Proximity limit of the imaginary part of the heavy ion optical potential due to nucleon transfer

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In an earlier work it was assumed that the imaginary part of the optical potential due to nucleon transfer could be described by a proximity-type formula. Here we derive such a formula starting from the transfer probabilities between specific quantum states and assuming leptodermous nuclei.

I. INTRODUCTION

In recent years some more progress has been made in understanding the imaginary part of the nucleus-nucleus optical potential from a microscopic point of view. According to Ref. 1 one can distinguish two major contributions to the imaginary potential: a volume part W_v due to the mean free path, and a surface part

$$W_s = W_{\text{inel}} + W_{\text{trans}} \tag{1.1}$$

due to the additional contribution of inelastic and transfer channels. The volume part has the shortest range and has been studied by solving the Bethe-Goldstone equation for two nuclear matter systems in relative motion.[2] Approximate methods based on complex energy functionals have also been investigated.[3] Quantitative studies of W_{inel} and W_{trans} based on the formalism of Ref. 1 have recently been performed[4] for ^{16}O colliding on various targets such as ^{28}Si, ^{40}Ca, ^{88}Sr, and ^{208}Pb. These studies involve rather elaborate numerical calculations depending on the details of nuclear structure.

In Ref. 5 we adopted a much simpler model for calculating the nucleon transfer contribution to the imaginary part of the optical potential. The nuclei were treated by the Fermi gas model, and the flux of nucleons from one nucleus to the other was calculated by taking into account the Pauli-allowed region in momentum space and the tunneling through the barrier formed between the single particle wells. As in Refs. 1 and 4, the resulting potential has a long range, reflecting the peripheral aspect of the process of depopulation of the entrance channel through single particle transfer.

The present work is entirely devoted to the derivation of W_{trans} and is intended to be a link between microscopic calculations[6,7] based on transition amplitudes between specific states and barrier penetrability models.[5] In other words, taking the sum of transition probabilities between various available states as a starting point we try to find out under which conditions a proximity-type formula based on nuclear current across the barrier can be derived for W_{trans}.

In Sec. II we define W_{trans} in a semiclassical approximation. In Sec. III we derive a general analytic form for the total transfer probability for peripheral collisions between large leptodermous nuclei. In Sec. IV we obtain a formula for W_{trans} which is compared to the proximity approximation of Ref. 5.

II. DEFINITION OF W

In the following we shall assume that nucleus 1 has a straight line trajectory and a uniform velocity v, with respect to nucleus 2 which is fixed. The geometry is shown in Fig. 1. The origin of coordinates is positioned at the center of nucleus 2 and nucleus 1 moves along the y axis, the distance of closest approach being $z=d$. The transition amplitude $A(2f,1i)$ of a nucleon from nucleus 1 to nucleus 2 can be written perturbatively as[1,8]

$$A(2f,1i) \simeq \frac{1}{i\hbar}\int_{-\infty}^{\infty}(\psi_{2f}, V_1\psi_{1i})dt\ , \tag{2.1}$$

where ψ_{1i} and ψ_{2f} are the wave functions of a bound state nucleon in nuclei 1 and 2, respectively. A similar expression can be written for transfer from 2 to 1.

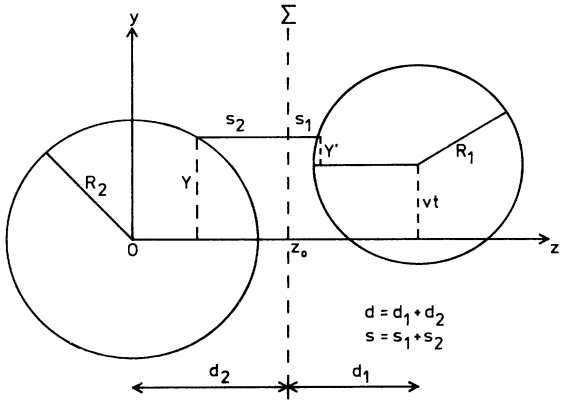

FIG. 1. The geometry of two colliding nuclei. The origin of coordinates is at the center of nucleus 2. Nucleus 1 moves uniformly on a straight line parallel to the y axis with velocity v. The distance of closest approach d is reached at $t=0$. The surface Σ is explained in the text.

Assuming that the depopulation of the entrance channel due to particle transfer between the colliding nuclei contributes to the imaginary potential W_{trans}, such a potential can be related to the total transfer probability

$$P_{ba}=\sum_{i,f}|A(bf,ai)|^2 \quad (a,b=1,2) \tag{2.2}$$

through the relation[1]

$$\frac{2}{\hbar}\int_{-\infty}^{\infty} W_{\text{trans}}[R(t)]dt=P_{21}+P_{12}\ , \tag{2.3}$$

where R is a point on the straight trajectory followed by the projectile.

