Four-body calculation of the four-nucleon system: Binding energies and scattering results

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Using a two-body separable t matrix between pairs, we solve the resulting four-body equations for four spinless bosons in the form that the $2+2$ subsystem contribution is treated exactly by the convolution method. The $3+1$ subamplitudes are represented as finite rank operators. We compare the utility of these methods both in four-body bound state and scattering calculations. We also develop an approximation that neglects four-body intermediate states in the $2+2$ subsystem contribution and compare it with exact results. Finally, we present the results for a four-nucleon calculation with spin dependent s-wave separable interactions between pairs. These formally exact four-body equations are subsequently used to develop one parameter models to describe low energy phase shifts and cross sections for the reactions $n\,^3H \to n\,^3H$, $dd \to dd$, and $dd \to p\,^3H$.

I. INTRODUCTION

In recent years, some progress has been made in the solution of the four-body equations of Yakubovsky[1] for the wave function or those of Alt, Grassberger, and Sandhas[2] (AGS) for the t-matrix components. In all calculations the two-body interaction is either separable or represented as a finite rank operator through any of the available techniques, such as the Hilbert-Schmit[3] (HS) expansion or the unitary pole expansion[4] (UPE). From there onward, the methods used to calculate four-body observables range from the solution of a two-continuous-variable integral equation to the expansion of all $3+1$ and $2+2$ subamplitudes in a separable form leading to a set of coupled one-variable integral equations. Although most of the work has been done in the negative energy region, where several numerically converged calculations of the four-nucleon binding energy already exist, there are a few threshold and low energy scattering results.[5] Since the analytic complexity of the four-body kernel grows as the center of mass energy goes beyond the three-body breakup threshold, the accuracy of calculations in that energy region remains to be confirmed. This is in general the underlying reason to develop approximations. The aim is to simplify the analytic structure of the kernel leading to four-body equations that may be solved with acceptable accuracy in a specified energy domain.[6] One hopes that by retaining the dominant features of the four-body kernel one may obtain results that are not too different from those one may get with the full kernel. So far nothing has really worked well, although some models may have come closer than others.

Recently, Haberzettl and Sandhas[7] formulated AGS four-body equations for the t-matrix components in a way that the $2+2$ subsystem contributions are treated exactly by the convolution method. In this approach all four-body amplitudes can be calculated from the solution of a single integral equation for the reaction $(3)+1 \to (3)+1$. They also went on to show that, using single term separable approximations for the two-particle and $3+1$ subsystem amplitudes, the driving term of the final four-body equation is reduced to one of the field theoretic models of Refs. 8 and 9, where the convolution method was first used to calculate the $2+2$ subsystem contributions. Their work also showed the shortcomings of the approximation used in the field theoretic model to formulate separable three-body subamplitudes and suggested instead an exact representation of all $3+1$ subsystem amplitudes as finite rank operators. At that time they proposed the generalized unitary pole expansion[10] (GUPE) because the effective three-body form factors are chosen there to be energy independent, but one may well consider other methods such as the energy dependent pole expansion[11] (EDPE) or any of the recently developed expansions[12] (SE1 and SE2) which represent the exact amplitudes with a fewer number of terms than GUPE or EDPE. Of all methods, SE2 is the only one that may be used at all energies since both EDPE and GUPE fail to represent the exact three-body amplitude above breakup threshold[12,13] and SE1 requires a larger number of terms than SE2 for the same accuracy.

In the present paper we return to the field theoretic approach of Refs. 8 and 9 and replace the model $3+1$ subamplitude by an exact representation of the three-body amplitude as an operator of rank N. The four-body equations we obtain are identical to those in Ref. 7 where, as mentioned above, the $2+2$ subsystem contributions are taken exactly by the convolution method. For a single term expansion of the three-body amplitude we essentially recover the equations of Ref. 8 but having now included a proper representation of the underlying $3+1$ subamplitude. Since we follow a field theoretic description of the two-body interaction, we have access to the parameter z, the wave function renormalization constant, that may take the range of values $0 \le z \le 1$. In the limit $z=0$, our approach yields the same results as AGS or Yakubovsky four-body equations with separable two-body interactions, much as the Aaron, Amado, and Yam (AAY)[14,15] three-body equations are identical to Faddeev equations with separable two-body interactions. As in the Haberzettl and

Sandhas modification of AGS equations, we treat the 2+2 subsystem contributions in an exact manner by the convolution method. This has three main advantages over the conventional procedure which consists of expanding the 3+1 and 2+2 subsystem amplitudes in a separable form: First, we deal with a smaller number of coupled four-body equations; second, we avoid convergency problems connected with the number of terms necessary to accurately represent the 2+2 subamplitude; and third, we save computing time if the programming is carefully done. This procedure has recently been attempted by Haberzettl and Sofianos[16] in a four-boson bound state calculation using a single term in the expansion of the $l=0$ 3+1 subamplitude.

In the first part of this paper we consider a system of four identical spinless "nucleons." Our aim here is to study how four-body observables, such as the binding energy and the low energy phase shifts for any 2→2 reaction, depend on the number N of terms in the expansion of the 3+1 subamplitude. Since any of the methods referenced above (GUPE, EDPE, SE1, and SE2) requires the calculation of three-body Sturmian functions at a chosen energy B, we also study which among the possible choices for B provides more accurate four-body results. For the chosen two-body interaction we get −89.74 MeV for the ground state which compares with the value −89.6 MeV obtained by Gibson and Lehman[17] and −90.1 MeV obtained by Narodetsky.[18] For the excited state we get −26.60 MeV compared with −26.64 MeV obtained by Narodetsky.[18] In the scattering region we take only the contribution of the s-wave 3+1 subamplitude and solve the resulting four-body equations by matrix inversion together with the contour deformation method.[19] The phase shifts for the reactions (3)+1→(3)+1 and (2)+(2)→(2)+(2) are calculated as a function of N. The GUPE, EDPE, SE1, and SE2 methods are tested in the reaction (2)+(2)→(2)+(2) with two different choices for the energy B at which the Sturmian functions are calculated in order to study the sensitivity of the four-body results to the correct off shell representation of the 3+1 subamplitude. The chosen values for B are $B=\epsilon_3=-25.53$ MeV and $B=\epsilon_3^*=-2.38$ MeV, which are the energies of the two $l=0$ three-body bound states.

In the present paper we also study the effect of an approximation previously developed[20] that neglects a certain class of four-body intermediate states in the calculation of the 2+2 subsystem contribution through the convolution method. We find that the four-body binding energy becomes −86.62 MeV for the ground state and −26.37 MeV for the excited state. This approximation is also tested in the scattering region.

In the second part of the paper we consider a four-nucleon calculation where as in Ref. 9 the two-body s-wave interaction is spin dependent. Again for the chosen two-body parameters we reproduce the results of previous calculations for the bound states of the α particle. Our result for the 0^+ ground state is $\epsilon_\alpha=-45.59$ MeV which compares with −45.7 MeV taken from the work of Gibson and Lehman[17] and −45.73 MeV obtained by Narodetsky.[18] For the 0^+ excited state we get $\epsilon_\alpha^*=-11.63$ MeV which compares with the value −11.69 MeV taken from Ref. 18.

Based on this four-nucleon calculation we develop simple models with which we attempt to describe low energy scattering data for the processses $n\,^3H\to n\,^3H$, dd→dd, and $dd\to p\,^3H$. Since, for the chosen two-body interaction, the three-body binding energy is $\epsilon_t=-11.01$ MeV, we change the wave function renormalization constant z_d away from zero to adjust the position of the three-body bound state pole with the triton binding energy. For $z_d=5.074\times10^{-2}$ we get $\epsilon_t=-8.48$ MeV together with $\epsilon_\alpha=-29.67$ MeV for the four nucleon 0^+ ground state and $\epsilon_\alpha^*=-8.65$ MeV for the 0^+ excited state.

We also develop an alternative way to generate the changes mentioned above without moving z_d away from zero. Instead we weaken the $l=0$ spin double three-nucleon kernel until it develops a unit eigenvalue at $E_3=-8.48$ MeV. Using the 3+1 subamplitude that emerges from this modified spin doublet three-nucleon kernel, one changes the four-body 0^+ ground state to $\epsilon_\alpha=-37.24$ MeV and the excited state to $\epsilon_\alpha^*=-9.07$ MeV.

These two models that are based on formally exact four-body equations are subsequently used to describe $n\,^3H\to n\,^3H$, $p\,^3H_e\to p\,^3H_e$, dd→dd, and $dd\to p\,^3H$ reactions below breakup threshold. The results of our calculations are particularly good for the $L=0$ and $L=2$ $I=1$ phase shifts. Nevertheless, the correct description of $n\,^3H\to n\,^3H$, dd→dd, and $dd\to p\,^3H$ reactions may require the inclusion of the $l=1$ 3+1 subamplitudes, at least in first order perturbation.

In Sec. II we describe the equations for four identical spinless bosons and in Sec. III we present the results of our calculations for the spinless four-body system. In Sec. IV we show the results of our four nucleon calculation and in Sec. V we draw some conclusions.

II. FORMALISM

In this section we write the equations for a system of four identical spinless nucleons and review some of the results that have been developed elsewhere. Although we follow a field theoretic description of the two-body interaction, the final equations are those of Haberzettl and Sandhas,[7] where the 2+2 subsystem contribution is taken exactly by the convolution method. We use the following units everywhere: $\hbar=2m=1$, where m is the mass of the nucleon.

A. Two-body interaction

As in Ref. 8 the two-body n-n interaction is mediated by the quasiparticle d that is coupled in s wave to d↔n+n. The two-body t matrix has a separable form in momentum space

$$\langle \vec{k}' \mid t(E) \mid \vec{k} \rangle = \frac{\gamma^2}{2} f(\vec{k}) \tau(E-\epsilon) f(\vec{k}') , \tag{1}$$

where τ is the d-particle propagator

$$\tau(X)=S(X)/X\ , \tag{2}$$

$$[S(X)]^{-1}=z-\frac{\gamma^2}{2}\int\frac{d^3n}{(2\pi)^3}\frac{f^2(\vec{n})}{(2\vec{n}^{\,2}-\epsilon)(X+\epsilon-2\vec{n}^{\,2})}\ , \tag{3}$$

and

$$1=z+\frac{\gamma^2}{2}\int\frac{d^3n}{(2\pi)^3}\frac{f^2(\vec{n})}{(2\vec{n}^{\,2}-\epsilon)^2}\ . \tag{4}$$

The two-body interaction is therefore characterized by the coupling strength γ, the vertex function $f(\vec{q})$, and the wave function renormalization constant z that is allowed to take on the range of values $0\le z\le 1$. If $z=1$, the d is an elementary particle uncoupled to n+n, while if $z=0$, a separable potential model is obtained in which the d is a bound state of two n's with binding energy equal to ϵ. For $z=0$, Eq. (4) shows that, like in a separable potential model, γ takes the value that normalizes the two-body bound state wave function to unity. Unless otherwise specified, we always consider $z=0$. For the vertex function $f(\vec{q})$ we chose the Yamaguchi form $f(\vec{q})=(q^2+\beta^2)^{-1}$, where β is the range parameter.

