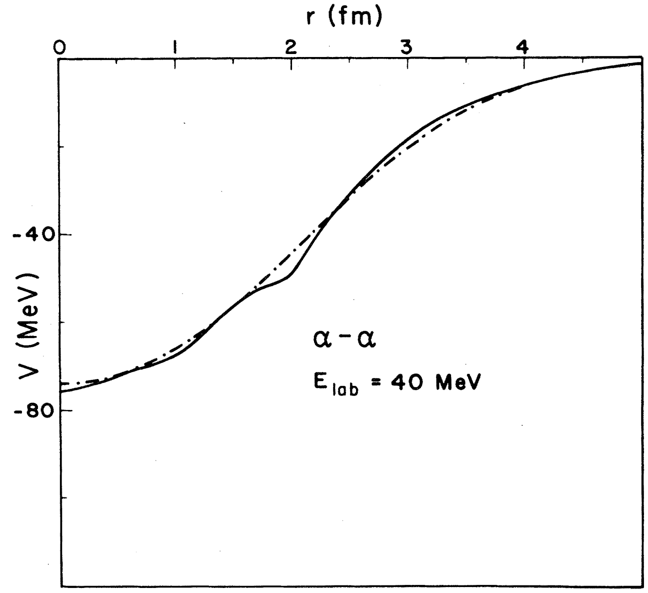

FIG. 4. Input potential (— · — · — ·) and its reconstruction by WKB inversion (——) for α-α scattering (Ref. 16) at $E_{lab}=40$ MeV.

tions at the energies 53.4 and 120 MeV exhibit a smoother behavior. From these deflection functions one would already in general predict a failure of the semiclassical inversion at 23.1 MeV and success at 120 MeV, with intermediate behavior at 53.4 MeV.

In Figs. 2(a)—(c) we plot $s=\sigma/k$ against the radial distance $r=\rho/k$. This function $s=s(r)$ is obtained by solving Eqs. (4) or (4′) for σ. For the 23.1 MeV case s is a nonmonotonic function of r in the region $1\leq r\leq 1.4$ fm, signaling a breakdown of the semiclassical inversion in that region. For the 53.4 MeV case we plot Res and Ims against r and find a monotonic behavior. The semiclassical inversion therefore works. We note that Ims is quite small so that the path $s(r)$ stays far away from the singularities α_n and β_n, cf. Eq. (17). At 120 MeV, Res and Ims are smooth functions of r. Although Ims is relatively large, the parameters α_n and β_n have such large imaginary parts that again they stay out of the way of the complex path $s=s(r)$.

In Figs. 3(a)—(c) we plot the real α-α potentials of Kukulin *et al.* and compare them to their quantal and semiclassical reconstructions. Turning first to the 23.1 MeV case, we note that the quantal reconstruction is excellent except for a minor discrepancy at $r\leq 0.5$ fm. The semiclassical reconstruction, however, leads to a triple-valued potential between 1.4 and 1.0 fm. For $r>1.4$ fm the semiclassical potential is not a very good reconstruction of the given input potential either. This indicates that when the semiclassical inversion really breaks down at a certain distance it tends to become unreliable even at much larger distances. In the inside region, $r<1.0$ fm, the semiclassically inverted potential oscillates wildly around the given input potential. At 53.4 MeV we find that the quantal inversion coincides with the input potential, while the semiclassical inversion produces a reasonably good result. The largest discrepancies occur at $r<2$ fm. Finally, at 120 MeV both the quantal and the semiclassical inversion results coincide with the input potential. The semiclassical inversion is just as good as the quantal inversion at this energy.

Except for relatively low energies the semiclassical inversion can therefore be regarded as satisfactory for α-α scattering. This is also confirmed by inversion of the phase shifts generated by the real direct α-α potential of Baldock *et al.*[16] at $E_{lab}=40$ MeV (see Fig. 4), which we represented as

TABLE II. Potential parameters for p-^{58}Ni.

