Bound states in pion-nucleus velocity-dependent potentials: Finite or infinite number

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The specific example suggested by Cariñena and Sesma is shown to yield an infinite number of bound states in agreement with the general results claimed earlier by Ericson and Myhrer and by Mandelzweig, Gal, and Friedman, and in accordance with a rigorous proof given here.

Ericson and Myhrer[1] (EM) observed that a velocity-dependent pion-nucleus optical potential induced by a zero range p-wave πN interaction,

$$2\mu V(r)=\vec{\nabla}\alpha(r)\cdot\vec{\nabla} \ , \tag{1}$$

gives rise to an infinite number of bound states provided $\alpha(r)>1$ over some nuclear region, say $0\leq r<r_0$. The underlying binding mechanism rests on the realization that the attractive potential term (1), reminiscent of kinetic energy, wins over the conventional kinetic energy term of the Schrödinger equation in the interior region $[0,r_0]$. Hence, the problem may be viewed as motion of a particle with a static reduced mass μ in a force field the effects of which culminate in a *negative* dynamical reduced mass $\mu/[1-\alpha(r)]$. Binding is a direct measure of how negative the expectation value of the resultant kinetic energy can become. Trial wave functions of an increasingly oscillatory nature in $[0,r_0]$ lead to an ever larger magnitude of this negative kinetic energy, and an infinite number of bound states emerges. While several elementary physical effects readily remove this infinite multiplicity (e.g., the finite range of the πN interaction) and render whatever remaining bound states unobservably wide [due to the imaginary part of $\alpha(r)$], this model problem merits a study of its own in view of the new quantum mechanical features it presents. Mandelzweig, Gal, and Friedman[2] (MGF) explored in some detail properties of the solutions to (1), particularly for Hermitian solutions ascribed to *real* $\alpha(r)$, i.e., real binding energies. In particular, their WKB construction (Appendix C of Ref. 2), as well as several soluble models worked out by them, confirmed the expectation that the number of bound states is *infinite.*

Cariñena and Sesma[3] (CS) have recently questioned the validity of this expectation, claiming within the example of the Schrödinger equation

$$\{\vec{\nabla}[1-\alpha(r)]\cdot\vec{\nabla}-\kappa^2\}\Psi(\vec{r})=0,\ \ \kappa^2=-2\mu E \ , \tag{2}$$

for the cut-off parabolic shape

$$\alpha(r)=A(1-r^2/R^2)\theta(R-r),\ \ A>1 \ , \tag{3}$$

in the $l=0$ partial wave to have derived a *finite* number of bound states. Here we explicitly refute their claim by reanalyzing this example. In terms of the $l=0$ radial wave function u, $\Psi=u(r)/r$ and $u(0)=0$, Eq. (2) assumes in the interval $r\leq R$ the form of the Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right]+\nu(\nu+1)u=0 \ , \tag{4}$$

where $x=r/r_0$, $r_0=R(1-A^{-1})^{1/2}$ and

$$\nu(\nu+1)=2+(\kappa R)^2/A \ . \tag{5}$$

The most general solution of Eq. (4) for real $\nu>1$ is a linear combination of the Legendre functions $P_\nu(x)$ and $Q_\nu(x)$. These functions can be made intrinsically real by applying the principal value definition as done, for example, in Chapter 3.4 of Ref. 4 for $x^2<1$. The resulting functions $P_\nu(x)$ are regular for $x>0$ while the functions $Q_\nu(x)$ are logarithmically singular at $x=1$:

$$Q_\nu(x)\underset{x\sim 1}{\sim}-\tfrac{1}{2}\ln(|x-1|/2)-\gamma-\psi(\nu+1) \ , \tag{6}$$

where γ is the Euler constant and ψ is the logarithmic derivative of the gamma function. A linear combination that satisfies $u(0)=0$ is given by[4]

$$u(x)\sim(\pi/2)\,\mathrm{tg}(\pi\nu/2)P_\nu(x)+Q_\nu(x) \ . \tag{7}$$

The bound-state condition is derived by matching the logarithmic derivative $\mathscr{R}_\nu$ of the inner solution (7) to that of the outer solution

$$u(x)\sim\exp(-\kappa r_0 x) \ , \tag{8}$$

at $x=R/r_0=[A/(A-1)]^{1/2}$:

$$\mathscr{R}_\nu\equiv\left.\frac{(\pi/2)\mathrm{tg}(\pi\nu/2)P_\nu'(x)+Q_\nu'(x)}{(\pi/2)\mathrm{tg}(\pi\nu/2)P_\nu(x)+Q_\nu(x)}\right|_{x=[A/(A-1)]^{1/2}}$$
$$=-(A-1)^{1/2}[\nu(\nu+1)-2]^{1/2} \ . \tag{9}$$