The main purpose of the present work is to relate the quantal approach for calculating W_{trans} from Eqs. (2.1)—(2.3) to the proximity approximation used in Ref. 5. As in the proximity method we start from Eq. (2.3) and derive an expression for W_{trans} in terms of the flux of nucleons Φ through an element of unit area of one nucleus separated by the distance s from the corresponding element of unit area of the other nucleus. According to Fig. 1 this distance can be written as

$$s=s_1(X,Y)+s_2(X,Y')\ ;\quad Y'=Y-vt\ , \tag{2.4}$$

where s_1 and s_2 are the distances between the surface elements of nuclei 1 and 2, respectively, and a surface Σ. This is arbitrarily chosen in the region where the potentials V_1 and V_2 describing each nucleus separately vanish. Accordingly, Φ becomes a function of X,Y,Y', or, alternatively, X,Y,t. We want to express the total transition probability

$$P=P_{21}+P_{12} \tag{2.5}$$

in the form

$$\begin{aligned}P&=\int\Phi(X,Y,t)dX\,dY\,dt\\&=\frac{1}{v}\int\Phi[s_1(X,Y)+s_2(X,Y')]dX\,dY\,dY'\ ,\end{aligned} \tag{2.6}$$

where

$$\Phi=\Phi_{21}+\Phi_{12}\ . \tag{2.7}$$

If we consider a point

$$\vec{\mathrm{R}}(t)=\vec{\mathrm{d}}+\vec{\mathrm{v}}t$$

on the trajectory of the projectile, Eqs. (2.3) and (2.6) give

$$W_{\text{trans}}[R(t)]=\frac{\hbar}{2}\int\Phi(X,Y,t)dX\,dY\ . \tag{2.8}$$

In the following we aim at finding an expression for Φ by assuming that the nuclei are leptodermous. We note that the surface Σ in Fig. 1 does not normally appear in the proximity method, but it plays an important role in the quantal treatment.[6] This point will be discussed in Sec. III.

III. THE TRANSITION AMPLITUDE

For deriving $W(R)$ from (2.3) we need to evaluate $A(2f,1i)$ of (2.1). We rely on the assumption that the transfer is a peripheral process so that between the nuclei there is a region where the potentials V_1 and V_2 both vanish. The plane Σ is situated at $z=z_0$ (see Fig. 1) in this region. Then the transition amplitude takes the form[6]

$$A(2f,1i)\simeq\frac{i\hbar}{2m}\int_{-\infty}^{\infty}dt\int_{\Sigma}dx\,dy\left[\frac{\partial\psi_f^*(2)}{\partial z_0}\psi_i(1)-\psi_f^*(2)\frac{\partial\psi_i(1)}{\partial z_0}\right], \tag{3.1}$$

where $\psi_i(1)$ and $\psi_f(2)$ are the initial and final wave functions of the nucleon which is transferred from the orbit i in nucleus 1 to the orbit f in nucleus 2. The single particle wave functions are

$$\psi_i(1)=\varphi_i[\vec{r}-\vec{\mathrm{R}}(t)]e^{i/\hbar\{m\vec{v}\cdot\vec{r}-[\epsilon_i+(1/2)mv^2]t\}}\ , \tag{3.2}$$

$$\psi_f(2)=\varphi_f(\vec{r})e^{-(i/\hbar)\epsilon_f t}\ , \tag{3.3}$$

where φ_i,φ_j are the eigenstates of each nucleus at rest. Equation (3.2) expresses the fact that nucleus 1 is moving uniformly on a straight line with respect to 2, i.e., $\vec{\mathrm{R}}(t)=\vec{\mathrm{d}}+\vec{\mathrm{v}}t$.

By using the notation

$$\begin{aligned}k_1&=\frac{1}{\hbar v}(\epsilon_f-\epsilon_i-\tfrac{1}{2}mv^2)\ ,\\k_2&=\frac{1}{\hbar v}(\epsilon_f-\epsilon_i+\tfrac{1}{2}mv^2)\ ,\end{aligned} \tag{3.4}$$

and making the change of variable

$$t=\frac{1}{v}(y-y')\ , \tag{3.5}$$

the transition amplitude takes the following form:

$$A(2f,1i)=\frac{i\hbar}{2mv}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy'e^{-ik_1y'}\int_{-\infty}^{\infty}dy\,e^{ik_2y}\left[\varphi_f^*(x,y,z_0)\frac{\partial}{\partial z_0}\varphi_i(x,y',z_0-d)-\frac{\partial}{\partial z_0}\varphi_f(x,y,z_0)\varphi_i(x,y',z_0-d)\right]. \tag{3.6}$$

By introducing the Fourier transform

$$\varphi_\alpha(x,y,z)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}e^{i(k_xx+k_yy)}f_\alpha(k_x,k_y,z)dk_xdk_y \tag{3.7}$$

for $\alpha=i,f$, we can rewrite

$$A(2f,1i)=\frac{i\hbar}{2mv}\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dk_x\left[f_f^*(k_x,k_2,z_0)\frac{\partial}{\partial z_0}f_i(k_x,k_1,z_0-d)-\frac{\partial}{\partial z_0}f_f^*(k_x,k_2,z_0)f_i(k_x,k_1,z_0-d)\right] . \tag{3.8}$$