E_{lab} (MeV)	V (MeV)	W (MeV)	W_D (MeV)	r_{0V} (fm)	r_{0W} (fm)	r_{0D} (fm)	a_V (fm)	a_W (fm)	a_D (fm)	Ref.
36	35.390	5.086	3.366	1.17	1.32	1.32	0.75	0.51	0.51	27
55	54.068	9.195	0	1.17	1.32		0.75	0.51		27
100.4	29.017	7.699	0	1.196	1.457		0.789	0.512		28

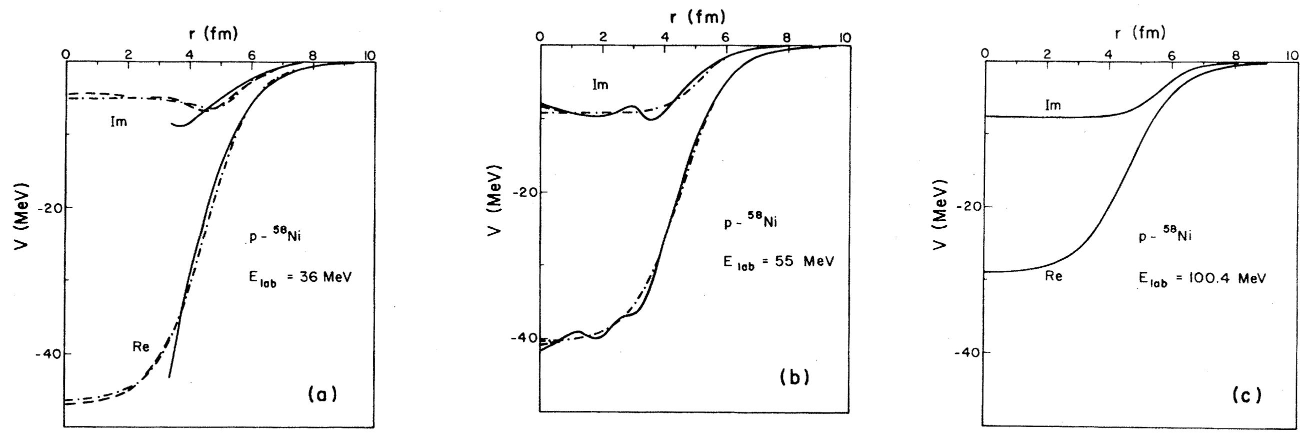

FIG. 5. Input potential (— · — · — ·) and its reconstructions by quantal (— — —) and WKB (——) inversion for p-^{58}Ni scattering at $E_{\rm lab}=36$ (a) and 55 (b) MeV (Ref. 27) and at $E_{\rm lab}=100.4$ MeV (c) (Ref. 28; here all three curves coincide).

$$V(r)=-119.85\exp[-(r/2.34)^2] +46.22\exp[-(r/1.8)^2]$$

in Ref. 20. The potential of Baldock *et al.* has a different shape, longer range, and smaller depth than the α-α potential of Kukulin *et al.* at 53.4 MeV [Fig. 3(b)], but the semiclassically inverted potential has similar characteristics. It also displays a hump at intermediate distances, and it deviates from the input potential not only in the interior but also further outside.

B. Inversion of p-^{58}Ni scattering

The potentials to be reconstructed by semiclassical inversion were in this case taken from Perey and Perey[27] and Kobos and Mackintosh.[28] The optical potentials are of complex Woods-Saxon shape with an additional surface derivative imaginary part. The depths, widths, and radii are given in Table II.

The results of the quantal and semiclassical inversions are compared to the input potentials in Figs. 5(a)—(c). For 36 MeV, the fit of the mixed S function to the physical values of the S function of the optical potential was not perfect. This shows up in the deviations of the quantally inverted potential from the input potential. The semiclassical inversion breaks down for $r<3.4$ fm. This evidently is the critical domain.

At 55 MeV the quantal inversion is nearly perfect except for a small region near the origin. Although the semiclassical inversion does not break down in this case, the deviations from the input potential are not negligible. The inverted potential is oscillatory but the oscillations are relatively small.

Finally, at 100.4 MeV, both the quantal and WKB inversions lead to perfect reproductions of the input potential.

C. Inversion of n-α and α-^{12}C scattering

The last examples refer to systems treated previously by other authors. In Fig. 6 the quantal and WKB inversions are compared with the input potential for the system n-α at 10 and 30 MeV, for which quantal inversions have been performed by Coudray[14] using the Newton-Sabatier method. Both types of inversion get worse as the energy is lowered. At 10 MeV, the WKB inversion reproduces

