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Correlation among low-energy three-nucleon observables

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We reconsider the N/D equations for the s-wave spin-doublet trinucleon system and provide an alternative analytic model solution to these equations, which clearly explains most of the low energy properties of this system. The present model solution, is simpler than the previous approximate solutions proposed by other authors, and it provides a better understanding of the various low-energy correlations for this system including the linear correlation between trinucleon energies and the neutron-deuteron scattering length.

I. INTRODUCTION

A strong correlation among the s-wave spin doublet neutron-deuteron (n-d) scattering length and the triton binding energy was first observed by Phillips[1] about fifteen years ago. [In this paper we shall simply use the names trinucleon or neutron-deuteron system to denote the s-wave spin doublet quantum state(s) of such a system.] Later on Barton and Phillips[2] studied this problem by using the partial wave dispersion theoretic (N/D) approach and concluded that once a correct value of the n-d scattering length is achieved, low energy n-d elastic scattering will not contain much qualitative new information. Brayshaw,[3] using his boundary condition model, showed that once correct values of triton binding energy and n-d scattering length are obtained in a numerical model calculation, the essential features of the breakup results automatically follow. The n-d effective range function $k\cot\delta$ has a pole[4] below the lowest scattering threshold, and this pole is expected to have a strong influence on low energy trinucleon observables. More recently, Whiting and Fuda[5] restudied the problem by the N/D approach and showed that the position of the pole of $k\cot\delta$ and its residue are correlated with the n-d scattering length a. Finally, Girard and Fuda[6] showed that an approximate linear correlation exists between the doublet n-d scattering length a and the energy of the excited virtual state of the triton. They also found out in a subsequent calculation[6] of the asymptotic normalization parameter (ANP) that it is important to use a potential which gives the experimental binding energy for the triton, and that the ANP and the binding energy of the triton are linearly correlated. As the triton binding energy and the n-d scattering length are linearly correlated, it means, in other words, that all the low energy spin doublet s-wave three nucleon observables are strongly correlated with the n-d scattering length.

In this work we shall consider a simple analytic model solution to the N/D equations for this problem and try to understand the various correlations among the low-energy three-nucleon observables. In particular, we shall study the linear correlation among the trinucleon energies and the n-d scattering length a, which are commonly known as the Phillips[1] (ground state) and the Girard-Fuda[6] (excited virtual state) plots. We shall also study in the present model the correlation among the scattering length a and various low-energy trinucleon observables, such as the position and the residue of the pole of $k\cot\delta$, and the ANP of the ground and the excited virtual states of the triton.

Recently, the present author in collaboration with Torreão[7,8] made an attempt to understand the Phillips and the Girard-Fuda plots using an "*ad hoc*" expression for $k\cot\delta$. The present approximate analytic solution is identical to the effective range function employed in Ref. 8. The present work provides a more fundamental derivation of the results of Ref. 8 and exploits the full content of the solution in order to understand the various correlations among the low-energy trinucleon observables.

The approximate analytic solution of the N/D equations we employ produces by construction the correct energies for the ground and the excited virtual states of the triton at the experimental value of the n-d scattering length. As a result, the present approximate solution not only explains the Phillips and the Girard-Fuda plots, but also sheds light on the understanding of various other low-energy correlations for this system.

In Sec. II we present the N/D equation and obtain its approximate analytic solution. In Sec. III we present a numerical study of the solution obtained in Sec. II and consider the various low-energy correlations for the trinucleon system. Finally in Sec. IV we present a brief summary.

II. THE MODEL

The spin doublet s-wave elastic n-d amplitude $f(k^2)$ satisfies

$$f(k^2)=e^{i\delta(k)}\sin\delta(k)/k\ , \tag{1}$$

where δ is the phase shift and k is the wave number defined by

$$E=-\alpha^2+3k^2/4\ . \tag{2}$$

In Eq. (2) E is the total three-particle energy, and α^2 is the deuteron binding energy in units of $\hbar^2/m=41.47$ MeV fm^2, where m is the nucleon mass. The amplitude f has right-hand unitarity cuts due to two- and three-body breakup processes starting at $k^2=0$, and also has left-hand potential cuts due to the exchange of nucleons and mesons starting at[2] $k^2=b\equiv-4\alpha^2/9$ and extending to ∞

and $-\infty$, respectively. Here we shall ignore the three-particle unitarity cut and take δ to be real for all positive energies. This could be justified[5] if one is interested in the low-energy process only.

