Pion form factor and the Klein-Gordon equation

M. Bawin and M. Jaminon
Institut de Physique, Université de Liège, B-4000 Liège 1, Belgium
(Received 28 November 1983)

We compute π^- ground state energy levels in the Coulomb field of nuclei as a function of the external charge Z, taking into account the pion electromagnetic form factor in the Klein-Gordon equation. We compare our results to those obtained by solving the Klein-Gordon equation *without* the form factor or by solving an approximate Klein-Gordon equation from which virtual pair terms have been removed. We find the approximate Klein-Gordon equation to be *a priori* a better approximation to the form factor Klein-Gordon equation for an external *point* Coulomb field. In contrast, the Klein-Gordon equation turns out to be an excellent approximation to the form factor Klein-Gordon equation for physical (finite size) nuclei, unlike the approximate Klein-Gordon equation.

I. INTRODUCTION

Although other forms of relativistic wave equations have also been studied,[1] the Klein-Gordon (KG) equation is the standard equation used to calculate pionic (π^-) bound states in the Coulomb field of nuclei.[2] In such calculations, the influence of the pion electromagnetic form factor $F_\pi(q^2)$ turns out to be very small,[1] so that neglect of the composite nature of the pion (and hence use of the KG equation, treating the pion as a point particle) is an excellent first approximation to the problem.[2] However, Brodsky recently argued[3] that composite particles like nucleons or pions should not be described by the Dirac or the KG equations, as their form factors actually suppress particle-antiparticle virtual pair (VP) creation, a process that is automatically included in the Dirac or KG equations.[4] Thus a relativistic spin zero wave equation without VP terms [approximate Klein-Gordon (PKG) equation] would then provide a better description of composite scalar particles than the KG equation itself, *without* inclusion of the particle form factor.

In this paper we wish to study whether the electromagnetic pion form factor indeed suppresses VP terms in the KG equation, as argued in Ref. 3. In order to do that, we have computed π^- ground state energy levels in the Coulomb field of a nucleus of charge $Z|e|$ for $0 \leq Z \leq 240$. Those π^- bound states were computed first by means of the KG equation with proper inclusion of the π^- electromagnetic form $F_\pi(q^2)$ factor. Our results were then compared to π^- bound states obtained by either neglecting $F_\pi(q^2)$ in the KG equation or by deleting VP terms in the KG equation, also without the $F_\pi(q^2)$ term (PKG equation). We carried out the above calculation both for a "pointlike" nucleus and for a physical (finite size) nucleus. In the first case our results show, in qualitative agreement with Ref. 3, that suppressing VP terms in the KG equation *decreases* the binding energy of π^- states, just as the inclusion of $F_\pi(q^2)$ in the KG equation. However, the repulsive effect of $F_\pi(q^2)$ is much greater than the repulsive effect introduced by pair suppression. In particular, we find that the critical value Z_c at which spontaneous pair production of $\pi^+\pi^-$ pairs occurs is now $Z_c \simeq 240$ for a physical pion in the Coulomb field of a point nucleus, compared to $Z_c \simeq 84$ for the PKG equation, and $Z_c = 69$ for the KG equation.

In the case of a "physical" (extended size) nucleus, however, we find that suppression of VP terms *increases* the binding energy, whereas inclusion of $F_\pi(q^2)$ in the KG equation yields (within the accuracy of our numerical computation) the *same* binding energies as those coming from the KG equation.

We are thus led to the following result: If one wishes to evaluate π^- bound states in a pure point Coulomb field for instance as a first approximation to the problem of the Coulomb field of a finite size nucleus, it is *a priori* better to use the PKG equation, which is obtained from the KG equation by suppressing VP terms, rather than to use the KG equation itself and ignore the pion form factor. Suppression of those VP terms plays the same qualitative role as the inclusion of the pion form factor $F_\pi(q^2)$ in the KG equation. For $0 < Z \leq 60$, the effect is even quantitatively reproduced. However, for physical nuclei, the PKG equation gives systematically too large binding energies while, because of the nucleus finite size, the effect of the form factor is practically negligible, so that the KG equation is definitely a better equation to use.

Our work is organized as follows. In Sec. II, for completeness sake, we briefly discuss VP terms in the KG equation. In Sec. III we discuss the form of $F_\pi(q^2)$ and of the Coulomb interaction for a finite size nucleus. Section IV discusses the PKG equation, while in Sec. V we discuss our results and state our conclusions.