We now show that, for any value $A>1$, the condition (9) is satisfied once in *any* of the intervals $2n-1<\nu<2n+1$, $n=1,2,\ldots$. The following properties of the Legendre functions for $x>1$ are helpful:

$$P_\nu(x)>0,\ \ P_\nu'(x)>0,\ \ Q_\nu(x)>0,\ \ Q_\nu'(x)<0 \ . \tag{10}$$

To prove these properties we generalize the proof of Theorem 1 of MGF. Multiply Eq. (4) for $P_\nu(x)$ by $P_\nu(x)$ and integrate from $x=1$ to an arbitrary $x_0>1$:

$$(x_0^2-1)P_\nu(x_0)P_\nu'(x_0)=\int_1^{x_0}(x^2-1)[P_\nu'(x)]^2dx+\nu(\nu+1)\int_1^{x_0}P_\nu^2(x)dx \ . \tag{11}$$

The right-hand side is positive definite; hence, P_ν and P_ν' never vanish and share the same sign for $x>1$. It is customary to accept a positive sign for $P_\nu(x)$, so that $P_\nu'(x)>0$. The logarithmic singularity of $Q_\nu(x)$ at $x=1$

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and its positive definite nature for $x > 1$ is best demonstrated by the integral representation

$$Q_\nu(x) = P_\nu(x) \int_x^\infty \frac{dt}{(t^2-1)P_\nu^2(t)}, \quad x > 1 \ , \tag{12}$$

which follows immediately from the Wronskian relationship $(1-x^2)(P_\nu Q_\nu' - P_\nu' Q_\nu) = 1$. The latter also shows that $Q_\nu' < 0$ at the very vicinity of $x = 1$. To show that $Q_\nu' < 0$ everywhere for $x > 1$ we assume to the contrary that $Q_\nu' > 0$ somewhere in this region. Therefore, a point $x_0 > 1$ must exist such that $Q_\nu'(x_0) = 0$; multiply Eq. (4) for $Q_\nu(x)$ by $Q_\nu(x)$ and integrate from x_0 to infinity:

$$\int_{x_0}^\infty (x^2-1)[Q_\nu'(x)]^2 dx + \nu(\nu+1) \int_{x_0}^\infty Q_\nu^2(x) dx = 0 \ , \tag{13}$$

which is impossible to satisfy in view of the positive definite nature of the integrands.

With the properties (10) thus proven it becomes clear that the denominator of $\mathscr{R}_\nu$, Eq. (9), vanishes at a value $\nu_n^{(1)}$ somewhere in the interval $(2n-1, 2n)$, remaining positive from there up to $\nu = 2n+1$, while the numerator vanishes at a value $\nu_n^{(2)}$ somewhere in the interval $(2n, 2n+1)$, keeping negative from there down to $\nu = 2n-1$. Therefore, $\mathscr{R}_\nu$ sweeps over *all* negative values when ν varies between $\nu_n^{(1)}$ and $\nu_n^{(2)}$ and provides a crossing there to the bounded negative definite right-hand side of (9). This situation is qualitatively depicted in Fig. 1 for any ν interval $[2n-1, 2n+1], n = 1, 2, \ldots$. Thus, there exists an infinity of mutually nondegenerate bound-state solutions. More explicitly, for $A = 1+\epsilon$ the matching point becomes $\sim \epsilon^{-1/2}$ and one is justified in using the asymptotic expansions[4] in x:

$$P_\nu(x) \to \frac{2^\nu}{\pi^{1/2}} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)} x^\nu \ ,$$
$$Q_\nu(x) \to \frac{\pi^{1/2}}{2^{\nu+1}} \frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)} x^{-\nu-1} \ . \tag{14}$$

The resulting points $\nu_n^{(1)}$ and $\nu_n^{(2)}$ are infinitesimally close to $\nu = 2n$. For example, for $A = 1.1$: $\nu_1^{(1)} \simeq 1.9999$ while $\nu_1^{(2)} \simeq 2.0002$, a delicately minute interval to search for numerically! Since $\mathscr{R}_{2n} \simeq -(2n+1)\epsilon^{1/2}$ whereas the right-hand side of Eq. (9) assumes the value $-[(2n+1)^2 - (2n+3)]^{1/2}\epsilon^{1/2}$ for $\nu = 2n$, the desired crossing of the $\mathscr{R}_\nu$ curve in the figure occurs to the right of $\nu = 2n$; for the case discussed above, $A = 1.1$, this occurs at $\nu = 2.0001$ for the first solution. Thus, the bound-state energies increase relative to their values $\nu = 2n$ appropriate to the parabolic model of MGF when a cutoff in $\alpha(r)$ is made for $r > R$. This feature appears natural because the kinetic energy assumes there its normal form rather than growing up indefinitely in consequence of the ever decreasing dynamical reduced mass in the parabolic model.

