Assessment of approximations made in breakup-fusion descriptions

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(Received 28 November 1983)

Three methods for calculating the breakup-fusion cross sections are compared; ours, and those of Baur and Trautmann and of Kasano and Ichimura. All three use the two-step and the spectator approximations. It will be shown first that, while ours does not make any further approximation, the other two make a few additional approximations. Errors caused by these additional approximations are assessed numerically. The errors are found rather large in general. A critical assessment of the formalism by Austern and Vincent, on which the work of Kasano and Ichimura was based, is also made.

I. INTRODUCTION

The breakup of both light and heavy ion projectiles is one of the major subjects of current nuclear reaction studies. In the simplest version, which may be called elastic breakup (EB), the projectile a is simply broken up into a pair, b and x, the target A remaining in its ground state. It often happens, however, that x is fused with A. This process, which may be called the breakup-fusion (BF) process, is known to contribute significantly (often dominantly) to the singles cross section of b. See Ref. 1 for a review of the BF reaction.

Baur and Trautmann (BT) were the first to study the BF reaction. BT and their collaborators[2–5] in fact analyzed a variety of BF-type processes. In doing these analyses, they used a formula which we shall henceforth call the BT formula. This formula was obtained by first writing the elastic breakup (one-step) DWBA amplitude in the *post* form, and then by replacing, in a somewhat *ad hoc* manner, the elastic breakup channel wave function by a more complicated one. An approximation, which may be called the surface approximation, was then introduced in making the resultant formula calculable.

Subsequently, more elaborate formulation of the BF process was done by Kerman and McVoy (KM) (Ref. 6), by Austern and Vincent (AV) (Ref. 7), and by Udagawa and Tamura (UT) (Ref. 8). All these formulations, as well as that of BT, use the two-step description of the BF process. All four formalisms further treat the particle b as a spectator, in that the motion of b is described in terms of the standard distorted wave. In other words, all four formalisms are characterized by the use of two approximations; the two-step and the spectator approximations. For simplicity, we shall henceforth call them together *the two-step approximation.*

Both KM and UT used the *prior form* (interaction) in describing the first step, i.e., the elastic breakup process, and arrived at the same BF formula, although somewhat different methods were used for its derivation. In the following, we shall call the KM and the UT formula *our* formula, for simplicity. Our formula has been used rather extensively and successfully in the past, in analyzing both massive[9–11] and light-particle[12] transfer-type BF processes, induced by both light and heavy ions.

Our BF formula is given[8] as (a constant times) the expectation value of the imaginary part of the optical potential of the $x+A$ system (which we shall call the x channel), with respect to the state function for the x channel. This x-channel wave function is given as a projection (onto the distorted wave state of b) of the one-step DWBA breakup wave function.

The formula obtained by AV has the same form as ours, but differs in one important regard. In place of the prior form for the interaction, the post form for the interaction appears. It is well known[13,14] that, while the post-prior equivalence holds for the one-step DWBA transition amplitudes, it does not hold for wave functions in general (like the x-channel wave function), unless the evaluation of the function is limited to the asymptotic region. It is also well known[13,14] that the nonorthogonality correction term must be present when the post interaction is used. This is because the (x-channel) wave function, with the post interaction, inevitably includes the projection of the incident wave function, which is unphysical. The nonorthogonality correction term serves to remove this unphysical component. The post form x-channel wave function is used in the BF formula of AV, without being corrected by the nonorthogonality term. Therefore, AV has to be regarded as an approximate theory in the sense that it simply neglects the nonorthogonality term correction (or accepts the unphysical contributions).

Recently, Kasano and Ichimura (KI) (Ref. 15) performed numerical calculations based on the AV formalism. They were thus done under the same AV approximation. [KI further made the zero-range (ZR) approximation.] In the present paper, we intend to assess numerically the KI approximations. As we shall also show, the BT is a theory which contains one more approximation to add to the KI approximations. We intend to assess the BT approximations as well. Since, as we noted above, all the theories (KM, UT, AV, KI, and BT) use the two-step approximation, and since our theory[6,8] does not make any further approximation, we may sometimes call our theory "exact" in what follows, for the sake of simplicity. By the same token, AV, KI, and BT theories will be called "ap-

proximate" theories.

In Sec. II A, we shall present our BF formula, which, as stressed above, is in the prior form. In Sec. II B, we show how to transform our BF formula exactly into the post form. It will be seen that a nonorthogonality correction term does appear. It will also be seen that, if this correction term is neglected, the BF formula of AV results. In Sec. II C, we shall then show how to obtain the approximate formulas of KI and BT. In Sec. III, we present results of a few typical numerical calculations, and assess the KI and BT approximations. Concluding remarks will be given in Sec. IV. In the Appendix, we review briefly the AV formalism. We do this because AV is not an *ad hoc*, but a rather carefully formulated theory. It is thus desirable to show explicitly where in AV the neglect of the nonorthogonality term was indeed committed.