B. Three-body equations

In the three-body sector, nd scattering proceeds by successive n exchanges, and the scattering amplitude satisfies the AAY three-body equation[14]

$$\langle\vec{k}'\,|\,T(E)\,|\,\vec{k}\rangle=\langle\vec{k}'\,|\,V(E)\,|\,\vec{k}\rangle+\int\frac{d^3n}{(2\pi)^3}\langle\vec{k}'\,|\,V(E)\,|\,\vec{n}\rangle\tau(E-\epsilon-\tfrac{3}{2}\vec{n}^{\,2})\langle\vec{n}\,|\,T(E)\,|\,\vec{k}\rangle\ , \tag{5}$$

where τ is the d particle propagator given by (2) and $V(E)$ is the effective potential that results from a single n exchange

$$\langle\vec{k}'\,|\,V(E)\,|\,\vec{k}\rangle=\gamma^2\frac{f(\vec{k}+\frac{1}{2}\vec{k}')f(\vec{k}'+\frac{1}{2}\vec{k})}{E-\vec{k}^{\,2}-(\vec{k}+\vec{k}')^2-\vec{k}'^{\,2}}\ . \tag{6}$$

The three-body amplitude T, once embedded in four-body space, accounts for the 3+1 subsystem contributions to the four-body sector. As shown in Ref. 7, a separable representation of all 3+1 subamplitudes is required to obtain a one vector variable four-body integral equation for the t-matrix components. Therefore we now review some of the methods we use as a means to express the t matrix T for nd→nd as a finite rank operator.

C. Separable expansions for 1+3 subamplitudes

Here we follow the work of Ref. 10 for GUPE, Ref. 11 for EDPE, and Ref. 12 for two recently developed separable expansion methods (SE1) and (SE2) that are most appropriate at energies above breakup threshold. Let T be the three-body subamplitude and

$$T(Z)=V(Z)+V(Z)\tau(Z)T(Z) \tag{7}$$

the operator version of Eq. (5) where Z is the complex energy parameter. Given a suitably chosen energy B, the eigenvalues ξ_n of the kernel $V\tau$ at that energy satisfy the equation

$$|\,\xi_n(B)\rangle=\frac{1}{\eta_n(B)}V(B)\tau(B)\,|\,\xi_n(B)\rangle\ , \tag{8}$$

where η_n is the correspondent eigenvalue. The eigenfunctions are normalized according to

$$\langle\xi_n(B)\,|\,\tau(B)\,|\,\xi_m(B)\rangle=-\delta_{mn}\ . \tag{9}$$

In any of the methods referenced above the T matrix $T(Z)$ may be written as an operator of rank N

$$T_N(Z)=\sum_{m,n=1}^{N}|\,g_m(Z)\rangle D_{mn}(Z)\langle g_n(Z)\,|\ , \tag{10}$$

where $g_n(Z)$ is an appropriate three-body form factor and $D_{mn}(Z)$ is the element of an $N\times N$ matrix. The expressions for g and D depend on the method we choose.

For GUPE,

$$[D^{-1}(Z)]_{nm}=\Delta_{nm}(Z)-\langle\xi_n(B)\,|\,\tau(Z)\,|\,\xi_m(B)\rangle\ , \tag{11}$$

where

$$[\Delta^{-1}(Z)]_{nm}=\langle\xi_n(B)\,|\,\tau(B)V(Z)\tau(B)\,|\,\xi_m(B)\rangle\ . \tag{12}$$

The form factor $g_n(Z)$ equals $\xi_n(B)$ for all Z. Therefore from the numerical point of view GUPE is the simplest method since it involves energy independent three-body form factors.

For EDPE,

$$[D^{-1}(Z)]_{nm}=[\Delta^{-1}(Z)]_{nm}-\langle g_n(Z)\,|\,\tau(Z)\,|\,g_m(Z)\rangle\ , \tag{13}$$

where $\Delta^{-1}(Z)$ is given by Eq. (12) and

$$|\,g_n(Z)\rangle=V(Z)\tau(B)\,|\,\xi_n(B)\rangle\ . \tag{14}$$

For SE1,

$$[D^{-1}(Z)]_{nm}=\langle\xi_n(B)\,|\,\tau^P(Z)V(Z)\tau^P(Z)\,|\,\xi_m(B)\rangle-\langle g_n(Z)\,|\,\tau(Z)\,|\,g_m(Z)\rangle\ , \tag{15}$$

where

$$|\,g_n(Z)\rangle=V(Z)\tau^P(Z)\,|\,\xi_n(B)\rangle\ , \tag{16}$$

and $\tau^P(Z)$ denotes the principal value part of $\tau(Z)$.

For SE2,

$$[D^{-1}(Z)]_{nm}=\langle\tilde{\xi}_n(Z)\tau^P(Z)V(Z)\tau^P(Z)\,|\,\tilde{\xi}(Z)\rangle-\langle g_n(Z)\,|\,\tau(Z)\,|\,g_m(Z)\rangle\ , \tag{17}$$

where

$$|\,\tilde{\xi}_n(Z)\rangle=V(Z)\tau(B)\,|\,\xi_n(B)\rangle \tag{18}$$

and

$$|\,g_n(Z)\rangle=V(Z)\tau^P(Z)\,|\,\tilde{\xi}(Z)\rangle\ . \tag{19}$$

The energy B at which the Sturmian functions $\xi_n(B)$ are

calculated depends on the properties of the original operator T one attempts to represent. The two common choices of B are either the bound state energy or the energy of the lowest scattering threshold.

D. Four-body equations

Having chosen a separable representation for the $3+1$ subamplitude T we now proceed to the four-body sector. The possible two-to-two reactions are $1+(3)\to 1+(3)$ and $1+(3)\to(2)+(2)$, as well as $(2)+(2)\to(2)+(2)$ and $(2)+(2)\to 1+(3)$, where (3) stands for a bound state of three n's and (2) for a bound state of two n's. As in Refs. 7 and 16 we treat the $2+2$ subsystem contributions exactly by the convolution method. Therefore all four-body amplitudes can be calculated from the solution of a single one vector variable integral equation. Letting $\mathscr{T}_1$ represent the full $1+3\to 1+3$ four-body amplitude, the specific form of the equation is

$$\langle \vec{k}' | \mathscr{T}_1^{ji}(E) | \vec{k} \rangle = \langle \vec{k}' | \mathscr{B}^{ji}(E) | \vec{k} \rangle + \sum_{m,n}^{N} \int \frac{d^3q}{(2\pi)^3} \langle \vec{k}' | \mathscr{B}^{jm}(E) | \vec{q} \rangle D_{mn}(E - \tfrac{4}{3}\vec{q}^{\,2}) \langle \vec{q} | \mathscr{T}_1^{ni}(E) | \vec{k} \rangle , \tag{20}$$

where

$$\mathscr{B}^{ji}(E) = B_1^{ji}(E) + U_1^{ji}(E) + U_2^{ji}(E) , \tag{21}$$

D_{mn} is an element of the $N\times N$ matrix shown in (10), and N is the number of terms in the expansion of the $3+1$ subamplitude. This equation is graphically represented in Fig. 1(a) where $B_1(E)$ is the d-particle exchange Born term, $U_1(E)$ is the box amplitude depicted first in Fig. 1(b), and $U_2(E)$ is the sum of the last two box amplitudes. Both $U_1(E)$ and $U_2(E)$ are responsible for the $2+2$ contributions to the four-body sector through the intermediate d+d propagator. The solution of Eq. (20) also provides a means of obtaining the rearrangement amplitude for $1+(3)\to(2)+(2)$. This amplitude has been depicted in Fig. 2 where we see that it may be written in terms of an integral over the half-off-shell elastic amplitude $\mathscr{T}_1(E)$. Letting $\mathscr{T}_2$ be the amplitude for $1+3\to 2+2$, the precise form of this relation is

$$\langle \vec{k}' | \mathscr{T}_2^{ai}(E) | \vec{k} \rangle = \langle \vec{k}' | B_2^{ai}(E) | \vec{k} \rangle + \sum_{m,n}^{N} \int \frac{d^3q}{(2\pi)^3} \langle \vec{k}' | B_2^{am}(E) | \vec{q} \rangle D_{mn}(E - \tfrac{4}{3}\vec{q}^{\,2}) \langle \vec{q} | \mathscr{T}_1^{ni}(E) | \vec{k} \rangle , \tag{22}$$

where a is the $(2)+(2)$ channel dd. The Born term $B_1(E)$ may be written as

$$\begin{aligned} &\langle \vec{k}' | B_1^{mn}(E) | \vec{k} \rangle = g_m(E - \tfrac{4}{3}\vec{k}'^{2}; \vec{k} + \tfrac{1}{3}k') \tau(W) g_n(E - \tfrac{4}{3}\vec{k}^{\,2}; \vec{k}' + \tfrac{1}{3}\vec{k}) , \\ &W = E - \epsilon - \vec{k}^{\,2} - \tfrac{1}{2}(\vec{k} + \vec{k}')^2 - \vec{k}'^{2} , \end{aligned} \tag{23}$$

where g_m is the three-body form factor of Eq. (10). Since τ is the full d-particle propagator, $B_1(E)$ contains both the d-particle exchange pole as well as the n-n continuum contribution. The Born term $B_2(E)$ is given by

$$\langle \vec{k}' | B_2^{am}(E) | \vec{k} \rangle = \frac{1}{\sqrt{2}} [\langle \vec{k}' | \bar{B}_2^{am}(E) | \vec{k} \rangle + \langle -\vec{k}' | \bar{B}_2^{am}(E) | \vec{k} \rangle] , \tag{24}$$

$$\langle \vec{k}' | \bar{B}_2^{am}(E) | \vec{k} \rangle = \frac{\gamma f(\vec{k} + \frac{1}{2}\vec{k}') g_m(E - \frac{4}{3}\vec{k}^{\,2}; \vec{k}' + \frac{2}{3}\vec{k})}{E - \epsilon - \frac{1}{2}\vec{k}'^{2} - (\vec{k} + \vec{k}')^2 - \vec{k}^{\,2}} . \tag{25}$$