II. VIRTUAL PAIR CREATION IN THE KG EQUATION

For completeness sake, we briefly discuss the interpretation of the KG equation.[4] The KG equation for a particle of mass m and total energy E in an external vector field V is

$$[(E-V)^2-\vec{p}^{\,2}-m^2]\psi=0 \,. \tag{1}$$

The meaning of Eq. (1) is best studied in the "free particle representation," where Eq. (1) is written in a "two-

component" form. One then gets the Hamiltonian form of Eq. (1) (Ref. 4), $[E_p=(\vec{p}^{\,2}+m^2)^{1/2}]$,

$$(E-E_p)u(\vec{p})=\int d^3q\, V(\vec{p}-\vec{q})\frac{E_p+E_q}{2(E_pE_q)^{1/2}}u(\vec{q}) + \int d^3q\, V(\vec{p}-\vec{q})\frac{E_p-E_q}{2(E_pE_q)^{1/2}}v(\vec{q}) , \quad (2)$$

$$(E+E_p)v(\vec{p})=\int d^3q\, V(\vec{p}-\vec{q})\frac{E_p+E_q}{2(E_pE_q)^{1/2}}v(\vec{q}) + \int d^3qV(\vec{p}-\vec{q})\frac{E_p-E_q}{2(E_pE_q)^{1/2}}u(\vec{q}) . \quad (3)$$

Let us record that the connection between ψ and

$$\begin{bmatrix} u \\ v \end{bmatrix}$$

is given by[4]

$$\psi=\frac{1}{\sqrt{2}}(\varphi+\chi) , \qquad \frac{E}{m}\psi=\frac{1}{\sqrt{2}}(\varphi-\chi) , \quad (4)$$

$$\begin{bmatrix} \varphi \\ \chi \end{bmatrix}=\frac{1}{2(mE_p)^{1/2}}\begin{bmatrix} E_p+m & m-E_p \\ m-E_p & E_p+m \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} . \quad (5)$$

In Eqs. (2) and (3), $v(\vec{p})$ describes virtual particle-antiparticle pair creation (VP terms) by the external field[4] associated with the existence of negative energy states in the original KG equation (1). An approximation to Eq. (1) which consists of ignoring VP terms is then given by

$$(E-E_p)u(\vec{p})=\int d^3q\, V(\vec{p}-\vec{q})\frac{E_p+E_q}{2(E_pE_q)^{1/2}}u(\vec{q}) . \quad (6)$$

Equation (6) will be referred to as the PKG equation. In particular, the s-wave PKG equation reads,

$$(E-E_p)u(p)=\int_0^\infty q^2\frac{E_q+E_p}{2(E_pE_q)^{1/2}}V_0(p,q)u(q)dq , \quad (7)$$

when we write,

$$V(\vec{p}-\vec{p}\,')=\sum_{l,m}V_l(p,p')Y_{lm}(\hat{p})Y_{lm}(\hat{p}') .$$

III. KLEIN-GORDON EQUATION FOR A COMPOSITE PION IN THE COULOMB FIELD OF A NUCLEUS

We now discuss the explicit form of Eq. (1) when the particle of mass m has some structure, i.e., is described by a form factor $F_\pi(q^2)$. We shall refer to (1) for such a composite particle as the form factor Klein-Gordon (FFKG) equation. As stated in the Introduction, we shall only be concerned with π^- bound states in the external Coulomb field of a nucleus $V^C(r)$ of mass number A and proton number Z. For $V^C(r)$, we take the form (α is the fine structure constant),

$$V^C(r)=-\frac{\alpha Z}{R} , \quad r<R , \quad (8)$$

$$V^C(r)=-\frac{\alpha Z}{r} , \quad r>R . \quad (9)$$

We took

$$R=r_0A^{1/3} , \quad (10)$$

$$r_0=1.2 \text{ fm} , \quad (11)$$

whereas

$$A=2Z , \quad (Z<100) , \quad (12)$$

$$A(Z)=63.6+1.30Z+0.00733Z^2 , \quad (Z\geq 100) . \quad (13)$$

Formula (13) is taken from Ref. 5.