The amplitude defined by (1) and having the above-mentioned cut structure can be written in the form[2]

$$f(k^2)=N(k^2)/D(k^2)\ , \tag{3}$$

where D is dimensionless and N possesses the dimension of f. The functions N and D are known to satisfy the equations[2,5]

$$N(k^2)=\frac{1}{\pi}\int_{-\infty}^{b} dk'^2\frac{D(k'^2)\mathrm{Im}f(k'^2)}{k'^2-k^2}\ , \tag{4}$$

$$D(k^2)=1-\frac{k^2}{\pi}\int_0^{\infty} dk'^2\frac{N(k'^2)}{k'(k'^2-k^2)}\ , \tag{5}$$

or equivalently

$$D(k^2)=1-\frac{ik}{\pi}\int_{-\infty}^{b}\frac{dk'^2D(k'^2)\mathrm{Im}f(k'^2)}{k'(k+k')}\ , \tag{6}$$

$$N(k^2)=-k^{-1}\mathrm{Im}D(k^2)\ . \tag{7}$$

These are N/D equations that we shall use and are essentially the same as those of Ref. 5 apart from a scaling of the energy variable.

Equations (3)–(7) are already approximate ones as we have neglected the three particle unitarity cut, and are not expected to produce the correct trinucleon energies unless they are built into the formalism. Further approximations about the left-hand cut are needed in order to solve Eq. (6). In the present work we shall not attempt an exact solution of Eq. (6) but will introduce a further approximation to it so that an analytic solution can be obtained into which we build in the correct trinucleon ground and excited virtual state energies at the experimental value of the n-d scattering length. This will "compensate" for all the approximations introduced in writing and solving the N/D equations.

Since it is not practical to include all of the left-hand cut we follow Whiting and Fuda[5] in order to introduce a systematic way of parametrizing the effect of the omitted portion of the cut. We treat the one nucleon exchange part of $\mathrm{Im}f(k^2)$ explicitly in Eq. (6), i.e., replace $f(k^2)$ by the Born term $B(k^2)$ in the integral in Eq. (6), because $B(k^2)$ is the only term in $f(k^2)$ that contributes to this cut, and the neglected portion of the left-hand cut is treated as a power series in k which is expected to converge at low energies. The integral over the one nucleon exchange cut is written as a contour integral C around this cut in the counterclockwise sense. Retaining only one term in the power series expansion, Eq. (6) is rewritten as

$$D(k^2)=1-ic_1k+\frac{k}{2\pi}\oint_C\frac{dk'^2D(k'^2)B(k'^2)}{k'(k+k')}\ . \tag{8}$$

Next we approximate the one nucleon exchange cut by a pole at $k^2=-L_0^2$ with residue d':

$$B(k'^2)=d'/(k'^2+L_0^2)\ ; \tag{9}$$

this will allow us to perform the contour integration in Eq. (8), and we get

$$D(k^2)=1-ic_1k+dk/(k+iL_0)\ , \tag{10}$$

where $d\equiv d'D(-L_0^2)/L_0$. Equation (10) is the approximate analytic solution of the N/D equations, which we use in the present work. This solution is similar to the one used by Whiting and Fuda.[5] In their work, however, d' and L_0^2 are derived from the one nucleon exchange Born term, whereas in the present work d' and L_0^2, and hence d and L_0, are adjustable parameters to fit the correct energies of the trinucleon system. Recalling that

$$k\cot\delta=\frac{\mathrm{Re}D(k^2)}{N(k^2)}=-k\frac{\mathrm{Re}D(k^2)}{\mathrm{Im}D(k^2)},\ \ k^2>0\ , \tag{11}$$

the unknown parameter c_1 of Eq. (10) can be eliminated in favor of the doublet scattering length a:

$$-a=c_1+d/L_0\ . \tag{12}$$

Using Eq. (12), Eq. (10) can be rewritten as

$$D(k^2)=1+iak-i\frac{(a+c_1)k^2}{k+iL_0}\ . \tag{13}$$

We shall parametrize Eq. (13) imposing the condition that it should produce the correct trinucleon ground and virtual state energies for the experimental n-d scattering length[9] $a\equiv\bar{a}=0.65$ fm. The trinucleon ground and virtual states appear as zeros of $D(k^2)$ for

$$k=iL=\pm i[4m(\epsilon-\epsilon_2)/(3\hbar^2)]^{1/2}\ ,$$

where $-\epsilon$ and $-\epsilon_2$ are triton and deuteron energies in MeV. L is positive for a ground state and negative for a virtual state.