The presence of a direct and an exchange term in $B_2(E)$ is due to the identity of the particles in the $(2)+(2)$ channel. As mentioned before, the box amplitudes $U_1(E)$ and $U_2(E)$ are calculated through the convolution method.

$$\langle \vec{k}' | U_\mu^{mn}(E) | \vec{k} \rangle = \int \frac{d^3k''}{(2\pi)^3} \langle \vec{k}'' | \bar{B}_2^{am}(E) | \vec{k}' \rangle G_\mu(E; \vec{k}, \vec{k}'', \vec{k}') \langle \pm\vec{k}'' | \bar{B}_2^{an}(E) | \vec{k} \rangle , \tag{26}$$

where the plus sign corresponds to $U_1(E)$ and the minus sign to $U_2(E)$. The G_1 and G_2 propagators are

$$G_1(E; \vec{k}, \vec{k}'', \vec{k}') = \tau(Y) - \frac{(Y-Q)(Y-Q')}{\pi} \int_0^\infty dx \frac{\mathrm{Im}[\tau(x-\epsilon)]\tau(Y+\epsilon-x)}{(Y-Q+\epsilon-x)(Y-Q'+\epsilon-x)} , \tag{27}$$

$$\begin{aligned} G_2(E; k, k'', k') = {} & \frac{(Y-Q'')}{-Q''}\tau(Y) + \frac{(Y-Q)(Y-Q'')}{Y-Q-Q''}\tau(Q'')\tau(Y-Q'') - \frac{(Y-Q)(Y-Q'')}{\pi} \\ & \times \int_0^\infty dx \frac{\mathrm{Im}[\tau(x-\epsilon)]\tau(Y+\epsilon-x)}{(x-\epsilon-Q'')(Y-Q+\epsilon-x)} , \end{aligned} \tag{28}$$

where

$$
\begin{aligned}
Y &= E - 2\epsilon - \vec{k}''^2 , \\
Y - Q &= E - \epsilon - \tfrac{1}{2}\vec{k}'' - (\vec{k}'' + \vec{k})^2 - \vec{k}^2 , \\
Y - Q' &= E - \epsilon - \tfrac{1}{2}\vec{k}''^2 - (\vec{k}'' + \vec{k}')^2 - \vec{k}'^2 , \\
Y - Q'' &= E - \epsilon - \tfrac{1}{2}\vec{k}''^2 - (\vec{k}'' - \vec{k}')^2 - \vec{k}'^2 .
\end{aligned}
\tag{29}
$$

Having written the equation for the $2\to2$ reactions initiated by the $1+(3)$ state we now present the equations for the incoming $(2)+(2)$ state dd. Again, all amplitudes may be calculated from the solution of a single integral equation. Naming $\mathscr{T}_3$ the amplitude for $2+2\to3+1$, the specific form of the integral equation is

$$
\langle \vec{k}' | \mathscr{T}_3^{ja}(E) | \vec{k} \rangle = \langle \vec{k}' | B_2^{ja}(E) | \vec{k} \rangle + \sum_{m,n}^{N} \int \frac{d^3q}{(2\pi)^3} \langle \vec{k}' | \mathscr{B}^{jm}(E) | \vec{q} \rangle D_{mn}(E - \tfrac{4}{3}\vec{q}^2) \langle \vec{q} | \mathscr{T}_3^{na}(E) | \vec{k} \rangle , \tag{30}
$$

where $B_2(E)$ is given by (24) and $\mathscr{B}(E)$ is the sum of $B_1(E)$, $U_1(E)$, and $U_2(E)$ given by Eqs. (23) and (26). Since the kernel $\mathscr{B}D$ in Eq. (30) is the same as in Eq. (20) one can calculate $\mathscr{T}_3$ once the resolvent $(1-\mathscr{B}D)^{-1}$ is obtained from (20). Finally, letting $\mathscr{T}_4$ be the amplitude for dd$\to$dd the precise form of the integral relation between $\mathscr{T}_3$ and $\mathscr{T}_4$ is

$$
\langle \vec{k}' | \mathscr{T}_4^{aa}(E) | \vec{k} \rangle = \sum_{mn}^{N} \int \frac{d^3q}{(2\pi)^3} \langle \vec{k}' | B_2^{am}(E) | \vec{q} \rangle D_{mn}(E - \tfrac{4}{3}q^2) \langle \vec{q} | \mathscr{T}_3^{na}(E) | \vec{k} \rangle . \tag{31}
$$

III. RESULTS: FOUR SPINLESS NUCLEONS

Here we present the results of our four-body calculation for a system of four spinless nucleons. The equations are from Sec. II where the form factor $f(\vec{q})$ of the two-body interaction is $f(q)=(q^2+\beta^2)^{-1}$ with $\beta=1.45$ fm^{-1}. The two-body binding energy is always $\epsilon=-2.226$ MeV and $z=0$. Since in the limit $z=0$ the two-body n-n interaction is equivalent to a separable potential between pairs, we reproduce the results of previous four-body work[17,18] where AGS or Yakubovsky equations are solved with two-body separable potentials of the Yamaguchi type. The aim of this work is threefold. First, we want to study how four-body observables such as the binding energy and the low energy phase shifts for all $2\to2$ processes depend on both the method we choose to represent the $3+1$ subamplitudes in a separable form and the rank N of the expansion. For that purpose we consider GUPE, EDPE, SE1, and SE2 methods with N varying from one to six. Second, we want to investigate how four-body results depend on the value chosen for the energy B at which the Sturmian functions $\xi_n(B)$ [see Eq. (8)] are calculated. As mentioned above, the functions $\xi_n(B)$ are used to express the three-body subamplitude T as a finite rank operator. The common choice for B whenever there is a three-body bound state is the energy of that same state. Since for the chosen two-body interaction we have two $l=0$ three-body bound states at $\epsilon_3=-25.53$ MeV and $\epsilon_3^*=-2.38$ MeV, there are two possible choices for the energy B. Therefore two different representations of the $l=0$ $3+1$ subamplitude may be used in the four-body sector and one would like to know which is best. Finally, we would like to show the utility of the formalism we use where the contribution of the $2+2$ subsystem is taken exactly by the convolution method.

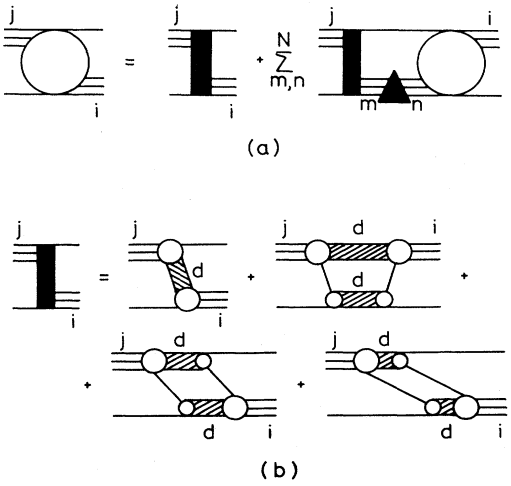

FIG. 1. Graphical representation of the integral equation for the $1+(3)\to1+(3)$ amplitude (circles).

A. Bound state results

The four-body equations of Sec. II C are solved by matrix inversion with 21 points in the momentum variable integration mesh. For all integrations leading to the partial wave expansion of the driving terms we use a 6 point Gauss-Legendre mesh, whereas in the solution of Eq. (8) we take a 28 point momentum variable integration mesh to calculate the Sturmian functions $\xi_n(B)$ corresponding to the N largest eigenvalues. In Tables I and II we show how the four-body binding energy converges as we in-

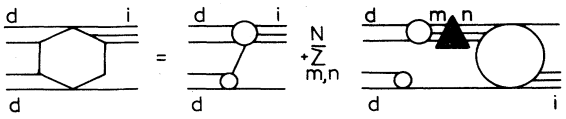

FIG. 2. Graphical representation of the integral relation for the $1+(3)\to(2)+(2)$ amplitude (hexagons).

TABLE I. Four spinless "nucleons" ground state energy as a function of the number N of terms in the expansion of the $3+1$ subamplitude for different methods and $B=\epsilon_3=-25.53$ MeV.

N	1	2	3	4
GUPE	−85.08	−89.65	−89.76	−89.74
EDPE	−88.96	−89.69	−89.74	−89.74
SE1	−89.60	−89.72	−89.74	−89.74
SE2	−89.00	−89.73	−89.74	−89.74

TABLE II. Four spinless "nucleons" excited state energy as a function of the number N of terms in the expansion of the $3+1$ subamplitude for different methods and $B=\epsilon_3=-25.53$ MeV.

N	1	2	3	4
GUPE		−26.47	−26.68	−26.63
EDPE		−26.16	−26.54	−26.60
SE1		−26.13	−26.53	−26.60
SE1		−26.06	−26.50	−26.60

crease the number N of terms in the representation of the $l=0$ $3+1$ subamplitude in a separable form for different expansion methods. Table I refers to the ground state energy and Table II to the excited state energy. The energy B at which the Sturmian functions are calculated is $B=\epsilon_3=-25.53$ MeV. The converged results are ~ 89.74 MeV for the ground state and -26.60 MeV for the excited state energy. These values agree within 0.5% with the results of previous calculations of Gibson and Lehman[17] and Narodetsky[18] which obtained, respectively, -89.6 MeV and -90.1 MeV for the ground state energy. Narodetsky also obtained -26.64 MeV for the energy of the excited state. As mentioned in Ref. 21 we also find that four-body results obtained with the GUPE method do not converge monotonically and that $N=3$ terms are required to get a converged ground state energy if one uses EDPE (the same is true for SE1 and SE2). For the excited state a larger number of terms ($N=4$) is required to obtain a converged result.

