Time scale of short time deviations from exponential decay

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It is discussed to what extent nonexponential decay or the so-called zenon effect may lead to a measurable slowing down of extremely rare decay processes as double beta decay or proton decay.

In recent papers,[1,2] we presented calculations of the two-neutrino double beta ($\beta\beta$) decay of ^{76}Ge, ^{82}Se, ^{128}Te, and ^{130}Te. Such calculations are needed for the analysis of geochemical $\beta\beta$-decay measurements[3] in terms of a B-L (baryon-lepton number) violation allowing for a neutrinoless decay mode, since it is not possible to distinguish in such experiments between the two decay modes.

The nuclear structure calculations of this decay, which include Δ-h excitations and collective effects arising from spin-isospin and quadrupole-quadrupole forces show a persisting discrepancy of a factor of ~ 10 with experiment. Because of the extraordinary large half-lives ($T_{1/2} > 10^{18}$ yr) as one among several possible explanations, the question was raised[4] whether here deviations from the simple perturbation treatment of a decay process in the form of nonexponential decay are observed. This question is also of interest and has been recently discussed with respect to proton decay.[5-8]

In this Brief Report we discuss critically the question of whether such quantum mechanically rigorously demanded deviations from the usual decay formulas may lead to observable effects and give estimates using the Heisenberg uncertainty relation.

It is easily seen that the exponential decay law following from a statistical ansatz is only an approximation in a quantum mechanical description. Consider an unstable state $|\Psi\rangle$ being prepared at an instant $t=0$. The probability of finding the system at some later time t still in the state $|\Psi\rangle$ is given by

$$P_0(t) = |\langle\Psi|e^{-iHt/\hbar}|\Psi\rangle|^2 \ . \tag{1}$$

Expanding this expression for small t up to order t^2 gives

$$\begin{aligned} P_0(t) &\cong \left(1+\frac{it}{\hbar}\langle\Psi|H|\Psi\rangle - \frac{t^2}{2\hbar^2}\langle\Psi|H^2|\Psi\rangle\right) \\ &\quad\times\left(1-\frac{it}{\hbar}\langle\Psi|H|\Psi\rangle - \frac{t^2}{2\hbar^2}\langle\Psi|H^2|\Psi\rangle\right) \\ &= 1-\frac{t^2}{\hbar^2}\langle\Psi|(H-\bar{E})^2|\Psi\rangle + 0(t^4) \ , \end{aligned} \tag{2}$$

with

$$\bar{E} = \langle\Psi|H|\Psi\rangle \ .$$

So for very small times, the decay rate is not constant as characteristic for an exponential decay law, but varies proportional to t. From Eq. (2), already the theorem[9]

$$\left.\frac{dP_0(t)}{dt}\right|_{t=0} = 0 \tag{3}$$

follows. From a detailed treatment, it can be seen that the mean square spread of the energy distribution $\Delta E^2 = \langle\Psi|(H-\bar{E})^2|\Psi\rangle$ appearing in Eq. (2) is the essential parameter which determines the onset of exponentiality. For the characteristic time t_0, before which strong nonexponential effects should occur, the following relation can be deduced:[7]

$$t_0 \cong \frac{\hbar^2}{\tau\Delta E^2} \ . \tag{4}$$

Here τ means the usual lifetime neglecting any deviations from exponential decay.

Equations (2) and (3) tell us that for sufficiently short times, the decay rate is whatever small. However, to make any quantitative estimate is very difficult. Peres[10] uses the threshold effect to get a quantitative estimate for the onset of the exponential decay:

$$t_0 = \frac{\hbar}{(\bar{E}-E_{\text{threshold}})} \ . \tag{5}$$

Applying this estimate to $\beta\beta$ decay yields $t_0 \cong 10^{-21}$ sec, which is much too small to give any measurable effect. On the other hand, this would require the rest mass distribution of the $\beta\beta$-decaying state $\langle E|\Psi\rangle$ to extend over a range of 10^{25} natural decay widths Γ in the case of ^{130}Te. The observed decay rates, which are roughly a factor of 10 smaller than calculated assuming exponential decay, would evolve, if $\Delta E = 10^4\ \Gamma$ for ^{130}Te, ^{82}Se and $10^6\ \Gamma$ for ^{128}Te, which means that the energy spectrum of $|\Psi\rangle$ had to be strongly suppressed[7] outside the region $|E-\bar{E}| < \Delta E^2/\Gamma$.