II. CROSS SECTION FORMULAS

A. BF formula in the prior form

The BF reaction that we consider here may be written as

$$a+A\rightarrow b+x+A\rightarrow b+B^* \ . \tag{1}$$

Equation (1) means that, in the first step, the projectile a breaks up into particles b and x. The particle x is then fused into the target nucleus A to form a compound nucleus B^*, while b is emitted and observed. We shall call the system that consists of x and A the x channel, and of b and B^* $(=x+A)$ the b channel.

The formula for the b singles cross section (via the BF process) is given[6,8] as

$$d^2\sigma/dE_b d\Omega_b=(2\pi/\hbar v_a)d(E_b) \times(\langle\Psi_{x,a}^{(+)}\mid W_x\mid\Psi_{x,a}^{(+)}\rangle/\pi) \ . \tag{2}$$

Here, v_a is the incident velocity, W_x is the (negative of the) imaginary part of the optical potential in the x channel, and $\Psi_{x,a}^{(+)}$ is the wave function of the x channel. $d(E_b)$ is the phase space volume of the emitted particle b, and is given as

$$d(E_b)=(\mu_b k_b)/(8\pi^2\hbar^2) \ , \tag{3}$$

μ_b and k_b being the reduced mass and the wave number associated with the b channel. The expectation value of W_x with respect to $\Psi_{x,a}^{(+)}$ in Eq. (2) describes the absorption in the x channel.

Within the two-step approximation we employ here, in which the first-step breakup is treated in the prior form DWBA, the x-channel wave function $\Psi_{x,a}^{(+)}$ is obtained as[6,8]

$$\Psi_{x,a}^{(+)}(\vec{r})=\int G_x^{(+)}(\vec{r},\vec{r}\,')\rho^{(a)}(\vec{r}\,')d\vec{r}\,' \ , \tag{4a}$$

with

$$\rho^{(a)}(\vec{r})=(\chi_b^{(-)}\phi_x\phi_b\phi_A\mid V_a\mid\phi_a\phi_A\chi_a^{(+)}\rangle \ . \tag{4b}$$

Here, the symbol $(\ |\ |\ )$ means that integrations are to be taken over all coordinates, excepting the x-channel coordinate $\vec{r}$. Also, $G_x^{(+)}(\vec{r},\vec{r}\,')$ in Eq. (4a) is the optical model Green's function in the x channel, while $\chi_a^{(+)}$ and $\chi_b^{(-)}$ in Eq. (4b) are the distorted waves in the a $(=a+A)$ and the b $(=b+B^*)$ channels, respectively. Furthermore, ϕ_i $(i=a,\ x,\ b,$ and $A)$ denotes the intrinsic wave function of the particle i.

The $\rho^{(a)}(\vec{r})$ defined by Eq. (4b) is often referred to as a source function. The superscript a was attached to signify that this source function was defined by using the prior form of the interaction, i.e., V_a. Later, we shall introduce another source function $\rho^{(b)}(\vec{r})$, which is constructed by using the post form of the interaction, i.e., V_b. (A precise definition of V_a and V_b will be given in the following.)

For later convenience, we shall give here also the partial-wave expanded forms of the formulas given above. If we expand $\Psi_{x,a}^{(+)}(\vec{r})$ as

$$\Psi_{x,a}^{(+)}(\vec{r})=(1/r)\sum_{l_x m_x}u_{l_x m_x}(r)Y_{l_x m_x}(\Omega) \ , \tag{5}$$

the radial function, $u_{l_x m_x}(r)$, is found to satisfy an inhomogeneous differential equation written as

$$\{(\hbar^2/2\mu_x)[d^2/dr^2-l_x(l_x+1)/r^2]+E_x-U_x\}u_{l_x m_x}(r) =\rho_{l_x m_x}^{(a)}(r) \ . \tag{6a}$$

Here, E_x, μ_x, and U_x are, respectively, the energy, reduced mass, and optical potential in the x channel. Note that E_x is related to the energies E_a and E_b of the incident and outgoing particles as

$$E_x=E_a-E_b+Q \ , \tag{6b}$$

where Q is the breakup reaction Q value. Also, $\rho_{l_x m_x}^{(a)}(r)$ is the $(l_x m_x)$ component of $\rho^{(a)}$, and is given by

$$\rho_{l_x m_x}^{(a)}(r)=r\int\rho^{(a)}(\vec{r})Y_{l_x m_x}(\Omega)d\Omega \ . \tag{6c}$$

Inserting Eq. (5) into (2), the latter can be rewritten as

$$d^2\sigma/dE_b d\Omega_b=\sum_{l_x}d^2\sigma_{l_x}/dE_b d\Omega_b \ , \tag{7a}$$

$$d^2\sigma_{l_x}/dE_b d\Omega_b=(2\pi/\hbar v_a)d(E_b) \times\sum_{m_x}(\langle u_{l_x m_x}\mid W_x\mid u_{l_x m_x}\rangle/\pi) \ . \tag{7b}$$