Considering now the other possible choice for $B=\epsilon_3^*=-2.38$ MeV, we show in Tables III and IV the values we get for the energies of the four-body bound states. Comparing these results with those of Tables I and II we find that they differ roughly by 0.5%, which is within the precision of the calculation. Nevertheless the choice $B=\epsilon_3^*$ gives rise to bound state results that are slightly more bound than those obtained with $B=\epsilon_3$ and in better agreement with the results of Narodetsky. Since both choices give rise to formally exact expansions for the $3+1$ subamplitude there should be no difference between the two calculations as long as the four-body results have converged with N. The small discrepancy we get, though, within the precision of the calculation may be attributed to the nature of the basis sets $\xi_n(\epsilon_3)$ and $\xi_n(\epsilon_3^*)$ for which the corresponding eigenvalues satisfy the relation $\eta_n(\epsilon_3)<\eta_n(\epsilon_3^*)$ for all n. For a finite number N of terms, the sets $\xi_n(\epsilon_3)$ and $\xi_n(\epsilon_3^*)$ may lead to expansions that differ slightly on the representation of the subamplitude. Due to a lack of numerical precision this difference may not disappear as we increase N. It is, however, interesting to know that with roughly the same number of terms one gets equally good results using values for B that are wide apart. It is with the choice $B=\epsilon_3^*$ that one realizes the strength of the SE2 method over EDPE or GUPE where for $N=1$ one gets a four-body ground state result that has essentially converged. Nevertheless for bound state calculations there is no real advantage in using SE2 since it leads to 40% more computing time than any of the other methods which with suitable programming are timewise equivalent.

B. Scattering results

In the scattering region we use matrix inversion together with the contour rotation method to calculate the low energy phase shifts for the reactions $1+(3)\rightarrow 1+(3)$ and $(2)+(2)\rightarrow(2)+(2)$. As shown in the Appendix the path of integration is different from the one which is usually employed in order to avoid crossing dangerous singularities of the kernel. For simplicity we have included only the contribution of the $l=0$ $3+1$ subamplitude and performed calculations at energies below breakup threshold. In the presence of two three-body bound states there are two different reactions initiated by $1+(3)$ states. Here we consider only the phase shifts for the scattering of a "nucleon" n from the ground state of three n's. For that reason we choose $B=\epsilon_3$ in order to associate the three-body ground state form factor with a single Sturmian function, in particular the one corresponding to a unit eigenvalue.

In Tables V, VI, and VII the $L=0$, 1, and 2 phase shifts for the reaction $1+(3)\rightarrow 1+(3)$ are shown as a function of N for different expansion methods and $E=-15$, -4, and -2.5, MeV, respectively. We find that, at all energies, $N=4$ terms are sufficient to obtain a converged four-body phase shift for $L=1$ and 2 partial waves and that results obtained with EDPE, SE1, and SE2 are in closer agreement with each other than with

TABLE III. Four spinless "nucleons" ground state energy as a function of the number N of terms in the expansion of the $3+1$ subamplitude for different methods and $B=\epsilon_3^*=-2.38$ MeV.

N	1	2	3	4	5	6
GUPE		−74.48	−90.45	−90.00	−90.12	−90.10
EDPE	−69.82	−89.53	−89.80	−89.81		
SE1	−83.92	−90.09	−90.10	−90.10		
SE2	−90.10	−90.10	−90.10	−90.10		

TABLE IV. Four spinless "nucleons" excited state energy as a function of the number N of terms in the expansion of the 3+1 subamplitude for different methods and $B=\epsilon_3^*=-2.38$ MeV.

N	1	2	3	4	5	6
GUPE			−27.00	−26.64	−26.68	−26.66
EDPE		−26.47	−26.65	−26.66		
SE1		−26.63	−26.66	−26.66		
SE2		−26.39	−26.63	−26.66		

those obtained through the GUPE method. In Tables VIII and IX the $L=0$ and 2 phase shifts for the reaction $(2)+(2)\rightarrow(2)+(2)$ are shown as a function of N for different expansion methods and $E=-4$ and -2.5 MeV, respectively. Here we find not only considerable discrepancies between results obtained with different expansion methods but also lack of convergence as one increases N up to six. Since the $(2)+(2)\rightarrow(2)+(2)$ observables depend only on the off shell 3+1 subamplitude there is no reason from the computational point of view to prefer the choice $B=\epsilon_3$ over the choice $B=\epsilon_3^*$. Therefore, using the experience gained in Ref. 12 and in Sec. II A we take $B=\epsilon_3^*$ and compare the new results shown in Table X for $E=-4$ MeV and in Table XI for $E=-2.5$ MeV with those obtained previously with $B=\epsilon_3$. We find that with $B=\epsilon_3^*$ one gets converged four-body phase shifts for all expansion methods. These results agree with those obtained with choice $B=\epsilon_3$, but only for the SE2 method. This again clearly indicates the strength of this method which independently of the choice of B leads to converged four-body results at energies away from the energy B with which the Sturmian basis set is calculated. This also shows that four-body observables can depend strongly on the correct representation of the off shell behavior of the 3+1 subamplitudes. Although we have not tested these methods at energies above breakup threshold, this indicates that in this energy domain one may only be able to use the SE2 method since it is the only one we know which can represent accurately the 3+1 subamplitudes.

C. Approximate calculation of the 2+2 contribution

We now test an approximate calculation of the 2+2 subsystem contribution that has been previously developed[20] but never studied in the framework of an exact four-body calculation. Starting with Eqs. (27) and (28) for the intermediate 2+2 propagator and using a dispersion relation for τ,

$$\tau(Z)=\frac{1}{Z}-\frac{1}{\pi}\int dx'\frac{\mathrm{Im}[\tau(x'-\epsilon)]}{Z+\epsilon-x'}\ , \tag{32}$$

one may express $G_1(E)$ and $G_3(E)$ as

$$G_1(E)=\bar{G}_1(E)+\frac{(Y-Q)(Y-Q')}{\pi^2}\int dx\int dx'\frac{\mathrm{Im}[\tau(x-\epsilon)]\mathrm{Im}[\tau(x'-\epsilon)]}{(Y-Q+\epsilon-x)(Y+2\epsilon-x-x')(Y-Q'+\epsilon-x')}\ , \tag{33}$$

$$G_2(E)=\bar{G}_2(E)+\frac{(Y-Q)(Y-Q'')}{\pi^2}\int dx\int dx'\frac{\mathrm{Im}[\tau(x-\epsilon)]\mathrm{Im}[\tau(x'-\epsilon)]}{(Y-Q+\epsilon-x)(Y+2\epsilon-x-x')(Y-Q''+\epsilon-x')}\ , \tag{34}$$

where Y, Q, Q', and Q'' are given by (29) and

TABLE V. $L=0$, 1, and 2 phase shifts (in deg) for the $1+(3)\rightarrow1+(3)$ reaction as a function of N for different expansion methods and $B=\epsilon_3=-25.53$ MeV. The four-body center of mass energy is $E=-15$ MeV.

	N	1	2	3	4	5	6
$L=0$	GUPE	−38.07	18.79	20.08	20.12	20.50	20.74
	EDPE	−12.42	17.97	20.02	20.30	20.59	20.77
	SE1	−9.06	17.87	20.20	20.31	20.59	20.78
	SE2	−5.15	17.54	20.11	20.32	20.60	20.75
$L=1$	GUPE	−29.55	−29.87	−29.98	−30.06	−30.11	−30.14
	EDPE	−32.45	−31.35	−30.60	−30.26	−30.15	−30.13
	SE1	−32.85	−31.47	−30.63	−30.27	−30.15	−30.13
	SE2	−33.32	−31.60	−30.69	−30.29	−30.15	−30.12
$L=2$	GUPE	9.27	9.10	9.03	9.00	8.97	8.96
	EDPE	8.58	8.75	8.87	8.93	8.95	8.96
	SE1	8.52	8.73	8.86	8.92	8.95	8.96
	SE2	8.43	8.69	8.83	8.90	8.94	8.95

TABLE VI. $L=0$, 1, and 2 phase shifts (in deg) for the $1+(3)\rightarrow 1+(3)$ reaction as a function of N for different expansion methods and $B=\epsilon_3=-25.53$ MeV. The four-body center of mass energy is $E=-4$ MeV.

	N	1	2	3	4	5	6
$L=0$	GUPE	−63.11	−15.29	−14.36	−13.81	−12.58	−11.69
	EDPE	−43.52	−15.50	−14.33	−12.99	−12.31	−12.34
	SE1	−40.56	−15.59	−14.26	−12.94	−12.44	−12.45
	SE2	−36.90	−16.01	−14.13	−12.87	−12.56	−12.49
$L=1$	GUPE	−46.25	−47.79	−48.42	−48.64	−48.68	−48.67
	EDPE	−52.10	−49.86	−48.87	−48.68	−48.66	−48.64
	SE1	−53.28	−50.05	−48.90	−48.68	−48.66	−48.64
	SE2	−54.62	−50.18	−48.94	−48.70	−48.64	−48.62
$L=2$	GUPE	20.58	19.95	19.73	19.66	19.66	19.66
	EDPE	19.11	19.48	19.62	19.65	19.65	19.66
	SE1	18.93	19.44	19.62	19.65	19.65	19.66
	SE2	18.66	19.36	19.58	19.63	19.64	19.65

$$\bar{G}_1(E)=\tau(Y)-\frac{1}{Y}+\frac{(Y-Q)(Y-Q')}{Q-Q'}\left\{\frac{1}{Q}[\tau(Y-Q)-\tau(Y)]-\frac{1}{Q'}[\tau(Y-Q')-\tau(Y)]\right\}, \tag{35}$$

$$\bar{G}_2(E)=\frac{Y-Q''}{Q''}[\tau(Y-Q'')-\tau(Y)]+\frac{Y-Q}{Q}[\tau(Y-Q)-\tau(Y)]-\frac{1}{Y}+\frac{(Y-Q)(Y-Q'')}{Y-Q-Q''}\tau(Y-Q)\tau(Y-Q'')\,. \tag{36}$$

The last term in Eq. (36) corresponds to the box diagram depicted last in Fig. 1. If one approximates $G_1(E)$ and $G_2(E)$ by $\bar{G}_1(E)$ and $\bar{G}_2(E)$, respectively, one neglects all four-body intermediate states in the first two box amplitudes depicted in Fig. 1. This not only speeds up the calculation by a factor of 30%, but also allows us to study the importance of four-body intermediate states in an exact four-body calculation. Since the full propagators $G_1(E)$ and $G_2(E)$ result from the convolution of the individual propagators for each pair, one may view this approximation as neglecting the part corresponding to the convolution of the continuum contribution in one pair with the continuum contribution in the other pair.