Now we show, that the Peres estimate already evolves from a time dependent perturbation treatment, as it is found in textbooks, if the transition to infinite times is not performed. As usual the total Hamiltonian H is divided in two parts, $H = H_0 + V$, H_0 including all interactions which do not lead to a decay, and V having nonvanishing matrix elements between the decaying nucleus and the decay products. The eigenstates of H_0 with energy E_i will be denoted by $|\Psi_i\rangle$. Let us assume that the nucleus has not yet decayed at $t=0$. Then in the perturbation approximation it is described by an eigenstate $|\Psi_1\rangle$ of H_0 at $t=0$. If we look at some time t for decay products, their state vector is also approximated by an eigenstate $|\Psi_2\rangle$ of H_0. We define $|\Psi_2\rangle$ to also include the radiated particles. By doing this we avoid the introduction of a time dependent V. In the time interval $|0,t|$ we allow V to act on $|\Psi_1\rangle$. The time evolution for a first order process is then given by (there is no essential difference between first and second order effects for the following discussion)

$$\langle\Psi_2|e^{-iHt/\hbar}|\Psi_1\rangle = -\frac{i}{\hbar}e^{-iE_2t/\hbar}\int_0^t\langle\Psi_2|V|\Psi_1\rangle\exp[i(E_2-E_1)t'/\hbar]\,dt' = e^{-iE_2t/\hbar}\langle\Psi_2|V|\Psi_1\rangle\frac{\exp[i(E_2-E_1)t/\hbar]-1}{E_1-E_2} \ . \tag{6}$$

The decay probability $P_{1\to 2}(t)$ into the state $|\Psi_2\rangle$ is then

$$P_{1\to 2}(t) = |\langle\Psi_2|e^{-iHt/\hbar}|\Psi_1\rangle|^2 = \frac{4|\langle\Psi_2|V|\Psi_1\rangle|^2}{(E_1-E_2)^2}\sin^2\left[(E_1-E_2)\frac{t}{2\hbar}\right] . \quad (7)$$

This result seems to be rather unphysical for two reasons. First, $P_{1\to 2}(t)$ is a periodic function, which means that the probability of finding $|\Psi_1\rangle$ having decayed into $|\Psi_2\rangle$ after having reached a maximum goes back to zero at some later time. Secondly, there is a finite transition probability for $E_2 \neq E_1$, which seems to violate the conservation of energy. The reason for the latter is that in the initial ($t' < 0$) and final ($t' > t$) states the interaction V is assumed to be switched off. The switching on and off of V results in a contribution to the energy balance of the order $\langle\Psi_2|V|\Psi_1\rangle$. The appropriate exponential behavior of the decay process follows from an integration over final states: The *total* decay probability into any final state $P_{\text{int}}(t)$ is given by

$$P_{\text{int}}(t) = \int_{-\infty}^{+\infty} \rho(E_f) P_{1\to f}(t)\, dE_f \quad (8)$$

with the density of final states $\rho(E)$. Assuming $\rho(E_f)$ = const and $V(E) \equiv \langle\Psi_f|V|\Psi_1\rangle$ = const leads to Fermi's golden rule

$$P_{\text{int}}(t) = |V(E_1)|^2 \rho(E_1) t 2\pi/\hbar \quad \text{for } t \ll \tau \quad (9a)$$

or in the usual differential form

$$\frac{dP_{\text{int}}(t)}{dt} = \frac{2\pi}{\hbar}|V(E_1)|^2\rho(E_1) . \quad (9b)$$

Now let us look at the various approximations that have been made. First, $V(E)\rho(E)$ is by no means constant, but $\rho(E)$ varies with some power law for not too large E. The simplest assumption beyond $\rho(E)$ = const is a sharp cutoff at the threshold energy E_{th}, which is given by the rest mass of the decay products. In this approximation, $P_{\text{int}}(t)$ is calculated by inserting Eq. (7) into (8), and thus

$$P_{\text{int}}(t) = 4\rho(E_1)|V(E_1)|^2 \times \int_{E_{\text{th}}}^{+\infty} \left[\frac{\sin[(E_1-E)t/2\hbar]}{E_1-E}\right]^2 dE . \quad (10)$$

It is seen that the missing part of the integral from $E = -\infty$ up to E_{th} involves the sin with frequencies higher than $(E_1 - E_{\text{th}})(4\pi\hbar)^{-1}$. Consequently deviations from Eq. (9a) occur for $t \lesssim \hbar/(E_1 - E_{\text{th}})$ as was found by Peres. The actual form of $V(E)\rho(E)$ may be quite different. However, from an evaluation of Eq. (8) for a definite time t, it can be seen that provided $V(E)\rho(E)$ is a "smooth" function (for example, some approximate power law), contributions from outside the region $|E_f - E_i| \lesssim \hbar/t$ can be neglected. This means that for times t orders of magnitude larger than t_0 in Eq. (5), $P_{\text{int}}(t)$ is determined by a small energy window around E_1, for which $V(E)\rho(E) = V(\bar{E})\rho(\bar{E})$ is a good approximation. So, in perturbation theory, the exponential decay law is a result of the smoothness of the function $V(E)\rho(E)$.

The crucial condition for the application of perturbation theory is that initial and final states can be described by eigenstates of the unperturbed Hamiltonian H_0 to a good approximation. Once this assumption is made, the dispersion ΔE of the parent state is fixed, and no room is left for a quantity as the preparation function introduced by Khalfin.[5] An *exact* treatment of the decay process, in contrast, includes the perturbing part V of H also in the initial and final state. But this involves the difficulty that then the initial state can no longer be described by an eigenstate of H and, in fact, is not a unique state, but its nature depends strongly on the process of formation. We want to discuss now the implications following from the uncertainty principle without going into details of the general very complicated quantum mechanical formalism of unstable systems.