B. BF formula in the post form

We shall now transform the BF formula of Sec. II A, which was given in the prior form, into that in the post

form. This is achieved by utilizing a relation, which is valid if used in the matrix element in Eq. (4b). The relation is that

$$V_a = H_a + V_a - E_a = H_b + H_x + V_b - E_a = H_x - E_x + V_b \ , \tag{8a}$$

where H_a, H_b, and H_x are defined, respectively, as

$$H_a = h_a + h_A + T_{aA} + U_{aA} \ ,$$
$$H_b = h_b + T_{bB} + U_{bB} \ , \tag{8b}$$
$$H_x = h_x + h_A + T_{xA} + U_{xA} \ ,$$

while

$$V_a = V_{aA} - U_{aA} \ ,$$
$$V_b = V_{bx} + V_{aA} - U_{bB} - U_{xA} \ . \tag{8c}$$

In the above equations, h_i ($i=a$, b, A, and x) is the intrinsic Hamiltonian of the particle i, while T_{ij}, V_{ij}, and U_{ij} are, respectively, the (relative motion) kinetic energy, the interaction potential, and the optical potential in the $i+j$ channel. (It will thus be clear that H_a, H_b, and H_x are the optical model Hamiltonians for the a, b, and x channels, respectively.)

The Green's function $G_x^{(+)}$ involved in (4a) is nothing but the inverse of $(E_x - H_x)$. Inserting (8a) into (4b), we thus see that (4a) is rewritten as

$$\Psi_{x,a}^{(+)}(\vec{r}) = \Psi_{x,b}^{(+)}(\vec{r}) - n(\vec{r}) \ , \tag{9a}$$

$$\Psi_{x,b}^{(+)}(\vec{r}) = \int G_x^{(+)}(\vec{r},\vec{r}\,')\rho^{(b)}(\vec{r}\,')d\vec{r}\,' \ , \tag{9b}$$

with

$$\rho^{(b)}(\vec{r}) = (\chi_b^{(-)}\phi_b\phi_x\phi_A \mid V_b \mid \phi_a\phi_A\chi_a^{(+)}\rangle \ , \tag{10a}$$

$$n(\vec{r}) = (\chi_b^{(-)}\phi_b\phi_x\phi_a \mid \phi_a\phi_A\chi_a^{(+)}\rangle \ . \tag{10b}$$

We emphasize here that $\Psi_{x,b}^{(+)}(\vec{r})$ of (9b) involves the x-channel wave function components that are projected out both from the EB wave function and the incident wave.[13,14] The former is physical and is precisely what is given by $\Psi_{x,a}^{(+)}(\vec{r})$ of (4a). The latter is, however, unphysical, and thus must be subtracted. Actually, this unphysical term is nothing but the nonorthogonality term $n(\vec{r})$, and the presence of the $n(\vec{r})$ term on the right-hand side (rhs) of (9a) does achieve the required subtraction. The presence of the $n(\vec{r})$ term in (9a) is thus crucial in making the calculated BF cross section free from unphysical contributions. (The situation is exactly the same as that appearing in the general formalism of two-step process calculations given in Refs. 13 and 14.)

To use $\Psi_{x,b}^{(+)}(\vec{r})$ only is thus physically unacceptable, or at best considered to result in an approximate theory, in which an unphysical contribution is accepted. Note that, if $\Psi_{x,b}^{(+)}(\vec{r})$ of (9b) is inserted into Eq. (2), we obtain the BF formula of AV [given by the second term on the left-hand side (lhs) of Eq. (17) of AV]. It is thus quite legitimate to understand that the AV (and KI) formula contains an approximation of neglecting the nonorthogonality term correction.

We may note at this stage that both $\rho^{(b)}(\vec{r})$ and $n(\vec{r})$ behave as $O(1/r^2)$ for $r\to\infty$, i.e., that both are long ranged. It thus makes the construction of $\Psi_{x,a}^{(+)}(\vec{r})$ via the use of (9) rather involved, offsetting the (seeming) advantage of being able to use the zero range approximation, when the post form is used in the light-ion induced BF reactions.

We may also note that, because of the $O(1/r^2)$ nature, the $n(\vec{r})$ term does not contribute to the asymptotic amplitude of $\Psi_{x,a}^{(+)}(\vec{r})$. The asymptotic form of $\Psi_{x,a}^{(+)}(\vec{r})$ is given, if (9a) is used, as

$$\Psi_{x,a}^{(+)}(\vec{r}) \overset{r\to\infty}{\longrightarrow} r^{-1}\exp(ik_x r)\langle\chi_x^{(-)} \mid \rho^{(b)}\rangle \ , \tag{10c}$$

where $\chi_x^{(-)}$ is the distorted wave in the x channel. If we use (4a), on the other hand, we will have

$$\Psi_{x,a}^{(+)}(\vec{r}) \overset{r\to\infty}{\longrightarrow} r^{-1}\exp(ik_x r)\langle\chi_x^{(-)} \mid \rho^{(a)}\rangle \ . \tag{10d}$$