Using the GUPE method with $B=\epsilon_3$ and $N=4$ for the $3+1$ subamplitude, and the approximation previously described for $G_1(E)$ and $G_2(E)$, we get −86.62 MeV for the four-body ground state energy and −26.37 MeV for the excited state. These results should be compared with the converged results in Tables I and II, respectively. This indicates that the correct description of four-body intermediate states in the $2+2$ subsystem is responsible for 4% more binding in the energy of the ground state and 1% in the excited state. In the scattering region the results of this approximation are shown in Table XII for the

TABLE VII. $L=0$, 1, and 2 phase shifts (in deg) for the $1+(3)\rightarrow 1+(3)$ reaction as a function of N for different expansion methods and $B=\epsilon_3=-25.53$ MeV. The four-body center of mass energy is $E=-2.5$ MeV.

	N	1	2	3	4	5	6
$L=0$	GUPE	−65.70	−18.49	−17.61	−17.06	−15.90	−15.40
	EDPE	−46.60	−18.76	−17.76	−16.69	−16.16	−16.09
	SE1	−43.68	−18.90	−17.83	−16.73	−16.22	−16.09
	SE2	−40.02	−19.43	−17.83	−16.63	−16.20	−16.00
$L=1$	GUPE	−47.84	−49.61	−50.31	−50.53	−50.57	−50.55
	EDPE	−54.06	−51.69	−50.72	−50.56	−50.54	−50.51
	SE1	−55.38	−51.89	−50.75	−50.57	−50.54	−50.51
	SE2	−56.88	−52.01	−50.80	−50.58	−50.52	−50.49
$L=2$	GUPE	21.69	21.01	20.77	20.71	20.71	20.72
	EDPE	20.16	20.54	20.68	20.70	20.70	20.70
	SE1	19.96	20.50	20.67	20.70	20.70	20.70
	SE2	19.67	20.42	20.64	20.69	20.69	20.69

TABLE VIII. $L=0$ and 2 phase shifts (in deg) for the $(2)+(2)\rightarrow(2)+(2)$ reaction as a function of N for different expansion methods and $B=\epsilon_3=-25.53$ MeV. The four-body center of mass energy is $E=-4$ MeV.

	N	1	2	3	4	5	6
$L=0$	GUPE	-25.81	-25.50	-24.35	-21.21	-15.45	-6.96
	EDPE	-30.51	-25.51	-14.63	0.20	12.90	20.92
	SE1	-32.82	-23.09	-5.93	10.55	20.15	24.81
	SE2	-34.08	-15.71	$+7.65$	20.72	25.54	27.21
$L=2$	GUPE	-2.83×10^{-2}	-2.92×10^{-2}	-2.43×10^{-2}	-1.49×10^{-2}	-1.07×10^{-2}	1.86×10^{-2}
	EDPE	-5.02×10^{-2}	-2.87×10^{-2}	4.73×10^{-3}	4.48×10^{-2}	8.35×10^{-2}	1.20×10^{-1}
	SE1	-6.14×10^{-2}	-2.02×10^{-2}	2.80×10^{-2}	7.22×10^{-2}	1.10×10^{-1}	1.44×10^{-1}
	SE2	-6.92×10^{-2}	-7.58×10^{-3}	6.08×10^{-2}	1.08×10^{-1}	1.45×10^{-1}	1.77×10^{-1}

$1+(3)\rightarrow1+(3)$ phase shifts at two different energies and for $L=0$, 1, and 2. Compared with the corresponding phase shifts of Tables V and VI we find that the discrepancies are larger in the $L=0$ partial wave than in $L=1$ or 2. Since the energy of these calculations is well below the four-body breakup threshold it is useful to know how important four-body off shell continuum contributions are in that energy region.

IV. FOUR-NUCLEON CALCULATION

Next we develop a four-nucleon calculation with a spin dependent s-wave two-body interaction. As in the three-nucleon calculation of Aaron, Amado, and Yam[15] the nucleon-nucleon interaction is mediated by the quasiparticles d and ϕ that are coupled in s wave to n+n. Two-body nn scattering proceeds through the d (deuteron) each time a spin triplet pair interacts and through the ϕ whenever a spin singlet pair interacts. Each interaction is characterized by a coupling constant γ and a vertex function $f(\vec{k})$. The triplet interaction is also characterized by the wave function renormalization constant z_d that takes on the range of values $0\leq z_d\leq1$. For $z_d=0$ this nonrelativistic field theoretic interaction is equivalent to a one term separable potential between pairs with $f(\vec{k})$ being the two-body form factor.[22] By setting z_d different from zero one weakens the nucleon-nucleon triplet interaction through the introduction of a bare d with probability z_d in the wave function of the physical deuteron. Therefore $(1-z_d)\times10^2$ is the percentage of s-wave nucleon-nucleon component in the deuteron wave function. Unless otherwise specified we always set $z_d=0$. The remaining parameters of the interaction in each spin channel are fit to the low energy triplet and singlet nucleon-nucleon data. For $f(\vec{k})=(k^2+\beta^2)^{-1}$ we take $\beta_t=1.45$ fm^{-1}, $\gamma_s^2=73.92$ fm^{-3}, and $\beta_s=1.165$ fm^{-1}. The value of γ_t^2 is related to the deuteron binding energy $\epsilon_d=-2.226$ MeV through an equation similar to Eq. (4). In the three-body sector both the total spin s and the total isospin i have two possible values, $\frac{1}{2}$ and $\frac{3}{2}$, and the dynamical equations for the three-body amplitudes are those of Ref. 15. Since no tensor or spin orbit force is included in the two-body interaction, both the total angular momentum l and the total spin s are conserved. For each value of l there are three independent amplitudes whose spin s and isospin i take on the values $(\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{1}{2})$, and $(\frac{1}{2},\frac{3}{2})$, where the first number refers to spin and the second to isospin. For $l=0$ there is a single three-nucleon bound state at $\epsilon_t=-11.01$ MeV with $s=\frac{1}{2}$ and $i=\frac{1}{2}$.

In the four-body sector we take only the contribution of the three $l=0$ three-body subamplitudes. Considering the importance of the p-wave contribution to neutron-deuteron scattering, this may be viewed as a very drastic approximation, particularly when we already know from the work of Tjon[23] that the inclusion of the $l=1$ $3+1$ subamplitude has a noticeable effect on four-body scattering observables such as phase shifts and cross sections for $n\,^3H_e\rightarrow n\,^3H_e$ or $n\,^3H\rightarrow n\,^3H$. Since computer size and time limitations prevented us from going any further we consider only the $l=0$ subamplitudes which are represented exactly as operators of rank N_r through the GUPE method [N_1, N_2, and N_3 are, respectively, the

TABLE IX. $L=0$ and 2 phase shifts (in deg) for the $(2)+(2)\rightarrow(2)+(2)$ reaction as a function of N for different expansion methods and $B=\epsilon_3=-25.53$ MeV. The four-body center of mass energy is $E=-2.5$ MeV.

	N	1	2	3	4	5	6
$L=0$	GUPE	-52.09	-51.61	-49.22	-42.61	-31.96	-20.47
	EDPE	-62.58	-51.27	-31.70	-16.49	-9.96	-7.33
	SE1	-69.47	-43.65	-18.43	-9.39	-7.27	-6.95
	SE2	-72.69	-27.19	-9.85	-7.45	-7.24	-7.14
$L=2$	GUPE	-6.76×10^{-1}	-6.90×10^{-1}	-6.05×10^{-1}	-4.10×10^{-1}	-1.09×10^{-1}	2.86×10^{-1}
	EDPE	-1.10	-6.98×10^{-1}	-5.92×10^{-2}	6.31×10^{-1}	1.23	1.71
	SE1	-1.33	-5.25×10^{-1}	3.89×10^{-1}	1.12	1.62	1.96
	SE2	-1.47	-7.57×10^{-2}	9.61×10^{-1}	1.58	1.95	2.17

TABLE X. $L=0$ and 2 phase shifts (in deg) for the $(2)+(2)\rightarrow(2)+(2)$ reaction as a function of N for different expansion methods and $B=\epsilon_3^*=-2.38$ MeV. The four-body center of mass energy is $E=-4$ MeV.

	N	1	2	3	4	5	6
$L=0$	GUPE	-52.38	27.47	27.06	27.01	27.83	27.97
	EDPE	-41.78	26.78	27.65	27.67	27.72	27.77
	SE1	-37.58	17.73	25.43	27.41	27.77	27.80
	SE2	-36.61	9.42	22.61	26.59	27.43	27.78
$L=2$	GUPE	-8.9×10^{-1}	5.02×10^{-1}	3.86×10^{-1}	3.20×10^{-1}	2.81×10^{-1}	2.62×10^{-1}
	EDPE	-1.94×10^{-1}	1.60×10^{-1}	1.86×10^{-1}	2.10×10^{-1}	2.27×10^{-1}	2.37×10^{-1}
	SE1	-1.07×10^{-1}	9.42×10^{-2}	1.40×10^{-1}	1.79×10^{-1}	2.07×10^{-1}	2.25×10^{-1}
	SE2	-9.07×10^{-2}	6.36×10^{-2}	1.14×10^{-1}	1.57×10^{-1}	1.91×10^{-1}	2.24×10^{-1}

number of terms in the $(\frac{1}{2},\frac{1}{2})$, $(\frac{3}{2},\frac{1}{2})$, and $(\frac{1}{2},\frac{3}{2})$ $l=0$ subamplitudes]. The Sturmian functions for the subamplitude $(\frac{1}{2},\frac{1}{2})$ are calculated at the energy of the three-body bound state pole $B=\epsilon_t=-11.01$ MeV and the resulting eigenvalues are both positive and negative. The absence of three-body bound states in the $(\frac{3}{2},\frac{1}{2})$, and $(\frac{1}{2},\frac{3}{2})$ channels leads to the choice $B=\epsilon_d=-2.226$ MeV.

The four-body equations we solve are similar to those of Sec. II D but are more complicated due to the inclusion of spin and isospin coupling coefficients.[24] For a single term in the expansion of the three $l=0$ $3+1$ subamplitudes we get the same equations as in Ref. 9 but having now included a suitable one term representation of the underlying three-body subamplitude. As in the spinless equations the contribution of the $2+2$ subsystem comes in through the box amplitudes in the driving term, the difference being that one may get in intermediate states both d and ϕ propagators. Due to the lack of tensor and spin orbit force in the chosen nucleon-nucleon interaction and the neglect of $l>0$ $3+1$ subsystem contributions, the total four-body angular momentum L and the total spin S are conserved. Both the total spin S and the total isospin I may take the values 0, 1, and 2.