These conclusions, namely, that an infinity of nondegenerate bound states exists for $A > 1$ in the cut-off parabolic model (3) and their relationship to those of the parabolic model[2] are diametrically opposed to the statements made by CS who claim that a minimal value of $A_{cr} = 62.4$ is necessary for inception of such states and that, for any value $A > A_{cr}$, there exist only a finite number of bound states. It is not clear what led them astray. In fact, the present starting expression for $u(x)$ Eq. (7) may easily be rewritten in the form

$$u(x) = -\frac{\pi^{3/2}[\mathrm{ctg}(\pi\nu/2) + \mathrm{tg}(\pi\nu/2)]}{\Gamma(-\nu/2)\Gamma(\nu/2+\frac{1}{2})} \times x F(\tfrac{1}{2}-\nu/2, 1+\nu/2; \tfrac{3}{2}; x^2) \ , \tag{15}$$

involving the same hypergeometric function advanced by CS for $x^2 < 1$. However, no details of what is actually done for $x^2 > 1$ are given by them, so it is impossible to reproduce their calculation. Furthermore, for some values of A, CS find degenerate bound-state eigenvalues which, obviously, are forbidden for a second order differential equation (this can be shown by working out the Wronskian of the corresponding wave functions and concluding that a mere proportionality must hold between these).

It is worth stressing that the behavior of $\alpha(r)$ in the region $\alpha < 1$ is of little interest as far as the phenomenon of binding in velocity-dependent potentials is qualitatively concerned. What does matter is the existence of a region of space in which $\alpha(r) > 1$ holds. As soon as this condition is satisfied, an infinity of bound states emerges, as shown in the next paragraph. In the framework of the originally proposed[2] parabolic model this is best illustrated by variations other than the introduction of cutoff, Eq. (3). Thus, suppose one confines the motion in the parabolic model to $r < \rho$, with $\rho > r_0$ (in particular, $\rho = R$). The appropriate bound-state condition

$$tg(\pi\nu/2)P_\nu(\rho/r_0) + Q_\nu(\rho/r_0) = 0 \tag{16}$$

yields, in view of the positive definite nature of P_ν, Q_ν, one

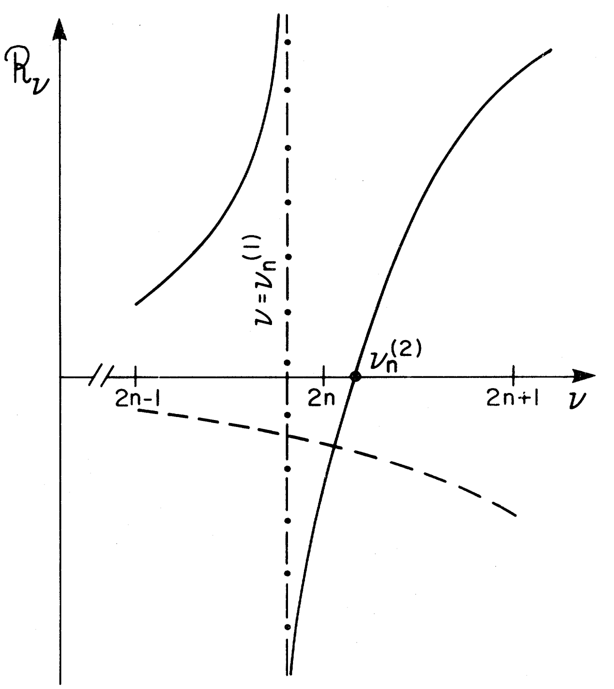

FIG. 1. Schematic representation of the function $\mathscr{R}_\nu$ (continuous line), Eq. (9), in the interval $(2n-1, 2n+1)$ for $n = 1, 2, \ldots$ together with its asymptote (the vertical dash-dotted line). The dashed line gives the right-hand side of Eq. (9). The continuous and dashed lines cross each other at a ν value between $\nu_n^{(1)}$ and $\nu_n^{(2)}$, to the right of $\nu = 2n$ for the CS model.