The EB cross section can be obtained as the absolute square of the asymptotic amplitude, that appears either in (10c) or (10d). However, since we have

$$|\langle\chi_x^{(-)} \mid \rho^{(b)}\rangle|^2 = |\langle\chi_x^{(-)} \mid \rho^{(a)}\rangle|^2 \ ,$$

because of the equivalence of (4a) and (9a), we see that the celebrated post-prior equivalence is prevailing for the EB cross section. This does not mean, however, that the post-prior equivalence also holds for the BF cross section [if one sets $n(\vec{r})=0$ in (9a)], because nonasymptotic values of $\Psi_{x,a}^{(+)}(\vec{r})$ are used in Eq. (2). This difference is the well-known difference, regarding the post-prior equivalence or nonequivalence between the one- and two-step processes;[13,14] one- and two-step processes are represented here by the EB and BF processes, respectively.

C. Introduction of approximations

Having obtained Eq. (9a), which is equivalent to (4a) (and thus is still exact after making the two-step approximation), we shall now begin to introduce three (additional) approximations one by one. It will be seen that the KI and BT formulas emerge in the course. The approximations are (1) to use the zero-range (ZR) approximation, (2) to suppress the nonorthogonality term $n(\vec{r})$, and (3) to introduce the so-called surface approximation. [If only approximations (1) and (2) are made, the KI formula emerges.]

Once (9a) has been obtained, the introduction of the ZR approximation (1) can be done fairly safely, at least in (10a) which has a short range interaction. [We see, e.g., in a (d,p) reaction, that $V_b \simeq V_{pn}$.] On the other hand, there is no way to justify the use of the ZR approximation for $n(\vec{r})$, but this problem did not bother either BT or KI. The $n(\vec{r})$ term did not appear in their formula from the beginning, a fact which shows that approximation (2) was made (implicitly) in both BT and KI. In BT, this was done in a somewhat *ad hoc* way, being included in their making of the surface approximation. The way this was done in the AV (and hence in the KI) formalism is, how-

ever, a little more subtle, and we shall discuss this in some detail in the Appendix. (See also the discussions given in Sec. II B.)

The use of approximation (1), i.e., of the ZR approximation [which we have to apply to the $n(\vec{r})$ term as well], is to replace (10a) and (10b) by

$$\rho^{(b)}(\vec{r}) \simeq D_0 \chi_b^{(-)}(B\vec{r}/A)\chi_a^{(+)}(\vec{r}) , \tag{11a}$$

$$n(\vec{r}) \simeq N_0 \chi_b^{(-)}(B\vec{r}/A)\chi_a^{(+)}(\vec{r}) , \tag{11b}$$

where D_0 and N_0 are given, respectively, by

$$D_0 = \int V_b(r)\phi_a(\vec{r})d\vec{r} , \tag{12a}$$

$$N_0 = \int \phi_a(\vec{r})d\vec{r} . \tag{12b}$$

The constant D_0 is what is usually referred to as the zero-range D_0 factor. N_0 is the corresponding factor for the nonorthogonality term. Note that under the above approximation, we have

$$n(\vec{r}) = (N_0/D_0)\rho^{(b)}(\vec{r}) . \tag{12c}$$

The second approximation, approximation (2), is simply to neglect the nonorthogonality term, i.e., to set

$$n(\vec{r}) \simeq 0 . \tag{13}$$

This approximation (2), combined with approximation (1), gives rise to the approximate formula of KI.[15]

In order to explain approximation (3), it is convenient to use the integral form of $u_{l_x m_x}$. This $u_{l_x m_x}$ satisfies (6a) if $\rho^{(b)}$, rather than $\rho^{(a)}$, appears on the rhs. Thus, we have

$$u_{l_x m_x}(r) = u^{(1)}_{l_x m_x}(r) + u^{(2)}_{l_x m_x}(r) , \tag{14a}$$

$$u^{(1)}_{l_x m_x}(r) = -(2\mu_x/\hbar^2 k_x)\chi_{l_x}(r) \times \int_r^\infty y_{l_x}(r')\rho^{(b)}_{l_x m_x}(r')dr' , \tag{14b}$$

$$u^{(2)}_{l_x m_x}(r) = -(2\mu_x/\hbar^2 k_x)y_{l_x}(r) \times \int_0^r \chi_{l_x}(r')\rho^{(b)}_{l_x m_x}(r')dr' , \tag{14c}$$

where χ_{l_x} and y_{l_x} are the regular and irregular solutions of the homogeneous part of (6a).