A. Bound state calculation

First we solve the four-body equations by matrix inversion with 21 points in the momentum variable integration mesh. The four-nucleon bound states are the 0^+ states with $\epsilon_\alpha=-45.59$ MeV and $\epsilon_\alpha^*=-11.63$ MeV with quantum numbers $S=0$, $I=0$, and $L=0$. The converged result is obtained with $N_1=4$ terms in the GUPE expansion of the $l=0$ $(\frac{1}{2},\frac{1}{2})$ $3+1$ subamplitude. Since we take $z_d=0$, these values may be compared with the results of previous work by Gibson and Lehman[17] and Narodetsky[18] which obtained -45.7 and -45.73 MeV, respectively, for the ground state energy. Narodetsky also obtained -11.69 for the energy of the 0^+ excited state. The agreement between calculations clearly indicates that one can trust the methods we use to calculate four-body observables, in particular those that depend on the alpha-particle wave function.

Since for the chosen two-body parameters the three-nucleon bound state is too deeply bound ($\epsilon_t=-11.01$ MeV), one cannot expect to describe low energy scattering data such as $n\,^3H\rightarrow n\,^3H$ phase shifts and cross sections with the present four-nucleon calculation. Therefore we develop a very simple model by changing z_d away from zero and keeping β_t, γ_s^2, and β_s fixed. As mentioned above this weakens the s-wave component of the triplet nucleon-nucleon interaction leading to $\epsilon_t=-8.48$ MeV for $z_d=5.074\times10^{-2}$. Having adjusted the three-nucleon bound state pole to the triton binding energy, the converged result for the energy of the four-nucleon 0^+ bound states is $\epsilon_\alpha=-29.67$ MeV and $\epsilon_\alpha^*=-8.65$ MeV. Further weakening of the effective four-body kernel may be obtained by retaining only two terms (one attractive and one repulsive) in the expansion of the $(\frac{1}{2},\frac{1}{2})$ $l=0$ subampli-

TABLE XI. $L=0$ and 2 phase shifts (in deg) for the $(2)+(2)\rightarrow(2)+(2)$ reaction as a function of N for different expansion methods and $B=\epsilon_3^*=-2.38$ MeV. The four-body center of mass energy is $E=-2.5$ MeV.

	N	1	2	3	4	5	6
$L=0$	GUPE	81.56	-17.71	-11.23	-7.90	-6.69	-6.54
	EDPE	-85.43	-10.29	-9.13	-7.72	-6.90	-6.55
	SE1	-79.66	-8.26	-7.80	-7.44	-6.96	-6.61
	SE2	-78.59	-9.26	-7.89	-7.49	-6.95	-6.60
$L=2$	GUPE	-9.61	3.96	2.90	2.49	2.40	2.40
	EDPE	-3.45	2.05	2.21	2.31	2.34	2.36
	SE1	-2.14	1.42	1.90	2.18	2.30	2.35
	SE2	-1.86	1.02	1.64	2.01	2.28	2.35

TABLE XII. $L=0$, 1, and 2 phase shifts (in deg) for the $1+(3)\rightarrow 1+(3)$ reaction with $N=4$ terms in the GUPE expansion of the $3+1$ subamplitude and an approximate calculation of the $2+2$ subsystem contribution.

	$E=-15$ MeV	$E=-4$ MeV
$L=0$	+15.65	−17.33
$L=1$	−30.55	−49.88
$L=2$	8.80	18.99

tude. This leads to $\epsilon_\alpha=-28.11$ MeV and no 0^+ excited state. Since $(1-z_d)\times 10^2$ is the percentage of s-wave component in the deuteron wave function, moving z_d away from zero may be physically interpreted as a means to compensate for the lack of tensor force in the chosen nucleon-nucleon interaction. For $z_d=5.074\times 10^{-2}$ we get approximately 95% for the s-wave component which corresponds to what one should have in the presence of tensor force. It is interesting to note that Gibson and Lehman[25] have calculated the four-nucleon binding energy with a two-body interaction between pairs that includes tensor force. They obtained values for ϵ_α that range from -30.5 to -26.6 MeV, depending on whether the percentage of d state is either 4% or 5%.

We have also developed an alternative way to generate the changes mentioned above without moving z_d away from zero. Although this model may be considered more phenomenological than the previous one, we use it for comparison. Instead of changing the two-body interaction, we weaken the $(\frac{1}{2},\frac{1}{2})$ $l=0$ subamplitude until we get a three-body bound state pole at the triton binding energy. For the chosen two-body interaction the $(\frac{1}{2},\frac{1}{2})$ $l=0$ three-nucleon kernel has a eigenvalue $\eta=1$ at the three-body energy $E_3=-11.01$ MeV, which becomes $\eta=1.105$ at $E_3=-8.48$ MeV. Therefore, dividing the $(\frac{1}{2},\frac{1}{2})$ $l=0$ kernel by $\lambda=1.105$ we generate a weaker kernel that sustains a unit eigenvalue at $E_3=-8.48$ MeV. Using the $3+1$ subamplitude that results from this modified $(\frac{1}{2},\frac{1}{2})$ $l=0$ kernel, one changes the four-body 0^+ ground state to $\epsilon_\alpha=-37.24$ MeV and the excited state to $\epsilon_\alpha^*=-9.07$ MeV. Again by retaining only two terms in the expansion of the modified $(\frac{1}{2},\frac{1}{2})$ $l=0$ $3+1$ subamplitude we get $\epsilon_\alpha=-33.6$ MeV and no 0^+ excited state. The first model is denoted A while the second is denoted B.

B. Scattering results

Having developed two simple four-nucleon models that are based on formally exact four-body equations, we now proceed to calculate phase shifts and cross sections for the $I=1$ reactions initiated by the $1+(3)$ state and the $I=0$ reaction initiated by the $(2)+(2)$ state dd. As in the spinless calculation, the equations are solved by matrix inversion together with the contour rotation method. Due to lack of computer time the number of points in the momentum variable integration mesh has been reduced to 14. For this reason the precision of this calculation has an upper limit of the order of 3%. The convergence of the calculation is shown in Table XIII for the $I=1$ $1+(3)\rightarrow 1+(3)$ phase shifts ${}^{SI}\delta_L$ at $E=-7$ MeV as a function of the number of terms N_r in each $3+1$ subamplitude. We find that, in general, two or three terms per $3+1$ subamplitude are sufficient to obtain a converged four-body result, although about 95% of the final result emerges with just a single term in each subamplitude. Therefore to save computing time and memory space we take $N_r=2$ (for all r) in the $I=1$ reactions, while in the $I=0$ reactions $N_1=2(3)$, $N_2=3(2)$, and $N_3=0$ since the isoquartet subamplitude does not contribute to $I=0$ four-body reactions.

In Figs. 3–8 the $n\,{}^3H\rightarrow n\,{}^3H$ phase shifts are shown as a function of the neutron laboratory energy for different values of S and L. The calculation (the crosses) corresponds to the parameters of model A where $z_d=5.704\times 10^{-2}$. For comparison, the predictions of a resonating group calculation[26] (dashed line) for $n\,{}^3H\rightarrow n\,{}^3H$ and the results of the phase shift analyses of Tombello[27] (open circles) are also shown together with the four-body results of Tjon[23] (black dots). With the exception of ${}^{11}\delta_1$, all other phase shifts conform with what is expected from previous calculations or phase shift analyses. The absence of a resonant behavior in ${}^{11}\delta_1$ may be attributed to the neglect of $l=1$ $3+1$ subsystem contributions as the work of Tjon[23] already suggests. For this reason the calculated cross sections for $n\,{}^3H\rightarrow n\,{}^3H$ and $p\,{}^3He\rightarrow p\,{}^3He$ are too flat when compared to existing data (see Figs. 9 and 10). Since we have neglected the Coulomb potential between protons in the four-body calculation, the Coulomb amplitude is added to the nuclear amplitudes multiplied by the appropriate Coulomb phases when compared with $p\,{}^3He\rightarrow p\,{}^3He$ data. With the exception of $E_n=1$ MeV results, all other cross sections at higher ener-

TABLE XIII. Four-body phase shifts ${}^{S1}\delta_L$ at $E=-7$ MeV for the reaction $1+(3)\rightarrow 1+(3)$ as a function of the number of separable terms N_r in each $3+1$ subamplitude. Here we set $z_d=0$.

N_1	N_2	N_3	${}^{01}\delta_0$	${}^{01}\delta_1$	${}^{11}\delta_0$	${}^{11}\delta_1$
1	1	1	−61.21	19.37	−66.05	15.77
1	1	2	−62.05	19.42	−62.27	15.82
1	1	4	−61.23	19.72	−61.46	16.10
1	2	1	−61.21	19.37	−58.37	16.43
1	4	1	−61.21	19.37	−60.17	16.19
2	2	2	−62.76	18.85	−59.59	16.46
3	2	1	−63.12	18.26	−59.84	15.95

TABLE XIV. Phase shifts ${}^{S0}\delta_L$ (in deg) at $E_d=6$ MeV for the reaction dd→dd resulting from different four-body calculations.