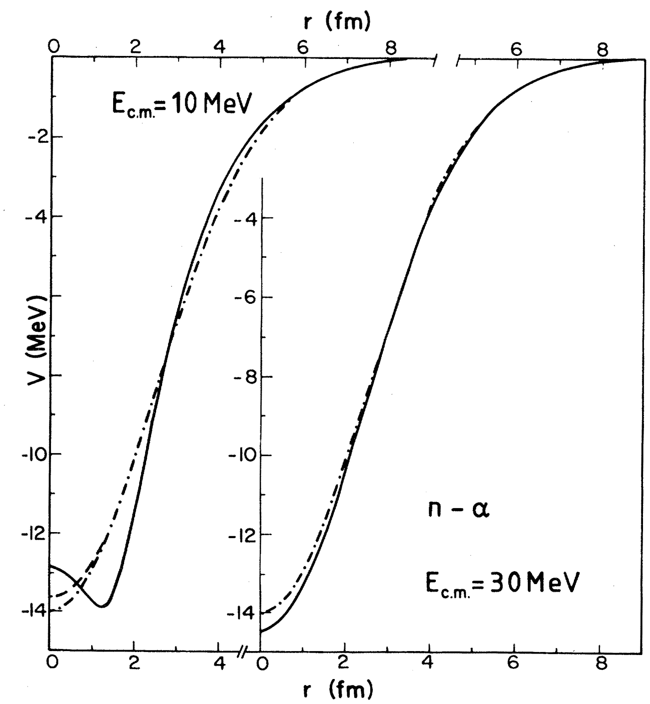

FIG. 6. Input potential (— · — · — ·) and its reconstruction by quantal (— — —) and WKB (——) inversion for n-α scattering (Ref. 14) at $E_{\rm c.m.}=10$ and 30 MeV.

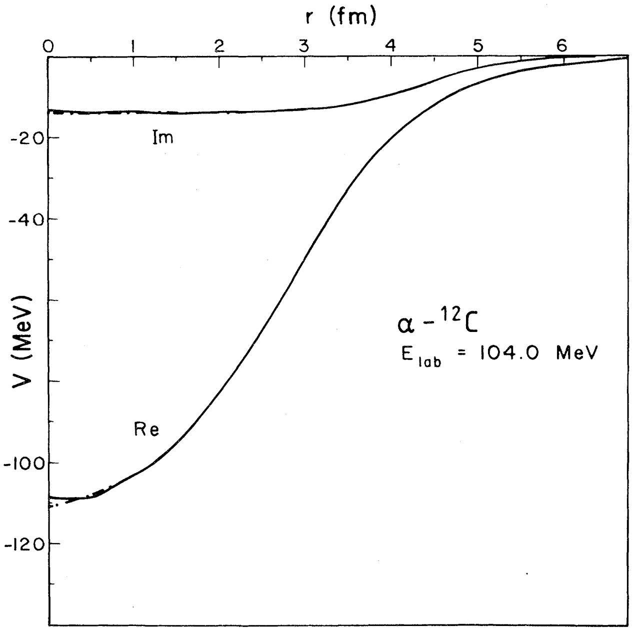

FIG. 7. Input potential (—·—·—·) and its reconstruction by WKB inversion (——) for α-^{12}C scattering at $E_{\rm lab}=104.0$ MeV (Ref. 10).

the input potential only qualitatively, but does not break down.

Finally, we reconsider the system $\alpha+^{12}$C at 104 MeV, for which Kujawski carried out a WKB inversion.[10] A quantal inversion[20] led to a near-perfect reproduction of the input potential. Figure 7 shows that a WKB inversion of the type considered in the present work also gives good results, except perhaps near the origin. Owing to the use of the background scattering function (12), no forbidden region $r<\eta/k$ appears where the inversion would break down, as in Ref. 10.

IV. CONCLUSION

In the present paper we have reconstructed a number of nuclear potentials by semiclassical (WKB) and quantal inversion at fixed energy of the associated scattering function. The input scattering function was represented as a rational function of angular momentum λ on the real λ axis. The potentials corresponding to this scattering function in the WKB and quantal inversion schemes were determined numerically. In this way the adequacy of the WKB inversion could be determined unambiguously.

Using the present method, the WKB inversion results of the earlier work of Kujawski[10] on α-^{12}C scattering could be improved. Generally it turns out that the WKB inversion works surprisingly well even for relatively light scattering systems at fairly low energy. At very low energies, the WKB inversion breaks down, i.e., the inverted potential becomes multiple valued in the critical (approximately equal to classically forbidden) region of space. If such a breakdown occurs, the potential is incorrect even in the remaining, classically allowed region. At energies slightly above the breakdown energy, the WKB inverted potential will be single valued at all distances, but it may differ markedly from the true, quantally-inverted potential.

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*On leave from the University of South Africa, Pretoria, South Africa.

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