The surface approximation of BT, i.e., our approximation (3), is to make the following three simplifications: (i) suppress $u^{(2)}_{l_x m_x}$, (ii) in $u^{(1)}_{l_x m_x}$ approximate y_{l_x} by $C_{l_x}^{-1}(\chi_{l_x} - F_{l_x})$ [where C_{l_x} is defined as $\chi_{l_x} \underset{r\to\infty}{\longrightarrow} F_{l_x} + C_{l_x}(G_{l_x} + iF_{l_x})$], and further (iii) set the lower limit r of the integral to 0. We thus have

$$u_{l_x m_x}(r) \simeq -(2\mu_x/\hbar^2 k_x)\chi_{l_x}(r) \times \int_0^\infty y^{\mathrm{SA}}_{l_x}(r')\rho^{(b)}_{l_x m_x}(r')dr' , \tag{15a}$$

$$y^{\mathrm{SA}}_{l_x}(r) \equiv C_{l_x}^{-1}(\chi_{l_x} - F_{l_x}) . \tag{15b}$$

(SA stands for surface approximation.) The above simplifications (i)—(iii) may be justified if the major contribution to the reaction comes from the peripheral region. This is why approximation (3) is called the surface approximation.

We remark here that we discussed the KI formula first, and then the BT formula. This is because the former was based on a general formalism of AV, and thus permitted us to single out the surface approximation from the other two approximations. In the way the formulation was done in BT, all the three approximations were unseparably mingled together.

III. NUMERICAL ESTIMATES OF THE ERRORS

In order to obtain quantitative estimates of the error caused by each of the approximations (1), (2), and (3), we have calculated cross sections with and without these approximations. We took as examples the ^{93}Nb(d,p) reaction with $E_d = 25.5$ MeV, the ^{58}Ni(α,t) reaction with $E_\alpha = 160$ MeV, and the ^{58}Ni(α,p) reaction with $E_\alpha = 80$ and 160 MeV. The optical potentials used are as follows. For the (d,p) reaction, the potentials used in Ref. 2 are employed, while for the (α,t) and (α,p) reactions we took them from Refs. 12 and 11, respectively.

With our formalism, we have to carry out finite-range calculations, and to do this we must have an explicit form of the internal wave function of the projectile. We used the Hulthen wave function for the deuteron, while for the α particle the wave function was generated by a Woods-Saxon potential with $V = 74.9$ MeV, $r_0 = 1.2$ fm, and $a = 0.67$ fm. (With these values of the parameters, the experimental binding energy was correctly reproduced.) Using these wave functions and the potentials used to generate them, the zero-range parameters D_0 and N_0 can be calculated. We found that $D_0 = 125$ MeV fm$^{3/2}$ and $N_0 = 56$ fm$^{3/2}$ for the (d,p) reaction, and $D_0 = 317$ MeV fm$^{3/2}$ and $N_0 = 13$ fm$^{3/2}$ for the (α,t) and (α,p) reactions. In the ZR calculations, we used these values as they stand, except that we set $N_0 = 8$ fm$^{3/2}$ for the (α,p) reaction. We found that, with this, the best cancellation took place between the contributions from the first and second terms in Eq. (9a) in the asymptotic region.

Even before carrying out detailed numerical analyses, we can predict the results to some extent. Generally speaking, the BT approximations are better for cases in which the first step, i.e., the breakup process, takes place predominantly in the peripheral region. Thus, they will be fairly good for (d,p) reactions; the deuteron is so loosely bound that it is broken up at large distances. On the other hand, the BT approximations would get poorer for the (α,p) and (α,t) reactions, because α is so tightly bound. This is more so the case for (α,p), because in this case the triton, that is to be eventually fused into the target, penetrates deep into the "deep peripheral" region (about 2 fm deeper than the usual peripheral region), which forces the center of mass (c.m.) of α to be in the deep peripheral region as well. (See Ref. 11 for a detailed numerical confirmation of this fact.) In the case of the (α,t) reaction, the proton must now be in the deep peripheral region, but the c.m. of the α stays outside. Therefore, the situation of (α,t) is expected to be intermediate between those of (d,p) and (α,p) reactions. As we shall see, these expectations are all borne out by the numerical calculations.

A. Singles cross sections

We present in Figs. 1–3 the various cross sections as functions of the energy of the outgoing particles. The angles at which these spectra were obtained are shown in each of the figures. The quantities σ_0, σ_1, σ_{1+2}, and σ_{1+2+3} presented there are, respectively, the cross sections obtained without any approximation (i.e., based on our formalism), with approximation (1) only, with approximations (1) and (2), and all of the approximations. Thus, σ_{1+2} and σ_{1+2+3} are the cross sections obtained with the KI and BT approximations, respectively.

In Fig. 1, where the results of the (d,p) reaction are shown, one sees that σ_1 is larger than σ_0 by about 40–60 % at the higher E_p region and by nearly an order of magnitude at the lower E_p. Thus, approximation (1), i.e., the ZR approximation, is found to let the theory overestimate the cross sections rather badly. Note that the ZR approximation, used in the analysis of the usual (d,p) reactions (that strip a neutron into a *bound orbit),* normally underestimates the cross section. In these cases the form factor is localized to the nuclear region, while in the present case of the *continuum state* transition, the form factor is not well localized as we have remarked above. This difference seems to be the origin of the differing behavior of the error due to the use of the ZR approximation.