Using the Wentzel-Kramers-Brillouin (WKB) expression of the wave function φ_α at distances z beyond the turning point, one can show that (see Ref. 6 for details)

$$f_i=C_{l_i}e^{-\gamma(d-z_0)}\frac{2\pi}{\gamma}Y_{l_im_i}(\hat{k}_1) ,$$
$$f_f=C_{l_f}e^{-\gamma z_0}\frac{2\pi}{\gamma}Y_{l_fm_f}(\hat{k}_2) , \tag{3.9}$$

where C_{l_α} are normalization constants and

$$\gamma^2=k_x^2+k_1^2+\gamma_{0i}^2=k_x^2+k_2^2+\gamma_{0f}^2 , \tag{3.10}$$

$$\hat{k}_1=\frac{1}{\gamma_{0i}}(k_x,k_1,i\gamma);\quad \hat{k}_2=\frac{1}{\gamma_{0f}}(k_x,k_2,i\gamma) , \tag{3.11}$$

with

$$\gamma_{0\alpha}^2=-\frac{2m}{\hbar}\epsilon_\alpha . \tag{3.12}$$

The z components of $\hat{k}_1$ and $\hat{k}_2$ are imaginary because they represent momentum components under the barrier. With the help of expressions (3.9) one can reduce the transition amplitude to a simpler form,

$$A(2f,1i)=\frac{i\hbar}{mv}\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_x\gamma(k_x)f_f^*(k_x,k_2,z_0)f_i(k_x,k_1,z_0-d) , \tag{3.13}$$

where γ of Eq. (3.10) has been written as a function of k_x.

Taking as quantum numbers $i=(\epsilon_i,\ l_i,\ m_i)$, $f=(\epsilon_f,\ l_f,m_f)$, and assuming that the single particle energy ϵ_α varies continuously, the total transfer probability from nucleus 1 to nucleus 2 becomes

$$P_{21}=\left[\frac{\hbar}{mv}\right]^2\left[\frac{1}{2\pi}\right]^2\int_{\epsilon_{F_2}}^{\infty}d\epsilon_f\int_0^{\epsilon_{F_1}}d\epsilon_i\int_{-\infty}^{\infty}dk_x\gamma(k_x)\int_{-\infty}^{\infty}dk_x'\gamma(k_x')F_{2\epsilon_f}^*(k_x,k_x',z_0)F_{1\epsilon_i}(k_x,k_x',z_0-d) . \tag{3.14}$$

Here,

$$F_{n\epsilon}(k_x,k_x',z)=\sum_\alpha f_\alpha(k_x,k_n,z)f_\alpha^*(k_x',k_n,z)\delta(\epsilon-\epsilon_\alpha)\quad(n=1,2) \tag{3.15}$$

takes into account the summation over all states $\alpha=(l,m)$ of energy ϵ. Note that in Eq. (3.14) the integration limits for ϵ_i and ϵ_f are consistent with the Pauli principle. On the left-hand side (lhs) of (3.15) we have omitted the argument k_n for simplicity.

At this stage we want to derive an approximate expression for $F_{n\epsilon}$ which is valid for leptodermous nuclei. If we use the inverse of the Fourier transform (3.7), the function $F_{n\epsilon}$ becomes

$$F_{n\epsilon}(k_x,k_x',z)=\int_{-\infty}^{\infty}dx\,dx'dy\,dy'e^{-i(k_xx+k_ny)}e^{i(k_x'x'+k_ny')}\rho_\epsilon(\vec{r},\vec{r}') , \tag{3.16}$$

where $\rho_\epsilon(\vec{r},\vec{r}')$ is the density matrix at fixed energy ϵ,

$$\rho_\epsilon(\vec{r},\vec{r}')=\sum_\alpha\delta(\epsilon-\epsilon_\alpha)\varphi_\alpha(\vec{r})\varphi_\alpha(\vec{r}') . \tag{3.17}$$

In Appendix A we derive an analytic expression for ρ_ϵ which is valid for points r and r' just outside the surface of a large leptodermous nucleus. The essential ingredients of the derivation are the WKB approximation and the use of phase space coordinates directly related to the angular momentum.[9] For finite nuclei this expression reads

$$\rho_\epsilon(\vec{r},\vec{r}')=\frac{1}{(2\pi)^2}\frac{m}{\hbar^2}\int_0^{K_{M_n}}dk_\perp\frac{k_\perp}{\gamma}e^{-w_\epsilon(r)-w_\epsilon(r')}\times J_0(k_\perp\theta R_n) , \tag{3.18}$$

where θ is the angle between $\vec{r}$ and $\vec{r}'$ and R_n $(n=1,2)$ is the nuclear radius. In Eq. (3.18)

$$k_\perp=(k_x^2+k_n^2)^{1/2} \tag{3.19}$$

represents the component of the nucleon momentum perpendicular to the z axis as can be seen from Eq. (3.11). The upper limit of integration on $k_\perp$ can be found from the energy balance in a region in the nucleus where the kinetic energy reaches a maximum, i.e., where V_n $(n=1,2)$ reaches the maximum depth V_{0n}. There we have

$$\frac{\hbar^2}{2m}(k_\perp^2+k_z^2)=V_{0n}-|\epsilon_\alpha| .$$

Hence,

$$K_{M_n}=\max k_\perp=\left[\frac{2m}{\hbar^2}(V_{0n}-|\epsilon_\alpha|)\right]^{1/2}. \tag{3.20}$$