	N_1	N_2	${}^{00}\delta_0$	${}^{00}\delta_2$	${}^{10}\delta_1$	${}^{20}\delta_0$	${}^{20}\delta_2$	$\sigma_{22\to22}$ (mb)
Exact four-body	3	2	−71.1	−3.13	−29.2	−69.2	−2.84	765
Model *A*	3	2	−80.9	−3.28	−36.2	−65.5	−2.47	774
	2	3	−78.5	−3.18	−36.4	−67.5	−2.32	765
Model *B*	2	3	−81.2	−3.28	−36.0	−72.1	−2.50	752

gies lack sufficient structure. In Fig. 9 we show the results for both models *A* and *B*. Except for the relative magnitude, the shape of the cross section is essentially the same for both models. This indicates that the strength of the $l=0$ subamplitudes has little effect on the $L=1$ four-body phases. Therefore any improvement on the results we show has to come primarily from the contribution of the $l=1$ 3+1 subamplitudes and to a lesser degree from the inclusion of tensor force in the nucleon-nucleon interaction. This is best understood by taking into account both the results depicted in Fig. 11 and those shown in Table XIV where we compare the dd→dd ${}^{S0}\delta_L$ phase shifts and total nuclear cross sections at $E_d=6.1$ MeV for different four-body calculations. The results in the first line of Table XIV (dotted line in Fig. 11) correspond to the exact four-body calculation with $z_d=0$. Therefore the underlying $(\frac{1}{2},\frac{1}{2})$ subamplitude has a pole at $E_3=-11.01$ MeV and the $S=0,L=0$ four-body kernel supports two 0^+ bound states at −45.59 and −11.63 MeV (see Sec. IV A). The second and third lines correspond to model *A* calculations where $z_d=5.704\times10^{-2}$ (full and full dotted lines in Fig. 11). The $(\frac{1}{2},\frac{1}{2})$ subamplitude now has a pole at $E_3=-8.48$ MeV, and the difference between $N_1=3$ and $N_1=2$ four-body kernels in $S=0,L=0$ is the number of 0^+ bound states (two with $N_1=3$ and one with $N_1=2$). Finally, the fourth line corresponds to model *B* calculations (dashed line in Fig. 11) where the pole position of the $(\frac{1}{2},\frac{1}{2})$ subamplitude has also been fit to the triton binding energy. Comparing the results of these calculations we find that by changing the nature of the underlying $l=0$ subamplitudes in different ways one cannot increase the nuclear part of the dd→dd total cross section or, in another way, the value of the dd→dd cross section at 90 deg. Since the dominant indirect effect of the tensor force is to reduce the strength of the triplet *s*-wave nucleon-nucleon potential which in turn changes the $l=0$ 3+1 subamplitudes, we think that one cannot attribute the shortcomings of this four-nucleon calculation to the absence of tensor force. As shown by Tjon[23] for the n ^{3}H→n ^{3}H observables, we expect the $l=0$ 3+1 subamplitudes to contribute strongly to dd→dd observables as well as to dd→p ^{3}H. In Fig. 12 we show the cross section for dd→p ^{3}H at $E_d=6.1$ MeV. Again, the results show a considerable lack of structure although the overall magnitude appears to be correct.

V. CONCLUSION

Using the field theoretic approach of Refs. 8 and 9 with an exact representation of the 3+1 subamplitude, we have shown the results of a four-nucleon calculation. The

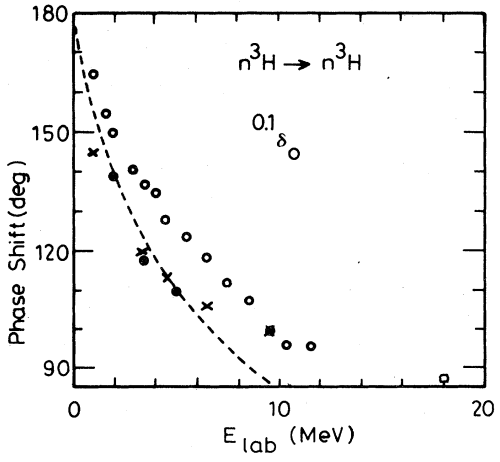

FIG. 3. Singlet *s*-wave phase shifts ${}^{01}\delta_0$ vs E_n. The crosses correspond to the results of model *A* with $z_d=5.704\times10^{-2}$. The dashed line corresponds to the resonating group calculation of Ref. 26, the dots to the four-body results of Tjon (Ref. 23) and the open circles to the phase shift analyses of Ref. 27.

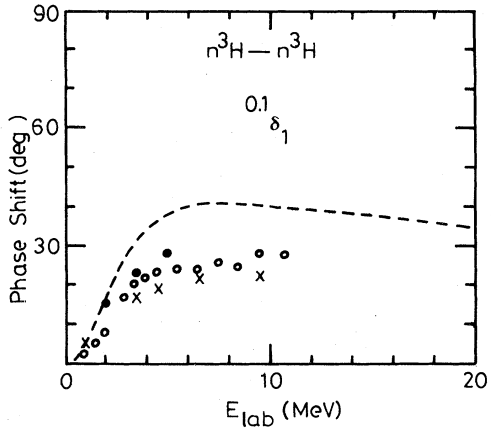

FIG. 4. Singlet *p*-wave phase shifts ${}^{01}\delta_1$ vs E_n. All symbols are as in Fig. 3.

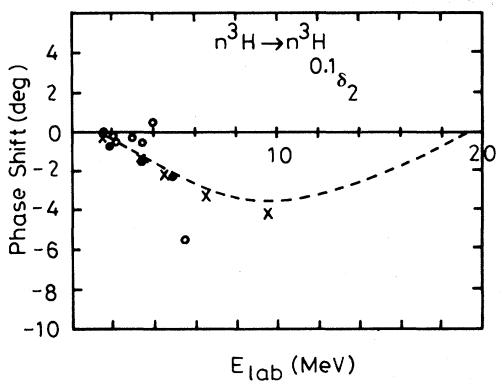

FIG. 5. Singlet d-wave phase shifts $^{01}\delta_2$ vs E_n. All symbols are as in Fig. 3.

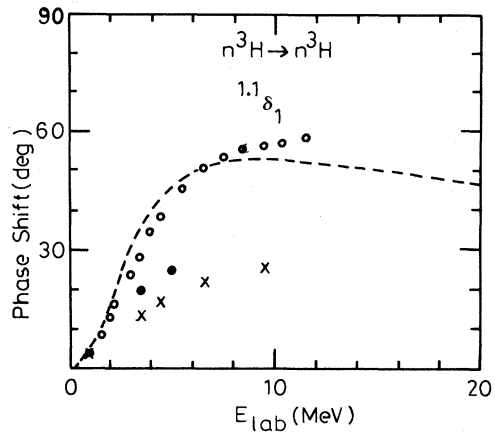

FIG. 7. Triplet p-wave phase shifts $^{11}\delta_1$ vs E_n. All symbols are as in Fig. 3.

equations are identical to AGS equations in the form that the 2+2 subsystem contribution is calculated exactly through the convolution method.[7] In the spinless calculation, the 3+1 subamplitude is represented as an operator of rank N through the GUPE, EDPE, SE1, and SE2 methods and the effectiveness of these methods studied for different N and the energy B, at which the Sturmian functions are calculated. If the four-body energy E is close to or below the energy B at which the Sturmian functions are calculated, we find that both GUPE and EDPE methods lead to four-body results that converge fast ($N \simeq 3$), but for $E \ll B$ or $E \gg B$ the results converge slowly ($N \gtrsim 6$) or not at all. It is in the $E \gg B$ domain that one needs SE1 or SE2 which are more powerful expansion methods than GUPE or EDPE. Nevertheless , a note of pessimism should be added concerning the use of separable expansion methods for the representation of three-body subamplitudes in a separable form. Since certain four-body results (see Sec. III B) are very sensitive to the correct representation of the underlying amplitudes, one has to be extremely careful as far as choosing the method and the energy B at which the Sturmian basis set is calculated. For this reason we expect the SE2 approach to be most useful in the scattering region, particularly above breakup threshold. If this procedure does not work then one is left with two variable integral equations as the only tool to obtain the solution of a four-particle problem. We also use the spinless calculation to test an approximation that involves neglecting four-body intermediate states in the calculation of the 2+2 subsystem contribution. We find that the off-shell four-body continuum contribution is responsible for 4% more binding in the ground state energy and affects by as much as 15% the $L=0$ $1+(3) \rightarrow 1+(3)$ phase shifts at energies well below breakup threshold.

In the four-nucleon calculation we confirm the results of Gibson and Lehman[17] and Narodetsky[18] for the 0^+ states. Since for the chosen two-body interaction the three-nucleon bound state is at $\epsilon_t = -11.01$ MeV, we use this four-body calculation to formulate one parameter models where the position of the three-nucleon subsystem bound state pole is moved to the triton binding energy. Subsequently, we calculate the four-body phase shifts and cross sections for the reactions $n\,^3H \rightarrow n\,^3H$, $p\,^3H_e \rightarrow p\,^3H_e$, $dd \rightarrow dd$, and $dd \rightarrow p\,^3H$. We find that triplet p waves are

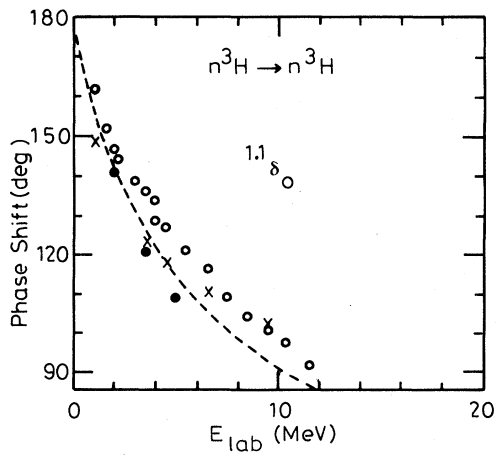

FIG. 6. Triplet s-wave phase shifts $^{11}\delta_0$ vs E_n. All symbols are as in Fig. 3.

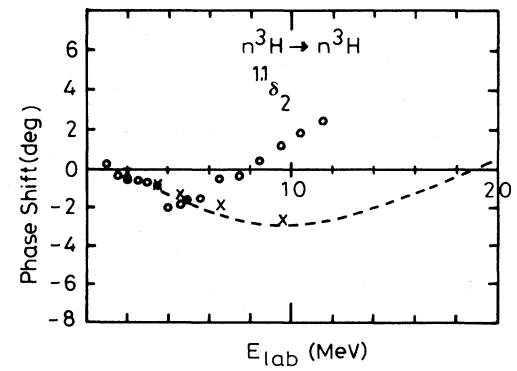

FIG. 8. Triplet d-wave phase shifts $^{11}\delta_2$ vs E_n. All symbols are as in Fig. 3.