If one switches on approximation (2), the large error at the lower E_p disappears, the resultant σ_{1+2} getting much closer to σ_0. However, the error at the higher E_p is now about a factor of 2. It is remarkable to see that σ_{1+2+3} is essentially the same as is σ_{1+2}, indicating that approximation (3) is good, at least for the (d,p) reaction. This is what we have expected.

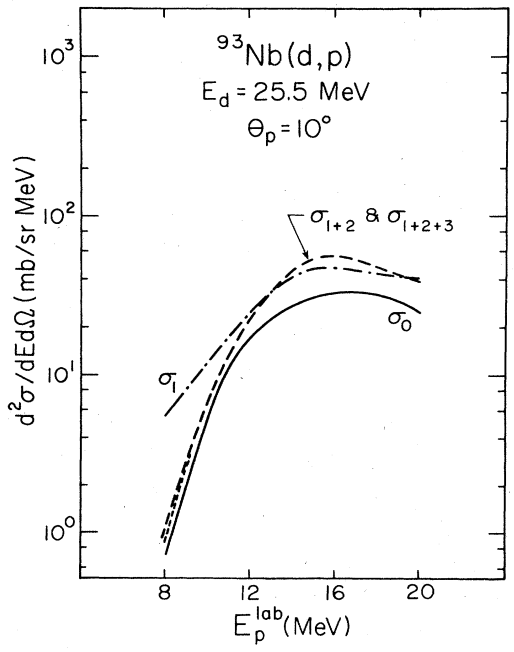

FIG. 1. Comparison of the cross sections obtained with and without approximations. Definition of the various cross sections σ_0, σ_1, σ_{1+2}, and σ_{1+2+3} is found in the text. This is the case of the ^{93}Nb(d,p) reaction with the incident energy E_d=25.5 MeV.

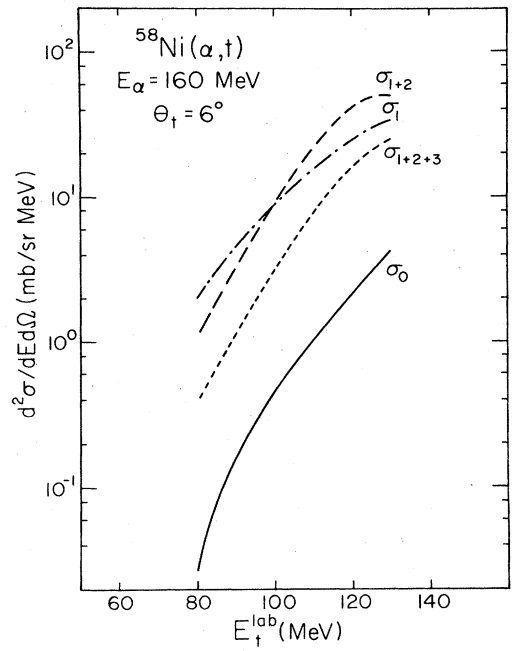

FIG. 2. Same as Fig. 1, except that this is for the ^{58}Ni(α,t) reaction with E_α=160 MeV.

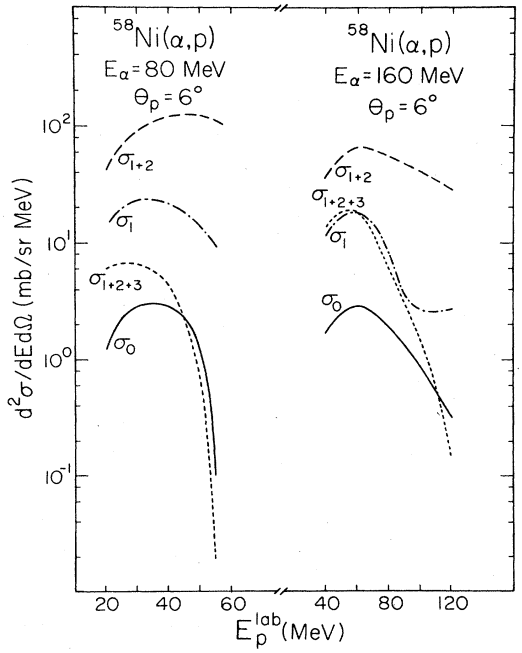

FIG. 3. Same as Fig. 1, except that this is for the ^{58}Ni(α,p) reaction with E_α=80 MeV (a) and 160 MeV (b).

The results for the (α,t) reaction are shown in Fig. 2. It is now seen that σ_1 and σ_{1+2} are fairly close to one another, both being about a factor of 10 too large compared with σ_0. It is also seen that the error is reduced somewhat in σ_{1+2+3}; although it is still a factor of about 5. [However, this does not mean that the approximation (3) is a good approximation here. It rather shows that, if approximation (3) is used by itself, it causes one to underestimate the correct cross section by a factor of 2.]