By analogy with the derivation made in Appendix A for finite nuclei, one can obtain a density matrix for semi-infinite nuclear matter described by the wave function

$$\varphi_{k_x,k_y\epsilon}(x,y,z)=\frac{1}{L}e^{i(k_x x+k_y y)}\varphi_\epsilon(z)\ ,$$

where L is the quantization box size along the x and y axis and $\varphi_\epsilon(z)$ is defined by the semiclassical expressions (A2) with r replaced by z. The result is very similar to (3.18),

$$\rho_\epsilon(\vec{r},\vec{r}')=\frac{1}{(2\pi)^2}\frac{m}{\hbar^2}\int_0^{K_M}dk_\perp\frac{k_\perp}{\gamma}e^{-w_\epsilon(z)-w_\epsilon(z')}J_0(k_\perp\rho)\ , \tag{3.21}$$

where the x,y dependence comes only through the vector $\vec{\rho}=(x-x',y-y')$. One can pass from (3.18) to (3.21) and vice versa by the replacements

$$r\leftrightarrow z\ ,\quad R_n\theta\leftrightarrow\rho\ . \tag{3.22}$$

The physical content of the similarity of the density matrices (3.18) and (3.21) is that the surface properties of a large finite nucleus are essentially the same as those of nuclear matter.

Substituting Eq. (3.18) into (3.16) and making the change of variables

$$X=\tfrac{1}{2}(x+x')\ ,\quad Y=\tfrac{1}{2}(y+y')\ ,$$
$$\rho_x=x-x'\ ,\quad \rho_y=y-y'\ ; \tag{3.23}$$
$$K_x=\tfrac{1}{2}(k_x+k_x')\ ,\quad \kappa_x=k_x-k_x'\ ;$$

we obtain at the surface Σ

$$F_{n\epsilon}(k_x,k_x',d_n)=\int_{-\infty}^{\infty}\tilde{F}_{n\epsilon}(K_x,X,Y,d_n)\times e^{-i\kappa_x X}dX\,dY\ , \tag{3.24}$$

where

$$\tilde{F}_{n\epsilon}(K_x,X,Y,d_n)=\frac{1}{2\pi}\frac{m}{\hbar^2}\theta(K_{M_n}^2-K_x^2-k_n^2)\times e^{-2w_\epsilon(d_n)}\frac{1}{\gamma}e^{-\gamma/R_n(X^2+Y^2)}\ . \tag{3.25}$$

The step function θ results from the integration over $k_\perp$. The exponential dependence on X and Y follows from the approximation

$$w_\epsilon(r)\simeq w_\epsilon(d_n)+\frac{\gamma}{2R_n}(x^2+y^2) \tag{3.26}$$

obtained for $w_\epsilon(r)$ in Appendix A. The X and Y dependence reflects the geometry of a spherical leptodermous nucleus.

Now we can express the total probability P_{21} of Eq. (3.14) in a convenient form by using Eq. (3.24) and introducing the approximation

$$\gamma(k_x)\simeq\gamma(k_x')\simeq\gamma(K_x)=\gamma\ . \tag{3.27}$$

The result is

$$P_{21}=\frac{1}{(2\pi)}\left[\frac{\hbar}{mv}\right]^2\int_{\epsilon_{F_2}}^{\infty}d\epsilon_f\int_0^{\epsilon_{F_1}}d\epsilon_i\int_{-\infty}^{\infty}dK_x\gamma^2(K_x)\int dX\,dY\,dY'\tilde{F}_{2\epsilon_f}(K_x,X,Y,d_2)\tilde{F}_{1\epsilon_i}(K_x,X,Y',d_1)\ . \tag{3.28}$$

This formula will be used in Sec. IV for deriving W_{trans}.

IV. FORMULA FOR W_{trans}

At present, we have all the ingredients for deriving an expression for W_{trans}. There are two steps in the arguments. Starting from Eqs. (2.6) and (3.28) for P_{21}, we obtain an equation for the flux Φ_{21} and compare it with the one-sided flux of Ref. 5. Then we calculate W_{trans} and discuss its relation to the proximity result of Ref. 5. Substituting (3.25) in (3.28) and using (2.6),

$$\Phi_{21}(X,Y,Y')=\int_{\epsilon_{F_2}}^{\infty}d\epsilon_f\int_0^{\epsilon_{F_1}}d\epsilon_i\int_{-\infty}^{\infty}dK_x M_{21}(\epsilon_f,\epsilon_i,K_x,s)e^{-\gamma[(1/\bar{R})X^2+(1/R_2)Y^2+(1/R_1)Y'^2]}\ , \tag{4.1}$$

where

$$M_{21}(\epsilon_f,\epsilon_i,K_x,s)=\frac{1}{(2\pi)^3}\frac{1}{\hbar^2 v}\theta(\Gamma^2-K_x^2)e^{-2Y(s)} \tag{4.2}$$