As usual,[1] the pion form factor has been taken to be the *three-dimensional* Fourier transform of the experimental elastic form factor $F_\pi(q^2)$. We chose the form of $F_\pi(q^2)$ as parametrized by Bebek *et al.*,[6] i.e., $q^2=q_0^2-\vec{q}^{\,2}$,

$$F_\pi(q^2)=\frac{1}{1-\frac{q^2}{\mu^2}}\simeq\frac{1}{1+\frac{|\vec{q}|^2}{\mu^2}} \quad (14)$$

with

$$\mu=680 \text{ MeV} . \quad (15)$$

In configuration space, (14) has a "Yukawa" form,

$$\chi^2(\vec{r})=\int d^3q\, e^{i\vec{q}\cdot\vec{r}}F_\pi(q^2) =\frac{\mu^2}{4\pi}\frac{e^{-\mu r}}{r} , \quad (16)$$

where $\chi^2(\vec{r})$ in (16) is normalized to unity. The effective interaction V for the pion to be used in (1) is then given by

$$V(\vec{r})=\int\chi^2(\vec{r}\,')V^C(\vec{r}\,'-\vec{r})d^3r' . \quad (17)$$

In the case of the point Coulomb field $[V^C(r)=-(\alpha Z/r)]$ we find from (16) and (17),

$$V(r)=-\frac{\alpha Z}{r}(1-e^{-\mu r}) . \quad (18)$$

In the extended nucleus case (8) and (9), we find from (16), (8), and (9),

$$V(r)=-\frac{\alpha Z}{R}\left[1-\frac{e^{-\mu R}}{\mu r}\sinh\mu r\right] , \quad (r<R) , \quad (19)$$

$$V(r)=+\frac{\alpha Z}{R}\frac{e^{-\mu r}}{\mu r}\sinh\mu R-\frac{\alpha Z}{r} , \quad (r>R) . \quad (20)$$

We solved the s-wave KG equation corresponding to (1) for the reduced radial wave function $u(r)$, $(\psi=u/r)$,

$$\frac{d^2u}{dr^2}+[(E-V)^2-m^2]u=0 , \quad (21)$$

where V is given by (18) or (19) and (20). In order to solve (21) numerically, we used a suitable modification of the BOSTAW code.[7]

IV. PKG EQUATION

In momentum space, the s-wave interaction V_0 corresponding to $V^C(r)$ given by (8) and (9) is

$$V_0(q)=-\frac{4\pi}{q}\int_0^\infty V^C(r)r\sin qr\,dr$$
$$=-\frac{\alpha Z}{\pi R}\int_{-1}^{+1}\frac{\sin qR}{q^3}d(\cos\vartheta)\,, \tag{22}$$

where

$$q=(p^2+p'^2-2pp'\cos\vartheta)^{1/2}\,. \tag{23}$$

For the point Coulomb case, one has, instead of (22),

$$V_0^C(p,p')=-\frac{\alpha Z}{2\pi pp'}\ln\left[\frac{p+p'}{p-p'}\right]^2. \tag{24}$$

In order to solve the PKG equation (7) with interactions (23) or (24), we found it convenient to introduce the T matrix associated with (7). For the s wave, $T_0(E,p,p')$ satisfies the following equation:

$$T_0(p',p)=\frac{E_{p'}+E_p}{2(E_{p'}E_p)^{1/2}}V_0(p,p')$$
$$+\int_0^\infty V_0(p',p'')\frac{E_{p'}+E_{p''}}{2(E_{p'}E_p)}\frac{T(p'',p)}{E_p-E_{p''}}p''^2dp''\,. \tag{25}$$

Bound state solutions to (25) were then found as poles of $T_0(p',p)$. The singularity at $p=p'$ was dealt with using Lande's subtraction method, which is discussed in detail in Ref. 1. This method works equally well for the pure Coulomb interaction (24) or the finite size Coulomb interaction (22).

V. RESULTS AND CONCLUSIONS

In Fig. 1, we show the variation of E/m for the ground state with Z for a π^- in the external Coulomb field of a point nucleus. The dashed curve is the KG result, taken from Ref. 8. It shows that for $Z>Z_c=69$, the bound state has "dived into the lower continuum," and the "nucleus" is unstable with respect to spontaneous $\pi^-\pi^+$ pair production. The continuous curve is the PKG result obtained by deleting VP terms from the KG equation. One can see that such VP terms increase the binding energy. The critical value is now $Z_c\simeq 84$. The short-dashed curve (FFKG) comes from including the pion form factor $\chi(r)$ in the KG equation with a point Coulomb interaction [formula (18)]. It shows, like the PKG curve, a repulsive effect. This repulsion, however, is much greater than for the PKG equation, as now we get $Z_c\simeq 240$. Figure 2 shows the variation of the binding energy $\epsilon\equiv m-E$ with Z for a physical extended size nucleus, as described by formulas (19) and (20). The KG and FFKG results are