Let L_B and L_v represent the L's corresponding to the correct trinucleon bound and virtual state energies, respectively, for $a=\bar{a}$. Then $D(-L_v^2)=D(-L_B^2)=0$ for $a=\bar{a}$, and we have from Eq. (13)

$$1-\bar{a}L_B+L_B^2(c_1+\bar{a})/(L_B+L_0)=0 \tag{14}$$

and

$$1+\bar{a}\,|L_v|+L_v^2(c_1+\bar{a})/(L_0-|L_v|)=0\ . \tag{15}$$

Equations (14) and (15) can be solved for c_1 and L_0, yielding

$$L_0=L_B\,|L_v|\,[L_B\,|L_v|\,\bar{a}+L_B-|L_v|\,]^{-1} \tag{16}$$

and

$$c_1=-[L_B\,|L_v|\,\bar{a}+L_B-|L_v|\,]^{-1}\ . \tag{17}$$

Next using Eqs. (11) and (13) we obtain the following relation for the effective range function $k\cot\delta$:

$$k\cot\delta=\frac{-A+B(1-Ca)k^2}{aA+k^2}\ , \tag{18}$$

where

$$A\equiv-L_0^2/c_1=L_B^2L_v^2[L_B\,|L_v|\,\bar{a}+L_B-|L_v|\,]^{-1}\ , \tag{19}$$

$$B\equiv(1-c_1L_0)/c_1\ , \tag{20}$$

$$C\equiv L_0/(1-c_1L_0)\ . \tag{21}$$

Now we would like to find the condition for having a

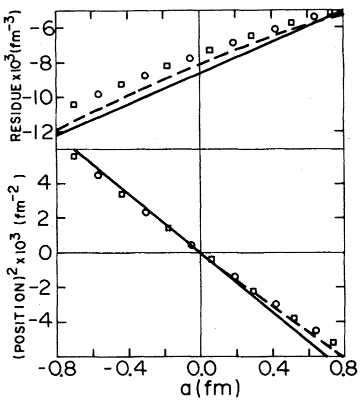

FIG. 1. Pole position and residue as a function of the doublet scattering length. Solid lines represent the present calculation. Circles and squares are from the Yamaguchi and Gaussian separable potential calculations, respectively, of Ref. 5. Dashed lines are from exact N/D calculations of Ref. 5. When the dashed and the solid lines overlap only the solid line is shown.

trinucleon bound or virtual state at $k=iL$. This condition, given by $D(-L^2)=0$, yields

$$1-aL+(c_1+a)L^2/(L+L_0)=0 \tag{22}$$

or,

$$aL=(L+L_0+c_1L^2)/L_0 \ , \tag{23}$$

which using Eqs. (16) and (17) can be rewritten as

$$aL=[\,|L_v|L_B+L(\bar{a}L_B|L_v|+L_B-|L_v|)-L^2]/(L_B|L_v|) \ . \tag{24}$$

Equation (24) yields the correlation among a and L, which will be used in the next section to explain the Phillips and the Girard-Fuda plots. Equations (13) and (16)—(24) constitute the analytic solution of the N/D equations which we will study numerically in the next section.

III. NUMERICAL RESULTS

In order to see how the present analytic solution works in practice we use the following input: $\bar{a}=0.65$ fm, $L_B=0.44849$ fm^{-1}, and $L_v=-0.12449$ fm^{-1}. This L_B corresponds to a triton binding energy of 8.48 MeV and

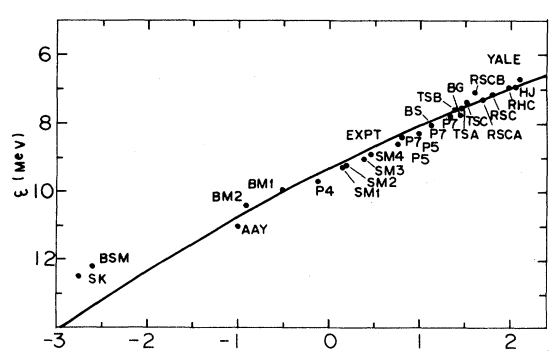

FIG. 2. Energy-scattering length plot for the triton ground state. The dark points are explained in the text.

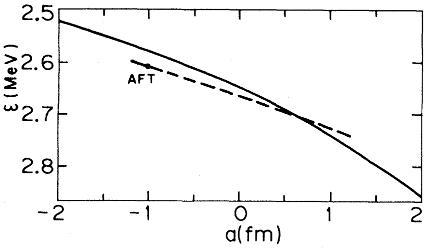

FIG. 3. Same as in Fig. 2 for the excited virtual state of the triton. The broken line is taken from Ref. 6.