As an illustrating example, we take a resonance scattering process. In a consistent description, without switching on and off additional fields, energy conservation must hold exactly. This means that the energy distribution of the resonance state is determined by the asymptotic ($t \to -\infty$) distribution of the forming components and the dynamics of the reaction mechanism. Therefore in such a process, the energetic composition of the wave function of the resonance can be restricted to a small energy window by restricting the kinematics of the forming components. But an energy spread ΔE_{form} is connected to a time spread Δt via the Heisenberg relation. This means that the state describing the forming components with energy spread ΔE_{form} and henceforth, also the formation process, cannot be restricted to a time interval Δt less than $\hbar/\Delta E_{\text{form}}$. The resulting delay of exponentiality from a cutoff of the Breit-Wigner resonance shape at $|E - \bar{E}| = \Delta E_{\text{form}}$ extends according to Eq. (4) until $t_0 \cong \hbar/\Delta E_{\text{form}}$ (provided $\Delta E_{\text{form}} < \bar{E} - E_{\text{th}}$). This means that the nonexponential regime in the decay of the resonance cannot continue much longer than the formation process. If one would apply the uncertainty relation to the energy spread ΔE of the formed state itself instead of ΔE_{form}, an even sharper restriction evolves. The discussed situation can be realized in a resonance fluorescence experiment using Laser techniques. The energy spread ΔE_f of the Laser beam is connected with a minimum time $\Delta t = \hbar/\Delta E_f$, which is needed for switching on and off the Laser.

Generalizing the above considerations, by relating any suppression of ΔE in any formation process by the energy-time uncertainty to the minimum time extension Δt of the formation process, we conclude that the nonexponential behavior cannot persist longer than Δt.

This would mean that formation process and nonexponential decay cannot be separated in time. This result, of course, holds already in classical mechanics. As long as the formation process is going on, there is also no exponential behavior assuming purely statistical decay laws. If the above generalization holds, the implications for $\beta\beta$ decay as well as for proton decay clearly are, that deviations from exponentiality are negligible nowadays. To our knowledge, protons have been formed in the big bang within 10^{-6} sec and $\beta\beta$-decaying nuclei are the products of β-decay chains in the r process with typical decay constants of the order of minutes.

There has been given an argument by Fleming,[7] which seems to contradict this conclusion. According to Fleming a mass spectrum of the proton, as assumed by Peres, would lead to a "kinematical fragmentation" within, at maximum, 10^7 yr. (For $\beta\beta$ decay of ^{130}Te, the corresponding time would be 10^5 yr.) To avoid this fragmentation with its unacceptable consequences, Fleming concludes that t_0 *must* be of the order of 10^{26} yr. However, for this argument to work he has to assume free propagation of the proton wave packets. The interactions of the proton with other particles

should destroy, however, the phase relations and hence mix the different mass components, and hence no fragmentation should occur.

Another effect, namely, the so-called zenon effect[11] has been discussed[12,6,8] as an amplification mechanism of the nonexponential behavior. The argument is that if during the nonexponential stage of the decay a measurement is performed probing the unstable system, the decay process is then restarted. If successive measurements are repeated in time intervals smaller than t_0, the system will never reach the exponential decay one would expect from usual perturbation theory. However, the zenon effect is, of course, automatically included in a complete quantum mechanical description of the whole system including also the measuring apparatus and all interactions. So the zenon effect, in principle, can only arise from the neglection of a part of the interactions. In this sense Valanju, Sudarshan, and Chiu[13] derived sizeable effects for pion production in hadron nucleus collisions. Recently Horwitz and Katznelson[6] suggested important corrections for the proton decay inside the nucleus from nucleon-nucleon interactions. However, their interpretation of nucleon-nucleon interactions as a measuring process in the sense of the zenon effect has been criticized.[14–16] We do not discuss here the question of how to define a measurement but give a simple argument using again the uncertainty principle. To have a significant influence on the decay process, the measuring process has to take place with a frequency of at least $1/t_0$. According to the Heisenberg principle, this frequency is related to an interaction energy $E_{\text{int}} \cong \hbar/t_0$ between the unstable system and the part which acts as measuring apparatus. If one accepts the Peres estimate, Eq. (5), this means the interaction energy has to be as large as the Q value of the decay. Of course, if there is such a strong interaction, it is nothing unexpected, that the decay is influenced. From this point one could consider the strong interactions of the nucleons inside a nucleus generating a zenon effect in β decay and also in $\beta\beta$ decay. However, such effects are, of course, automatically included in nuclear structure calculations. Interactions not completely included in such calculations are the electromagnetic interactions with the electron cloud. But these are according to the above criterion much too weak to lead to sizeable effects. In the case of proton decay already the nucleon-nucleon interaction energy is more than one order of magnitude smaller than the decay Q value. Consequently a slowing down of the decay by the zenon effect can hardly be expected either for $\beta\beta$ or proton decay.

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