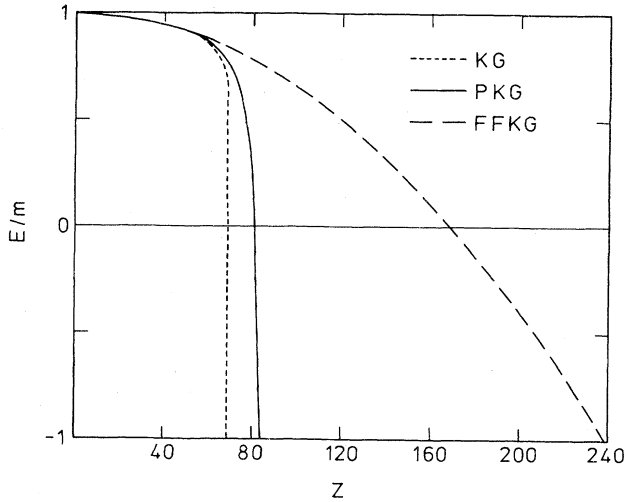

FIG. 1. Z dependence of E/m calculated with the KG (dashed curve), PKG (full curve), and FFKG (short-dashed curve) equations for the ground state level of a π^- in the point Coulomb field of a nucleus.

indistinguishable within the numerical accuracy of our computations. The PKG curve is now *below* the KG curve, showing that VP terms now have a *repulsive* effect for finite size nuclei, in contrast with the point Coulomb case, where VP terms were attractive.

The main goal of this paper was to study whether the KG equation was *a priori* the best equation to be used to describe the interaction of pions. We chose to investigate a simple model with *a priori* known forces, and calculated π^- bound states in the Coulomb field of a nucleus. Our conclusions are then as follows.

If, as a first approximation, one wishes to compute π^- bound states, taking the pion to be a point particle in the external *point* Coulomb field of the nucleus, the KG equation does *not* offer *a priori* the best framework for such a calculation. The PKG equation, where VP terms are deleted, is indeed a better approximation to the FFKG equa-

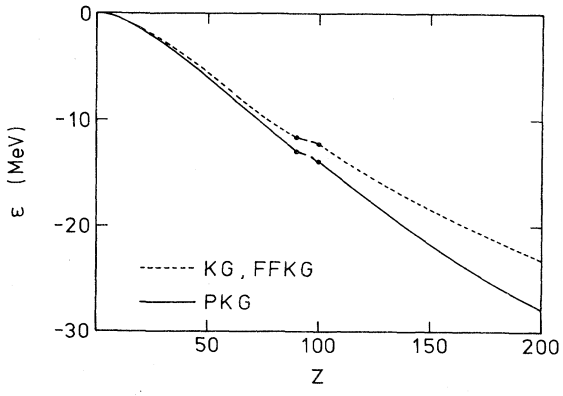

FIG. 2. Z dependence of the binding energy ϵ calculated with the KG, FFKG (dashed curve), and PKG (full curve) equations for the ground state level of a π^- in the Coulomb field of a finite size nucleus. The discontinuity in the curves results from a discontinuity at $Z=100$ in the relation between the mass number A and the proton number Z [formulas (12) and (13)].

tion (Klein-Gordon equation with full inclusion of the pion form factor) than the KG equation itself. This result is in agreement with Brodsky's arguments.[3] On the other hand, if one includes nonperturbatively the effect of the finite size of the nucleus right at the beginning of the calculations, the KG equation is *a priori* better than the PKG equation, as the effect of VP terms is in opposite direction for the point or finite size Coulomb problem.

ACKNOWLEDGMENTS

We wish to thank Dr. J. Cugnon for invaluable assistance with our numerical calculations. The work of M.B. was supported by the National Fund for Scientific Research, Belgium, while the work of M.J. was supported by the Inter-University Institute for Nuclear Science, Belgium.

[1]Yong Rae Kwon and F. Tabakin, Phys. Rev. C 18, 932 (1978).
[2]G. Backenstoss, Annu. Rev. Nucl. Sci. 20, 467 (1970).
[3]S. Brodsky, in *New Horizons in Electromagnetic Physics*, edited by J. V. Noble and R. R. Whitney (University of Virginia, Charlottesville, 1983), pp. 170—208. See also the discussion session, pp. 168 and 169.
[4]H. Feshbach and F. Villars, Rev. Mod. Phys. 30, 24 (1958).
[5]W. Pieper and W. Greiner, Z. Phys. A 218, 327 (1969).
[6]C. J. Bebek *et al.*, Phys. Rev. D 17, 1693 (1978).
[7]F. Fabbri and A. M. Saruis, University of Bologna Technical Report, 1967.
[8]M. Bawin and J. Cugnon, Phys. Lett. 107B, 257 (1981).