The results shown in Fig. 3 for the (α,p) reaction tell us that all the BT approximations are very wild here. The σ_1 overestimates σ_0 by an order of magnitude, and σ_{1+2} by nearly two orders of magnitudes. This large error is then drastically compensated by the similarly large but opposing error of approximation (3), letting σ_{1+2+3} be fairly close to σ_0. In fact, we see that $\sigma_{1+2+3}\simeq\sigma_0$, at least for large E_p, although it should be kept in mind that this good agreement may very well be a simple accident, and further that σ_{1+2+3} predicts the spectrum rather erroneously.

B. Partial wave cross sections

We have seen above that the BT approximations become worst for the (α,p) reaction. In order to see the origin of these errors somewhat more closely, we present in Fig. 4 the partial (wave) cross sections defined by Eq. (7b). They are plotted as functions of l_x, by taking the case of $E_\alpha=160$ MeV and $E_p=60$ MeV.

In Fig. 4, we notice that σ_0 has a sharp peak at around $l_x=20$, with a small hump at the region of small l_x, where σ_0 is very small. On the other hand, in σ_1, σ_{1+2}, and σ_{1+2+3}, this small hump has dramatically grown up so as to form another peak at around $l_x=5$.

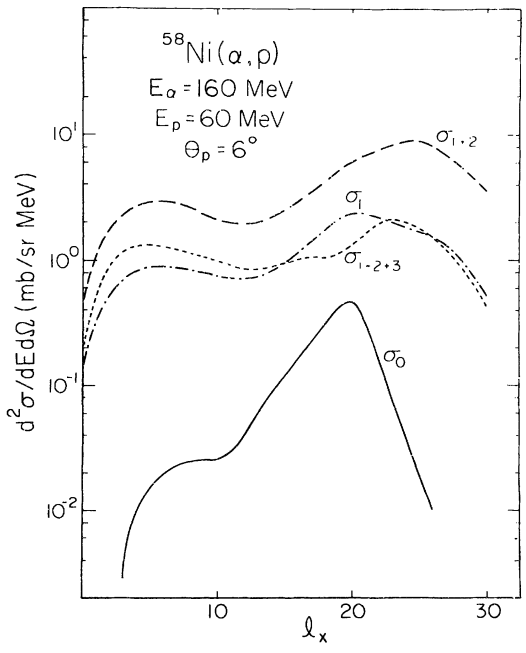

FIG. 4. Comparison of the partial wave cross sections obtained with and without approximations. The rest is the same as in Fig. 1, except that this is for the ^{58}Ni(α,p) reaction with $E_\alpha=160$ MeV and $E_p=60$ MeV.

Note that the grazing angular momenta are $(l_\alpha)_{gr}\simeq 36$ and $(l_p)_{gr}\simeq 10$ in the present case, and l_x is given by $\vec{l}_x=\vec{l}_\alpha+\vec{l}_p$. Therefore, to have a peak at small l_x $(\simeq 5)$ means that small l_α and l_p, and hence the nuclear interior, are contributing significantly. The fact that σ_0 is small for $l_x\simeq 5$ may mean that the cancellation of various amplitudes, described properly by the use of the finite range treatment, is taking place there. Such cancellations appear to fail to take place if the BT approximations are used.

C. Surface approximation

We have seen in Fig. 3, for the case of the (α,p) reaction, that the surface approximation helped to decrease drastically the cross section, so that σ_{1+2+3} became fairly close to σ_0, while σ_{1+2} was very large compared with σ_0. We thus wanted to find out which of three simplifications (see the end of Sec. II C) that constituted this approximation was responsible for the above drastic behavior, and made a somewhat detailed investigation on this point.

We found that the first simplification, i.e., the neglect of $u^{(2)}$ of (14) was acceptable; $u^{(2)}$ was indeed small. On the other hand, we found that $y_{l_x}^{SA}(r)\ll y_{l_x}(r)$ for the interior r, a fact that is easy to understand: $y_{l_x}(r)$ is an irregular wave function, while $y_{l_x}^{SA}(r)$ is regular. Therefore, if $\rho_{l_x}^{(b)}(r)$ is significant in the interior, as is the case in the (α,p) reaction, one obtains a much smaller value for $u_{l_x}^{(1)}$, if $y_{l_x}^{SA}(r)$ is used in place of $y_{l_x}(r)$. This explains why the surface approximation helped in decreasing the cross section.

IV. CONCLUDING REMARKS

We showed that the KI (Ref. 15) [BT (Refs. 2–5)] formula for the BF cross section can be derived by starting from our "exact" formula[6,8] and then by making two (three) successive approximations. We then investigated numerically how these approximations made the resultant cross sections deviate from those obtained based on our formula.

We expected from the beginning that the KI and BT approximations were acceptable for reactions in which the (first-step) breakup took place dominantly in the peripheral region. And this expectation was confirmed by finding that the approximate results of KI and BT agreed fairly well with our "exact" results, when (d,p) reactions were considered. We found, however, that the approximate results differed significantly from ours, for the nonperipheral (α,t) and (α,p) reactions, in particular for the latter.