solution in any of the ν intervals $[2n-1,2n]$ for $n=1,2,\ldots$. Hence, infinite binding occurs in spite of the definite increase in the kinetic energy due to confinement. This increase is reflected by the repulsive shift of the eigenvalues from the values $2n$ of the unconfined parabolic model. Furthermore, one could choose a confining radius $\rho < r_0$ in which case no logarithmic singularity is met at all. Employing $\cos\theta = r/r_0$ with $0<\theta_0=\cos^{-1}(\rho/r_0)$, the following asymptotic expansions[4] in ν are useful in the segment $\theta_0 \leq \theta \leq \pi/2$:

$$P_\nu(\cos\theta)=\frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}\left(\frac{2}{\pi\sin\theta}\right)^{1/2}\cos[(\nu+\tfrac{1}{2})\theta-\pi/4]+O(\nu^{-1}) ,$$
$$Q_\nu(\cos\theta)=\frac{\Gamma(\nu+1)}{\Gamma(\nu+3/2)}\left(\frac{\pi}{2\sin\theta}\right)^{1/2}\cos[(\nu+\tfrac{1}{2})\theta+\pi/4]+O(\nu^{-1}) . \quad (17)$$

The boundary condition (16) is readily satisifed by the infinite sequence of ν values

$$\nu=[(\pi/2)/(\pi/2-\theta_0)]2k-\tfrac{1}{2},\quad k=1,2,3,\ldots . \quad (18)$$

For $\theta_0 << \pi/2$ ($\rho \rightarrow r_0$) one recovers $\nu \sim 2k-\frac{1}{2}$, again a repulsive energy shift.

In conclusion, the existence of an infinity of bound-state solutions in the general case is made rigorous by following three steps. First, one proves that the energy spectrum of Eq. (2) is unbounded from below by considering the energy functional

$$I(\Psi)\equiv\frac{1}{2\mu}\int[1-\alpha(r)][\vec{\nabla}\Psi(\vec{r})]^2d^3r \Big/ \int\Psi^2d^3r . \quad (19)$$

An infinite sequence of trial wave functions Ψ_{nl}, $n=1,2,\ldots$ for any l, is served by the bound-state solutions[1] in a square well $\tilde{\alpha}(r)=\alpha_0\theta(R-r)$ with $1<\alpha_0<\alpha(0)$, confined to a radius R, $R<R_0$ where $\alpha(R_0)=\alpha_0$. Thus,

$$I(\Psi_{nl})<\frac{(1-\alpha_0)}{2\mu}\int_0^R(\vec{\nabla}\Psi_{nl})^2d^3r\Big/\int_0^R\Psi_{nl}^2d^3r ,$$
$$=-\frac{(\alpha_0-1)\pi^2\beta_{nl}^2}{2\mu R^2},\quad \beta_{nl}\underset{n\rightarrow\infty}{\longrightarrow}(n+l/2) , \quad (20)$$

and the energy functional (19) can be made arbitrarily negative by taking n sufficiently large or reducing R for a given n. If the spectrum were bounded from below by E_0 it would be impossible to drive the energy functional (19) below E_0, contrary to the construction. Hence, for any l, the energy spectrum is unbounded from below and its negative energy part is nonempty. In the second step one shows that any one of the negative-energy eigenfunctions Ψ_l corresponds to a *finite* energy. To this end multiply Eq. (2) for Ψ_l by wave function Φ_l belonging to an arbitrarily chosen point of the positive-energy continuous spectrum, subtract similar terms obtained by interchanging Ψ_l and Φ_l and integrate within a sphere of radius R to obtain

$$[1-\alpha(R)]\oint_{|\vec{r}|=R}[\Phi_l(\vec{r})\vec{\nabla}\Psi_l(\vec{r})-\Psi_l(\vec{r})\vec{\nabla}\Phi_l(\vec{r})]\cdot d\vec{S}$$
$$-(\kappa_{\Psi_l}^2-\kappa_{\Phi_l}^2)\int^R\Psi_l\Phi_l d^3r=0 . \quad (21)$$

Here all quantities involving the wave functions Ψ_l and Φ_l are finite, and so must be the difference $(\kappa_{\Psi_l}^2-\kappa_{\Phi_l}^2)$. Therefore, $\kappa_{\Psi_l}^2$ is finite. Since a finite sequence of negative-energy eigenstates for a given l is necessarily bounded from below, the existence of an *infinite* sequence of discrete eigenstates of (2), with increasingly negative energies, is thus established for any l. The addition of a velocity independent potential to (2) does not alter these results.

In the third, last step one derives the asymptotic distribution of the eigenvalues by repeating the WKB construction of MGF with the result [their Eq. (2.30)]:

$$\kappa_{nl}\simeq\frac{(n+l/2+1/4)\pi}{\int_0^{r_0}dr/[\alpha(r)-1]^{1/2}} . \quad (22)$$

This expression may also be obtained by generalizing the discussion of Sec. 6.2.3 in Ref. 5, in particular Eq. (19a) therein.

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