Reply to "Bound states in pion-nucleus velocity-dependent potentials: Finite or infinite number"

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The discrepancy between the results obtained by Gal and Mandelzweig and those published by us is shown to be due to an ambiguity in the definition of the radial wave function occurring whenever the velocity-dependent potential satisfies the criticality condition of Ericson and Myhrer. The non-Hermitian solutions predicted by these authors are closely related to that ambiguity.

In the preceding Comment,[1] Gal and Mandelzweig (GM) show, by means of convincing qualitative arguments, that the Schrödinger equation for a real velocity-dependent potential of finite-range spherically symmetric parabolic shape $\alpha(r)$ presents, provided the criticality condition of Ericson and Myhrer[2] (EM) is satisfied, an infinite number of negative energy solutions to be interpreted as bound states of indefinitely high binding energy. In a previous paper[3] we reported, after numerical analysis of the wave function written in terms of hypergeometric functions, that only a finite number of negative energy solutions appear. We are going to show in this reply that the discrepancy is due to the fact that the wave function for such a potential is not uniquely defined.

The radial wave equation presents at r_0, where $\alpha(r)$ equals one, a singular point that prevents the connection along the real axis of the expressions of the wave function for $r < r_0$ and $r > r_0$. Therefore, analytic continuation into the complex r plane is needed and, in order to deal with a single-valued function, the plane has to be cut either along $[-r_0, r_0]$, as done by GM, or along $(-\infty, -r_0]$ and $[r_0, +\infty)$, as taken by us. A possible wave function, $\psi_+(r)$, can be defined on the real axis by considering this as part of the upper half plane. Another possible definition $\psi_-(r)$ is obtained by regarding the real axis as belonging to the lower half plane. Any linear combination $\beta_+\psi_+(r) + \beta_-\psi_-(r)$ could, in principle, be taken as a wave function on the real axis. If one selects, for instance, $\beta_+ = 1$, $\beta_- = 0$, then one obtains[3] an infinity of *decaying* bound states of indefinitely high binding energy, in accordance with the predictions of EM. If, instead, $\beta_+ = 0$ and $\beta_- = 1$ are chosen, the resulting eigenvalues of the Hamiltonian are the complex conjugate of the previous ones, that is, an infinity of *growing* bound states are obtained. Other prescriptions for β_+ and β_- would presumably lead to Hamiltonian spectra intermediate between those ones.

As we are going to show, the requirement of the wave function being real, in the case of bound states, is not sufficient to eliminate the ambiguity. Let us assume that a real solution $\psi(r)$ has been found on the uncut part of the real axis. Then, ψ_+ and ψ_- could be the analytic continuations of $\psi(r)$ into the upper and lower complex r half planes, respectively. Obviously, $\psi_+(r)$ and $\psi_-(r)$ coincide on the uncut part of the real axis, but not necessarily on the cut. They adopt there complex conjugate values. A real wave function on the cut is obtained if it is defined as the mean of the solutions just above and below the cut:

$$\psi(r) = [\psi_+(r) + \psi_-(r)]/2 = \mathrm{Re}[\psi_+(r)] \ . \tag{1}$$

Nevertheless, usual boundary conditions on the wave function, like being regular at the origin, determine a solution only up to an arbitrary (complex) factor. Therefore, equally valid solutions in the upper and lower half planes could be

$$\phi_+ = \beta\psi_+, \quad \phi_- = \bar{\beta}\psi_- \ , \tag{2}$$

and on the real axis

$$\phi(r) = [\phi_+(r) + \phi_-(r)]/2 = \mathrm{Re}[\phi_+(r)] \ . \tag{3}$$

The two solutions $\psi(r)$ and $\phi(r)$ are essentially the same on the uncut part of the real axis, since $\psi_+(r)$ and $\psi_-(r)$ are real and coincident and, therefore, $\phi(r) = (\mathrm{Re}\beta)\psi(r)$. But they turn out to be different on the cut, provided $\mathrm{Im}\beta \neq 0$.

In order to illustrate the above considerations, let us particularize them to the problem discussed in Refs. 1 and 3. The reduced radial wave function $u(r)$ must vanish at the origin and obey a differential equation of the Legendre type, namely, Eq. (4) of GM. Equivalently, the radial wave function $\psi(r) = u(r)/r$, written in terms of the dimensionless variable $z = r^2/r_0^2$, must be regular at the origin and satisfy a hypergeometric equation

$$z(1-z)d^2\psi/dz^2 + [c - (a+b+1)z]d\psi/dz - ab\psi = 0 \ , \tag{4}$$

where

$$a = 1 + \nu/2, \quad b = \tfrac{1}{2} - \nu/2, \quad c = \tfrac{3}{2}, \quad z = x^2 \ , \tag{5}$$

with ν and x as given by GM. Both conditions are fulfilled[4] by the hypergeometric Gauss series $F(a,b;c;z)$ convergent in the circle $|z| < 1$. This solution essentially coincides, in the interval $0 \leq x < 1$ $(0 \leq r < r_0)$, with that found by GM, as explicitly shown in Eq. (15) of Ref. 1. The hypergeometric series can be analytically continued into the complex r plane, cut along the real axis from 1 to $+\infty$, by using[4]

$$F(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F(a,1-c+a;1-b+a;z^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F(b,1-c+b;1-a+b;z^{-1}) \ ,$$
$$|z| > 1, \quad |\arg(-z)| < \pi \ . \tag{6}$$