with

$$\Gamma^2=\max(K_{M_n}^2-k_n^2)\ , \tag{4.3}$$

$$\bar{R}=\frac{R_1R_2}{R_1+R_2}\ , \tag{4.4}$$

and

$$Y(s)=w_{\epsilon_f}(d_2)+w_{\epsilon_i}(d_1)\ . \tag{4.5}$$

The quantity $Y(s)$ can be easily recognized as the penetrability integral (B12) if one uses the definition (A3) for $w_{\epsilon_f}(d_2)$ and $w_{\epsilon_i}(d_1)$ and replaces r by z. In agreement with Fig. 2, one has

$$s=d_1+d_2-R_1-R_2\ . \tag{4.6}$$

We note that in writing (4.1) we have made use of the approximations (3.27) and (A14) which imply $\gamma(d)\simeq\gamma(K_x)$.

Of particular interest is the quantity

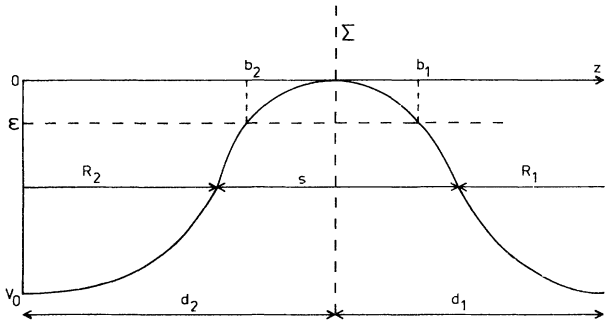

FIG. 2. The nuclear barrier along the z axis. The distance between the centers of the nuclei is $d_1+d_2=s+R_1+R_2$, where d_i ($i=1,2$) is the distance to the surface Σ explained in the text, R_i the nuclear radii, and s the distance between surfaces. b_i are the turning points of the single particle level.

$$\Phi_{21}^0=\Phi_{21}(0,0,0)\ . \tag{4.7}$$

This is just the flux between two semi-infinite slabs of nuclear matter separated by the distance s between surfaces. It can be obtained from (3.21) in a similar way to $\Phi_{21}(X,Y,Y')$.

For two identical nuclei the fluxes from both sides are equal,

$$\Phi_{21}^0=\Phi_{12}^0\ . \tag{4.8}$$

In Ref. 5 we calculated $W_{\rm trans}$ by assuming a proximity-type formula for the nucleon current. This reduced the calculation of the potential to the knowledge of the one-sided current between two semi-infinite slabs. In Appendix B we rewrote the one-sided flux of Ref. 5 in the form of (B15). This is identical to Φ_{21}^0 if the penetrability $P(k_x,s)$ defined in (B2) is approximated by the WKB formula

$$P(k_x,s)\simeq e^{-2Y(s)} \tag{4.9}$$

with $Y(s)$ given by (B12)—(B14).

The last step towards obtaining an expression for the imaginary potential from the definition (2.8) is to change the variable $Y'=Y-vt$ and integrate $\Phi_{21}+\Phi_{12}$ over X and Y. At the distance of closest approach when $t=0$ and $R=d$, one has

$$W_{\rm trans}(d)=\pi\hbar\bar{R}\int_{\epsilon_{F_2}}^{\infty}d\epsilon_f\int_0^{\epsilon_{F_1}}d\epsilon_i\int_{-\infty}^{\infty}dK_x\frac{1}{2\gamma(K_x)}[M_{21}(\epsilon_f,\epsilon_i,K_x,s)+M_{12}(\epsilon_f,\epsilon_i,K_x,s)]\ . \tag{4.10}$$

For values of s beyond which $(2m/\hbar^2)(V_1+V_2)\approx 0$ on the surface Σ, and for fixed energies ϵ_i,ϵ_f, the distance b_1-b_2 varies linearly with s. Hence,

$$\frac{dY(s)}{ds}\sim\gamma\ , \tag{4.11}$$

which leads to the identity

$$\frac{1}{2\gamma}e^{-2Y(s)}=\int_s^{\infty}e^{-2Y(s')}ds'\ . \tag{4.12}$$

By permuting integrals in (4.10) and using Eq. (4.12), we obtain

$$W_{\rm trans}=\pi\hbar\bar{R}I(s) \tag{4.13}$$

with

$$I(s)=\int_s^{\infty}ds'[\Phi_{21}^0(s')+\Phi_{12}^0(s')]\ . \tag{4.14}$$

Equation (4.13) for $W_{\rm trans}$ is essentially the same as Eq. (4) of Ref. 5. When the separation between the nuclei is large enough, the fluxes Φ_{21}^0 and Φ_{12}^0 derived here are the same as the one-sided fluxes between two semi-infinite slabs as introduced in Ref. 5. For smaller separations there is a difference because the present approach leads to a penetration factor $P=e^{-2Y}$, which is the WKB limit of Eq. (B2).