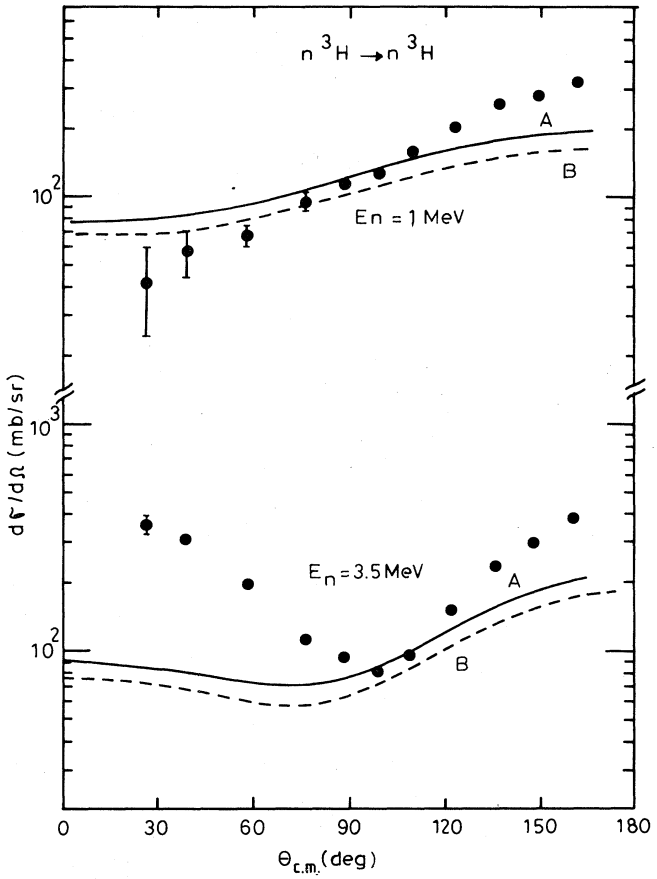

FIG. 9. Angular distribution for ^{3}H(n,n)^{3}H at different neutron laboratory energies. The solid line corresponds to model *A* calculations whereas the dashed line corresponds to model *B*. The dots are the experimental points of Ref. 28.

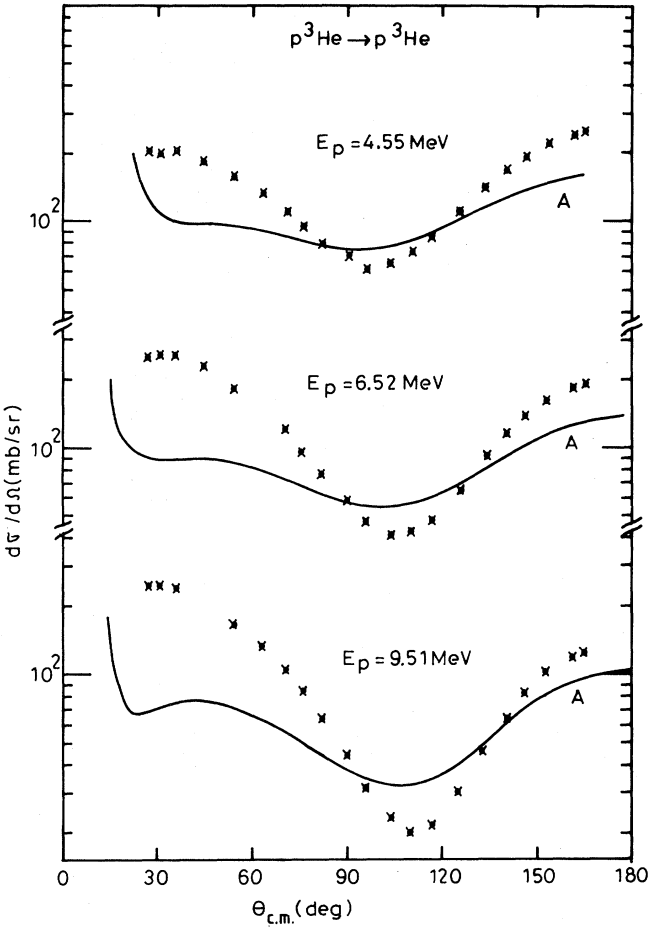

FIG. 10. Angular distribution for ^{3}H$_e$(p,p)^{3}H$_e$ at different proton laboratory energies. The solid curve corresponds to model *A* calculations and the crosses to the experimental points from Ref. 29.

too small and that the calculated cross sections are either too flat (see Figs. 9 and 10) or lack sufficient structure (see Fig. 12). This may be attributed to a lack of four-body *p*-wave strength due to the neglect of the $l=1$ subsystem amplitudes. By studying the behavior of dd→dd and dd→n ^{3}H$_e$ observables to changes in the $l=0$ $3+1$ subsystem amplitudes we conclude that the shortcomings of this calculation can only be attributed primarily to the neglect of $l=1$ three-body subsystem amplitudes and to a lesser degree to the absence of tensor force in the nucleon-nucleon interaction. Therefore any future four-nucleon scattering calculation has to include both $l=0$ and $l=1$ $3+1$ subamplitudes and the *s*-wave modified nucleon-nucleon *t* matrix due to the presence of the tensor force.

We would like to thank the computer center of the Electricidade de Portugal for making their computing facilities available to us.

APPENDIX

The solution of the four-body equation (20) by matrix inversion together with the contour rotation method re-

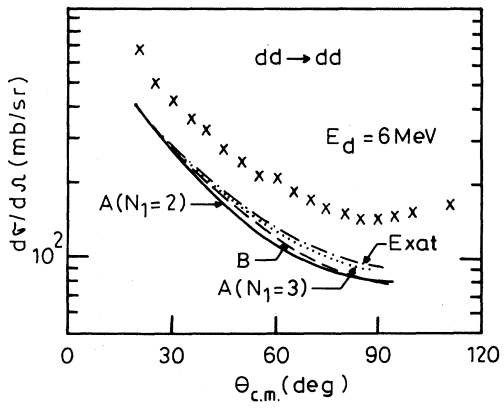

FIG. 11. Angular distribution for ^{2}H(d,d)^{2}H at $E_d=6.1$ MeV. The dotted line corresponds to the exact four-body results with $z_d=0$, the solid and solid-dotted line to model *A* calculations, and the dashed line to model *B*. The crosses are the experimental points from Ref. 30.

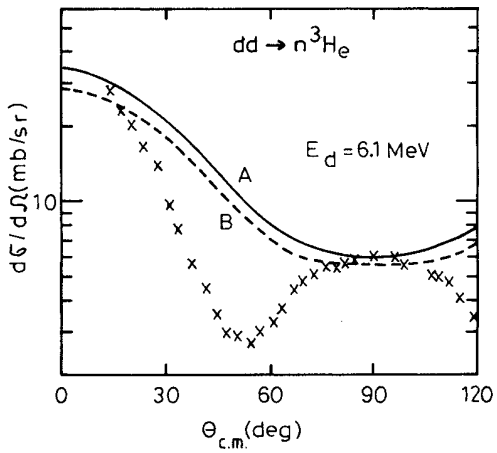

FIG. 12. Angular distribution for ^{2}H(d,p)^{3}H at $E_d=6.1$ MeV. The solid line corresponds to model A calculations and the crosses are experimental points from Ref. 31.

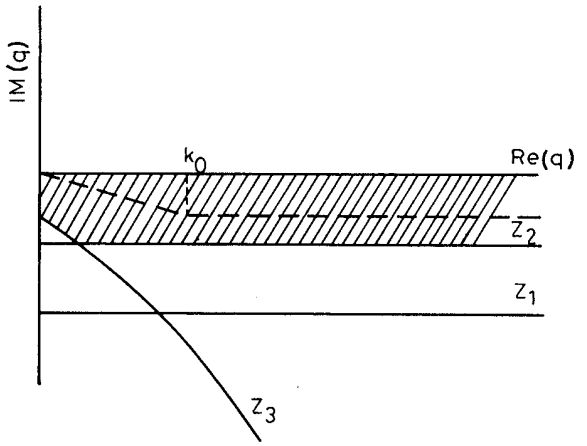

FIG. 13. Singularities of $V_l(E_3,q',q)$ in the fourth quadrant of the complex q plane. The dashed line represents the path of integration used to solve the four-body equations by matrix inversion together with the contour rotation method.

quires the calculation of the driving term $\langle \vec{k}' \mid \mathscr{B}^{jj}(E) \mid \vec{k} \rangle$ and effective propagator $D_{mn}(E-\frac{4}{3}k^2)$ for complex momentum k and k'. As long as one restricts oneself to the calculation of on-shell $2\rightarrow 2$ amplitudes, the limitations of deforming the path of integration in Eq. (20) are similar to those encountered in the three-body problem.[24] The only difference involves the constraints that emerge from the calculation of the three-body form factors $g_i(E_3;q)$ for complex E_3 and q where E_3 is the energy of the underlying three-body system [see Eqs. (23) and (25)] and q depends on $\vec{k}$ and $\vec{k}'$. Since $g_i(E_3;q)$ is either equal to or related to the Sturmian functions $\xi_i(B;q)$, the most severe limitation comes from the calculation of $\langle \vec{q} \mid V(E_3)\vec{q}' \rangle$ for complex E_3 and q and real q' running from zero to infinity. In a momentum space representation this matrix element appears either in Eq. (8) for $E_3=B$ or in Eqs. (14), (16), and (19) for $Z=E_3$. In each partial wave l the partial wave projection of the effective potential (6) is given by

$$V_l(E_3;q,q')=\tfrac{1}{2}\int_{-1}^{+1} P_l(x)\frac{f(u)f(u')}{E_3-q^2-q'^2-qq'x}dx \ , \quad \text{(A1)}$$

where

$$u=(\tfrac{1}{4}q^2+q'^2+qq'x)^{1/2} \ ,$$
$$u'=(\tfrac{1}{4}q'^2+q^2+qq'x)^{1/2} \ , \quad \text{(A2)}$$

and $P_l(x)$ is the Legendre polynomial of order l. The only limitations on the allowed complex values of q stem from the singularities of $V_l(E_3;q,q')$ which depend on E_3,q' and the range β of the Yamaguchi form factors $f(n)=(n^2+\beta^2)^{-1}$. In the fourth quadrant of the complex q plane the branch points of V_l move with q' according to

$$z_1(q')=2q'-i2\beta \ ,$$
$$z_2(q')=\tfrac{1}{2}q'-i\beta \ , \quad \text{(A3)}$$
$$z_3(E_3;q')=\tfrac{1}{2}q'-\tfrac{1}{2}(2E_3-3q'^2)^{1/2} \ .$$

Since in Eqs. (8), (14), (16), and (19) one integrates over q' from zero to infinity, the complex momentum q can only take on values in the hatched region shown in Fig. 13. For $R_e(E_3)<0$ (E below four-body breakup threshold), one can always find a path of integration for the four-body momentum variables k and k' that conforms with both $\mathrm{Im}(k)<0$, $\mathrm{Im}(k')<0$ and q within the allowed region shown in Fig. 13. Our final choice for the path of integration in variables k and k' is given by the dashed line ($k_0=[\frac{3}{4}(E-\epsilon_t)]^{1/2}$ or $k_0=(E-2\epsilon_d)^{1/2}$ are the elastic on-shell momenta in channels $1+(3)$ and $(2)+(2)$, respectively).

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