On the cut, the above mentioned functions $\psi_+(r)$ and $\psi_-(r)$ are, therefore, given by

$$\psi_\pm = \frac{\Gamma(3/2)\Gamma(-1/2-\nu)}{[\Gamma(1/2-\nu/2)]^2}\exp[\pm i(2+\nu)\pi/2]x^{-2-\nu}F(1+\nu/2,\tfrac{1}{2}+\nu/2;\tfrac{3}{2}+\nu;x^{-2})$$
$$+\frac{\Gamma(3/2)\Gamma(1/2+\nu)}{[\Gamma(1+\nu/2)]^2}\exp[\pm i(1-\nu)\pi/2]x^{-1+\nu}F(\tfrac{1}{2}-\nu/2,-\nu/2;\tfrac{1}{2}-\nu;x^{-2}),\quad x>1\ . \tag{7}$$

Consequently, the wave function on the cut, as defined in Eq. (1), results in

$$\psi = \frac{\Gamma(3/2)\Gamma(-1/2-\nu)}{[\Gamma(1/2-\nu/2)]^2}\cos[(2+\nu)\pi/2]x^{-2-\nu}F(1+\nu/2,\tfrac{1}{2}+\nu/2;\tfrac{3}{2}+\nu;x^{-2})$$
$$+\frac{\Gamma(3/2)\Gamma(1/2+\nu)}{[\Gamma(1+\nu/2)]^2}\cos[(1-\nu)\pi/2]x^{-1+\nu}F(\tfrac{1}{2}-\nu/2,-\nu/2;\tfrac{1}{2}-\nu;x^{-2}),\quad x>1\ . \tag{8}$$

This expression can be compared with the solution given by GM in terms of Legendre functions by means of[5]

$$Q_\nu(x) = 2^{-\nu-1}\pi^{1/2}\frac{\Gamma(1+\nu)}{\Gamma(3/2+\nu)}x^{-\nu-1}F(1+\nu/2,\tfrac{1}{2}+\nu/2;\tfrac{3}{2}+\nu;x^{-2})\ ,$$

$$P_\nu(x) = 2^{-\nu-1}\pi^{-1/2}\frac{\Gamma(-1/2-\nu)}{\Gamma(-\nu)}x^{-\nu-1}F(1+\nu/2,\tfrac{1}{2}+\nu/2;\tfrac{3}{2}+\nu;x^{-2})$$
$$+2^{\nu}\pi^{-1/2}\frac{\Gamma(1/2+\nu)}{\Gamma(1+\nu)}x^{\nu}F(\tfrac{1}{2}-\nu/2,-\nu/2;\tfrac{1}{2}-\nu;x^{-2}),\quad x>1\ . \tag{9}$$

By making use of the reflection and duplication formulae[6] for the gamma function, one obtains finally

$$\psi = \frac{\Gamma(3/2)\Gamma(1+\nu)}{[\Gamma(1+\nu/2)]^2}\pi^{-1/2}2^{-\nu}x^{-1}[\pi\sin(\nu\pi/2)P_\nu(x)+2\cos(\nu\pi/2)Q_\nu(x)],\quad x>1\ , \tag{10}$$

which is essentially the same solution proposed by GM in their Eq. (7). It becomes therefore evident that the GM solution is obtained if one selects $\beta_+=\beta_-=\frac{1}{2}$ for the coefficients in the linear combination of ψ_+ and ψ_- mentioned above.

In our solution of the problem[3] we chose, instead, $\beta_+=\bar{\beta}_-=\exp[i(\nu-1)\pi/2]/2$. At that moment, we were unaware of the remaining ambiguity even in the case of real wave function. The selection of such coefficients was fortuitous, motivated only by the fact that simplified expressions, easier to be numerically evaluated, were obtained for $\phi_\pm=\beta_\pm\psi_\pm$. In this way, the wave function on the cut becomes

$$\phi = \frac{\Gamma(3/2)\Gamma(-1/2-\nu)}{[\Gamma(1/2-\nu/2)]^2}\cos[(\nu+\tfrac{1}{2})\pi]x^{-2-\nu}F(1+\nu/2,\tfrac{1}{2}+\nu/2;\tfrac{3}{2}+\nu;x^{-2})$$
$$+\frac{\Gamma(3/2)\Gamma(1/2+\nu)}{[\Gamma(1+\nu/2)]^2}x^{-1+\nu}F(\tfrac{1}{2}-\nu/2,-\nu/2;\tfrac{1}{2}-\nu;x^{-2}),\quad x>1\ , \tag{11}$$

that, written in terms of Legendre functions, turns out to be

$$\phi = \frac{\Gamma(3/2)\Gamma(1/2+\nu)}{[\Gamma(1+\nu/2)]^2}2^{-\nu}\pi^{-1/2}x^{-1}[\pi P_\nu(x)+\sin(\nu\pi)Q_\nu(x)],\quad x>1\ . \tag{12}$$

A qualitative analysis of the logarithmic derivative of this wave function, along the same line of that made by GM, allows one to conclude that the equation obtained by matching, at the edge of the potential, the inner and outer wave functions presents only a finite number of negative energy solutions.

The ambiguity in the definition of the wave function inherent to the problem under consideration is entirely due to the presence of a branch point on the range $[0,+\infty)$ of the radial variable. As we have seen above, that ambiguity explains the existence of EM non-Hermitian solutions. The requirement, for bound states, of the wave function being real eliminates these solutions, but does not remove completely the ambiguity. It is necessary, in addition, to give a specific prescription about the combination of the upper and lower continued wave functions to be taken. The GM prescription leads to quite plausible results and could, therefore, be adopted, but not before finding arguments for eliminating all other prescriptions.

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[5]I. A. Stegun, in *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. Stegun (Dover, New York, 1965), p. 331.
[6]P. J. Davis, in *Handbook of Mathematical Functions,* edited by M. Abramowitz and I. Stegun (Dover, New York, 1965), p. 253.