The original aspect of the present work is that we have proved the validity of the proximity method for deriving $W_{\rm trans}$ from the one-sided currents between interacting nuclei. Our evaluation is based on a quantum mechanical expression of the transition amplitude and we obtain the same result as in Ref. 5, which relied on the classical concept of Swiatecki and Randrup.[10] An essential step is the similarity of the density matrices (3.18) and (3.21). The physical content of this similarity is that a large finite nucleus has the same surface properties as nuclear matter. This is a generalization of the assumption about densities made in deriving proximity potentials. Off-diagonal matrix elements are required for calculating proximity currents.

Pollarolo *et al.*[4] obtained $W_{\rm trans}$ by directly evaluating (2.2). The method involves extensive computation but it includes shell and curvature effects. In our method such effects are averaged out by Eq. (3.18), which is an approximation of the exact density matrix. It would therefore be interesting to make some quantitative comparisons of these two methods.

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APPENDIX A

In this appendix we use a semiclassical approach to derive a formula for the density matrix of a nucleus

$$\rho_\epsilon(\vec{r},\vec{r}')=\sum_\alpha\delta(\epsilon-\epsilon_\alpha)\varphi_\alpha(\vec{r})\varphi_\alpha^*(\vec{r}') \tag{A1}$$

at a given energy ϵ. We assume that the nuclei are spheri-

cal and first concentrate on the radial part $f_l(r)$ of $\varphi_\alpha(r)$. We recall that the WKB approximation[11] gives

$$f_{\epsilon l}(r)\sim\frac{\bar{C}_l}{\sqrt{k_l}}\sin\left[\int_b^r k_l(r)dr+\frac{\pi}{4}\right] \quad a<r<b$$
$$\sim\frac{1}{2}\frac{\bar{C}_l}{\sqrt{\gamma_l}}e^{-w_\epsilon(r)} \quad r>b\ , \tag{A2}$$

where $k_l=i\gamma_l$, a and b are the turning points, and

$$w_\epsilon(r)=\int_b^r\gamma_l(r)dr \tag{A3}$$

with

$$\gamma_l(r)=\left[\gamma_0^2+\frac{2m}{\hbar^2}V(r)+\frac{(l+\frac{1}{2})^2}{r^2}\right]^{1/2}, \tag{A4}$$

$$\gamma_0^2=-\frac{2m}{\hbar^2}\epsilon\ . \tag{A5}$$

The normalization constant $\bar{C}_l$ can be related to the period $2\tau_l$ of the specific orbit ϵ,l. If we assume that the main contribution to the norm comes from the interval (a,b) and average over the periodical function, we have

$$1=\int_0^\infty f_l^2dr\simeq\tfrac{1}{2}\bar{C}_l^2\int_a^b\frac{dr}{k_l}=\tfrac{1}{2}\bar{C}_l^2\frac{\hbar}{m}\tau_l\ . \tag{A6}$$

Hence,

$$\bar{C}_l^2=\frac{2m}{\hbar}\frac{1}{\tau_l}\ . \tag{A7}$$

In our discussion we are interested in points r,r' situated just outside the nuclear surface. If the nuclei are leptodermous this region is narrow and we can approximate (A4) by

$$\gamma_l(r)\simeq\left[\gamma_0^2+\frac{2m}{\hbar^2}V(r)+\frac{(l+\frac{1}{2})^2}{R^2}\right]^{1/2}, \tag{A8}$$

where R is the nuclear radius.

The points of interest r or r' are located on the surface Σ perpendicular to the z axis at $z=d$. Due to the sphericity of the nuclei, only points close to d bring important contributions. In such a region, $w_\epsilon(r)$ of Eq. (A3) can be approximated by

$$w_\epsilon(r)\simeq w_\epsilon(d)+(r-d)\gamma(d)\ , \tag{A9}$$

where

$$r-d\simeq\frac{1}{2d}(x^2+y^2) \tag{A10}$$

and

$$\gamma(d)\simeq\left[\gamma_0^2+\frac{(l+\frac{1}{2})}{R^2}\right]^{1/2} \tag{A11}$$

because the surface Σ is defined such as

$$V(d)=0\ . \tag{A12}$$

In the classical limit we can take

$$l+\tfrac{1}{2}=k_\perp R\ , \tag{A13}$$

and from (3.10) and (3.19) we obtain

$$\gamma(d)\simeq\gamma\ , \tag{A14}$$

i.e., independent of d. Throughout this paper we use expression (A9) for $w_\epsilon(r)$ where $w_\epsilon(d)$ is given by (A3) and $\gamma(d)$ by (A14).