L_v corresponds to a virtual state[6] 0.482 MeV below the elastic scattering threshold. The deuteron binding is assumed to be 2.224 MeV. Then Eqs. (18) and (24) become

$$k\cot\delta=\frac{-0.00865-0.51525(1-0.10836a)k^2}{k^2+0.00865a} \tag{25}$$

and

$$aL=\frac{L^2-0.36029L-0.05583}{-0.05583} \ . \tag{26}$$

We verified that for $a=0.65$ fm $k\cot\delta$ given by Eq. (25) reproduces the low energy doublet phase shifts very well. In this work, however, we are interested in correlation among various trinucleon "observables" and the n-d scattering length a.

First, we consider the correlation among the pole parameters of $k\cot\delta$ and the n-d scattering length. In Fig. 1 we plot the position of the pole in $k\cot\delta$ and its residue versus the scattering length a and compare it with the "exact" separable potential and N/D calculation of Whiting and Fuda. The work by Whiting and Fuda does not have the triton pole built in at the correct energy, and hence the present numerical result is expected to be quantitatively different from theirs and to be more close to the actual "experimental" situation. Hence, Fig. 1 is aimed at a qualitative comparison of the general trend of the correlations and not at a quantitative comparison. From Fig. 1 we find that the present result agrees reasonably well with the calculation of Whiting and Fuda; or in other words, the general trend of the correlation exhibited in Fig. 1 is independent of the details of the model used in the calculation.

In order to see how Eq. (26) works in practice, we calculated, using this equation, various points on the trinucleon energy ϵ versus n-d scattering length a curve for the ground and the excited virtual states, and the results are plotted in Figs. 2 and 3. In both cases the plots are approximate straight lines, as expected, for small a. The dark points on Figs. 2 and 3 are results of various calculations and are taken from the work by Afnan and Read,[10] except the ones marked AAY (Ref. 11); RSCB (Ref. 12); BM1 and BM2 (Ref. 13); SM1, SM2, SM3, and SM4 (Ref. 14); BSM (Ref. 15); SK (Ref. 16); P4, P5, and P7 (Ref. 1); and AFT (Ref. 17). The broken line on Fig. 3 is taken from the work by Girard and Fuda. From Figs. 2 and 3 we conclude that the present approximate analytic solution of the N/D equations explains the Phillips plot and the Girard-Fuda plot for the trinucleon system.

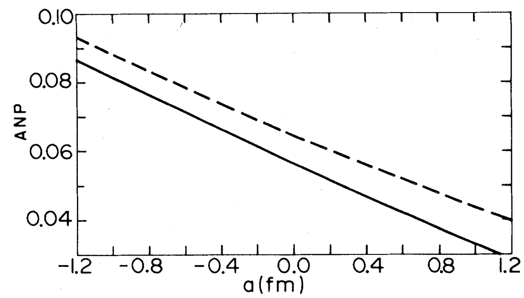

FIG. 4. The ANP of the excited virtual state as a function of the doublet scattering length. The solid line represents the present calculation. The dashed line is taken from Ref. 6.

Next we calculated the ANP of the trinucleon system using $k\cot\delta$ defined by Eq. (25). The ANP C^2 is defined by

$$\lim_{k\to iL} f(k) = -3LC^2(k^2+L^2)^{-1} . \tag{27}$$

We recall that L is positive for the bound state and negative for the virtual state. For the ground state the ANP obtained was much too small—smaller by about a factor of 4 compared to $C^2 \cong 3.3$ obtained by Girard and Fuda. The ANP of the virtual state was, however, reasonable. In Fig. 4 we plot the ANP of the virtual state for various values of the scattering lengths, and comparing it with the theoretical calculation of Girard and Fuda[6] we find that the agreement is qualitatively reasonable. As the triton pole is situated far away from the domain of validity of the present model, the poor value obtained for the ANP of the triton can be understood. Clearly, the three-body breakup cut must be included in the analysis if one expects to extract a good value of the ANP of the triton. This requires coupling to the three-body channel.

IV. SUMMARY

We presented a simple analytic solution of the N/D equations for the s-wave spin doublet trinucleon system. The present solution explains the linear correlations among n-d scattering length and trinucleon energies and sheds light on the understanding of various other low-energy correlations for the trinucleon system.

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