Let us now discuss the angular part of the density matrix. In Eq. (A1) the index α runs over all occupied degenerate states lm of energy ϵ. Summing over m gives

$$\sum_m Y_{lm}(\Omega)Y_{lm}^*(\Omega')=\frac{(2l+1)}{4\pi}P_l(\cos\vartheta)\ ,$$

where ϑ is the angle between $\vec{r}$ and $\vec{r}'$. We replace $P_l(\cos\vartheta)$ by the asymptotic form[12] (large l)

$$P_l(\cos\vartheta)\sim J_0[(l+\tfrac{1}{2})\vartheta]\ , \tag{A15}$$

where $J_0(x)$ is the zeroth order Bessel function, and take the continuous limit

$$\sum_l\rightarrow\int dl\,n(\epsilon,l)\ , \tag{A16}$$

where $n(\epsilon,l)$ is the density of states. Then we have

$$\rho_\epsilon(\vec{r},\vec{r}')=\frac{1}{2\pi rr'}\int dl\,l\,n(\epsilon,l)f_{\epsilon l}(r)f_{\epsilon l}(r')$$
$$\times J_0[(l+\tfrac{1}{2})\vartheta]\ . \tag{A17}$$

To be consistent with the semiclassical form (A2) for $f_{\epsilon l}$, we shall derive in the following an expression for $n(\epsilon,l)$ in the classical limit.

We follow the approach used by Horiuchi (see, e.g., Ref. 9) and write the volume element of the momentum space in terms of the variables E, α_ϑ, and α_φ, where

$$\alpha_\vartheta=\hbar(l+\tfrac{1}{2})>0\ , \tag{A18}$$

$$\alpha_\varphi=\hbar m\ ;\quad -\alpha_\vartheta\le\alpha_\varphi\le\alpha_\vartheta\ , \tag{A19}$$

which are related to the momentum components p_r, p_ϑ, and p_φ through the relations

$$\epsilon=\frac{1}{2m}\left[p_r^2+\frac{1}{r^2}p_\vartheta^2+\frac{1}{r^2\sin^2\vartheta}p_\varphi^2\right], \tag{A20}$$

$$\alpha_\vartheta=\left[p_\vartheta^2+\frac{p_\varphi^2}{\sin^2\vartheta}\right]^{1/2}, \tag{A21}$$

$$\alpha_\varphi=p_\varphi\ . \tag{A22}$$

This amounts to writing the volume element in the phase space as

$$dn=\frac{1}{(2\pi\hbar)^3}dr\,d\vartheta\,d\varphi\,dp_r dp_\vartheta dp_\varphi$$
$$=\frac{1}{(2\pi\hbar)^3}J^{-1}dr\,d\vartheta\,d\varphi\,d\epsilon\,d\alpha_\vartheta\,d\alpha_\varphi\ , \tag{A23}$$

where J is the Jacobian

$$J=\frac{\partial(\epsilon,\alpha_\vartheta,\alpha_\varphi)}{\partial(p_r,p_\vartheta,p_\varphi)}=\frac{p_r p_\vartheta}{m\alpha_\vartheta}\ . \tag{A24}$$

Integrating over ϑ and φ gives $2\pi^2$. An extra factor of 2 appears from taking into account both the positive and negative values of p_r at a fixed energy ϵ. The result is

$$\frac{4\pi^2}{(2\pi\hbar)^3}\frac{m\,dr}{|p_r|}d\epsilon\, d\alpha_\vartheta\, d\alpha_\varphi\ . \tag{A25}$$

The integration over α_φ gives $2\alpha_\vartheta$. By introducing the radial velocity $v_r=|p_r|/m$ and recalling that

$$\int_a^b \frac{dr}{v_r}=\tau_l\ , \tag{A26}$$

where the period τ depends on l through $\alpha_\vartheta=l+\frac{1}{2}$, one obtains for the density of states $n(\epsilon,l)$

$$n(\epsilon,l)=\frac{\tau_l}{\pi\hbar}\ . \tag{A27}$$

By using (A2), (A7), and (A27), we obtain for the density matrix (A1) outside the outer turning point

$$\rho_\epsilon(\vec{r},\vec{r}\,')=\frac{1}{(2\pi)^2 rr'}\frac{m}{\hbar^2}\times\int\frac{(l+\frac{1}{2})dl}{\gamma_l}e^{-w_\epsilon(r)-w_\epsilon(r')}J_0[(l+\tfrac{1}{2})\vartheta]\ . \tag{A28}$$

For points $\vec{r},\vec{r}\,'$ situated near the nuclear surface we can take

$$r\approx r'\approx R\ ,$$

and using (A13) we obtain

$$\rho_\epsilon(\vec{r},\vec{r}\,')=\frac{1}{(2\pi)^2}\frac{m}{\hbar^2}\int_0^{K_M}\frac{k_\perp dk_\perp}{\gamma_l}e^{-w_\epsilon(r)-w_\epsilon(r')}J_0(k_\perp\vartheta R)\ . \tag{A29}$$

This expression is used in Sec. III for deriving the flux $\Phi(X,Y,t)$. The upper limit K_M is defined by Eq. (3.20). The index l of γ can be removed if one bears in mind that γ depends on $k_\perp$ through (A13).

APPENDIX B

In this appendix we bring the formula for the one-sided flux used in Ref. 5 to a form comparable to the one derived in the present paper.

In Ref. 5 the one-sided flux from slab 1 to slab 2 was defined as

$$\Phi_{21}=\frac{1}{(2\pi)^3}\int_{k_z>0}d^3k\,P(k_z,s)\frac{\hbar k_z}{m}n_1(|\vec{k}|)\times[1-n_2(|\vec{k}|)]\ , \tag{B1}$$

where z is the axis perpendicular to the nuclear surfaces, n_1 and n_2 are the occupation probabilities of a state of momentum $\hbar\vec{k}$ in 1 and 2, respectively, and $P(k_z,s)$ is the transmission coefficient for a nucleon with momentum component $\hbar k_z$ in slab 1. The following semiclassical approximation was chosen to describe P:

$$P(k_z,s)=\frac{1}{1+\exp[2Y(s)]}\ , \tag{B2}$$

where Y is the penetrability integral

$$Y(s)=\int_{b_2}^{b_1}\left\{\frac{2m}{\hbar^2}[V(z,s)-k_z^2]\right\}^{1/2}dz \tag{B3}$$

between the turning points of the barrier

$$V(z,s)=V_0+V_1(z,s)+V_2(z,s) \tag{B4}$$

with Woods-Saxon forms for V_1 and V_2 having surfaces separated by a distance s. The formula (B2) reduces to the standard WKB expression when Y is large. The flux Φ_{12} is obtained from (B1) by interchanging 1 and 2 and by integrating over $k_z<0$. A factor of 2×2 is necessary if the spin and isospin degrees of freedom are included.

In order to make contact with the present work we wish to make a change of variables in Eq. (B1). The new variables ϵ_i and ϵ_f are the binding energies of a nucleon in slabs 1 and 2, respectively. If $V_0>0$ is the potential depth of both V_1 and V_2, ϵ_i and ϵ_f can be related to $\vec{k}$ through the energy balance in each slab. If slab 1 moves parallel to slab 2 with the momentum $\hbar q$ along the y axis, this reads

$$\epsilon_i+V_0=\frac{\hbar^2}{2m}[k_x^2+(k_y-q)^2+k_z^2] \tag{B5}$$

and

$$\epsilon_f+V_0=\frac{\hbar^2}{2m}(k_x^2+k_y^2+k_z^2)\ , \tag{B6}$$

which indicates that $\epsilon_i<0$ and $\epsilon_f<0$.

For changing variables, we need the Jacobian

$$J=\left|\frac{\partial(\epsilon_i,\epsilon_f)}{\partial(k_y,k_z)}\right|=\frac{\hbar^4k_zq}{m^2}\ , \tag{B7}$$

which allows us to write

$$dk_x dk_y dk_z=\frac{m^2}{k_zq\hbar^4}d\epsilon_i d\epsilon_f dk_x\ . \tag{B8}$$

On the other hand, by substracting Eqs. (B5) and (B6) and introducing the relative velocity

$$v=\frac{\hbar q}{m}\ , \tag{B9}$$

one obtains

$$k_y=\frac{1}{\hbar v}(\epsilon_f-\epsilon_i+\tfrac{1}{2}mv^2)\ . \tag{B10}$$

This expression is identical to k_2 of (3.4) and one can also see that $k_1=k_y-q$, which gives a physical significance to k_1 and k_2 through conservation laws. Using (B6), we rewrite in the integrand of (B3)

$$\frac{2mV_0}{\hbar}-k_z^2=k_x^2+k_y^2-\frac{2m}{\hbar^2}\epsilon_f=\gamma^2\ . \tag{B11}$$

In writing the second equality, we have made use of (3.10) and (3.12). Hence,

$$Y(s)=\int_{b_2}^{b_1}\left\{\gamma^2+\frac{2m}{\hbar^2}[V_1(z,s)+V_2(z,s)]\right\}^{1/2}dz\ . \tag{B12}$$

Replacing (B10) in (B11), we obtain

$$\gamma=\left[-\frac{2m}{\hbar^2}\left[\bar{\epsilon}-\frac{\hbar^2}{2m}k_x^2\right]\right]^{1/2}\ . \tag{B13}$$

The energy

$$\bar{\epsilon}=\tfrac{1}{2}(\epsilon_i+\epsilon_j)-\frac{1}{4}\left[\frac{2(\epsilon_i-\epsilon_f)^2}{mv^2}+\tfrac{1}{2}mv^2\right] \tag{B14}$$

is negative because $\epsilon_i<0$, $\epsilon_f<0$.

Equation (B13) shows that Y and hence P of (B1) can be expressed in terms of the integration variable k_x only. By using (B8) one can finally write

$$\Phi_{21}=\frac{1}{(2\pi)^3}\frac{1}{\hbar^2 v}\int_0^{\epsilon_{F_1}}d\epsilon_i\int_{\epsilon_{F_2}}^{\infty}d\epsilon_f\int_{-\infty}^{\infty}dk_x\theta\left[\frac{2m}{\hbar^2}(V_0+\bar{\epsilon})-k_x^2\right]P(k_x,s)\ , \tag{B15}$$

where the occupation probabilities n_1 and n_2 have imposed the above limits for ϵ_i and ϵ_f, and the step function θ fixes the limits of k_x at fixed $\bar{\epsilon}$. The arguments of the step functions in Eqs. (4.2) and (B15) are the same. This can be proved by using Eqs. (3.4), (3.20), and (4.3).

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