To be noted here is the fact that the error caused by the first two approximations in BT is often opposite to that caused by the third approximation. Therefore, it sometimes happens that the net BT result agrees rather well with ours. Nevertheless, the cancellation of the above errors is rather fortuitous, and is expected to vary drastical-

ly from one case to another. Therefore, one would have to be very careful in drawing definite conclusions, even when good fits to data are achieved with the BT-type calculations.

Concerning the BT fits to data, there is another important remark to make. Let us look at the σ_0 and σ_{1+2+3} curves (which represent, respectively, our and BT cross sections) in Fig. 3 for the $E_\alpha = 80$ MeV case. We note that this is a rather typical discrepancy we experience in comparing two kinds of cross sections. We further note that it also happens sometimes that σ_{1+2+3} fits data rather well, while σ_0 underestimates the data at lower E_p. This discrepancy between the data and σ_0 is not unexpected, however. The discrepancy is to be filled by considering higher order contributions, going beyond the two-step approximation which is common to our and BT theories. In fact, our very recent investigation[11] did show that σ_0 plus such higher order contributions did fit the data at lower E_p as well. Also there exists experimental evidence[16] which shows the presence of the higher order contributions. To be satisfied with the fit to data achieved by the use of σ_{1+2+3}, and to conclude that there is no need to consider any higher order effects, could thus be too premature. If the higher order corrections are added to σ_{1+2+3}, the resultant cross section would overshoot the data.

ACKNOWLEDGMENTS

We are indebted to Professor M. Ichimura for useful discussions. This work was supported in part by the U.S. Department of Energy.

APPENDIX: DISCUSSION OF THE AUSTERN-VINCENT FORMALISM

In the text, we have stated repeatedly that the AV formalism[7] contained an approximation, in that it neglected the nonorthogonality term correction. We shall explain in this appendix where in AV the above-mentioned approximation was in fact committed.

AV presented two methods; the wave function derivation and the operator derivation. We shall discuss here only the latter, because it is somewhat more self-contained than is the former, and because KI (Ref. 15) based their arguments upon it. In quoting formulas from AV below, we shall attach a prefix AV to the equation number in AV. We shall also quote them after transcribing the notation of AV into ours.

The operator derivation of AV begins with writing an equation called Eq. AV(20), which we shall reproduce here as

$$d^2\sigma/dE_b d\Omega_b = (2\pi/\hbar v_a) d(E_b) \times \sum_x |T_{bx,a}|^2 \delta(E_x - E_{x,A}) , \tag{A1}$$

where

$$T_{bx,a} = \langle \Phi^{(-)}_{x,A} \chi^{(-)}_b \phi_b \,|\, V_b \,|\, \Psi^{(+)} \rangle . \tag{A2}$$

The transition amplitude given by (A2) is exact,[17] in that (A1) together with (A2) gives the exact b singles cross section. As seen, (A2) is described in terms of $\Psi^{(+)}$, the exact wave function of the total system, and of $\Phi^{(-)}_{x,A}$, the exact wave function of the $x+A$ system. We may introduce an exact Hamiltonian $H_{x,A}$ for this $x+A$ system, so that

$$H_{x,A}\Phi^{(-)}_{x,A} = E_{x,A}\Phi^{(-)}_{x,A} . \tag{A3}$$

Equation (A3) gives a precise definition of the quantity $E_{x,A}$ that appears in (A1). It also defines $\sum_{(x+A)}$. It is a sum over the eigenstates $\Phi^{(-)}_{x,A}$.

The next step followed by AV was to go from AV (20) to AV (21), i.e., to go from (A1) to

$$d^2\sigma/dE_b d\Omega_b = -(2\pi/\hbar v_a) d(E_b) \mathrm{Im}(1/\pi) \sum_{(x+A)} \frac{|T_{bx,a}|^2}{E_x^+ - E_{x,A}}$$
$$= -(2\pi/\hbar v_a) d(E_b) \mathrm{Im} \sum_{(x+A)} \left[\langle \Psi^{(+)} \,|\, V_b \,|\, \chi^{(-)}_b \phi_b \Phi^{(-)}_{x,A} \rangle \frac{1}{E_x^+ - E_{x,A}} \langle \Phi^{(-)}_{x,A} \phi_b \chi^{(-)}_b \,|\, V_b \,|\, \Psi^{(+)} \rangle / \pi \right] . \tag{A4}$$

In (A4), we may replace the energy denominator $(E_x^+ - E_{x,A})$ by the operator $(E_x^+ - H_{x,A})$ by using (A3). Then we can carry out the sum over $(x+A)$ by using the completeness relation

$$\sum_{(x+A)} |\Phi^{(-)}_{x,A}\rangle\langle\Phi^{(-)}_{x,A}| = 1 ,$$

reducing Eq. (A4) to Eq. AV (22), i.e., to

$$d^2\sigma/dE_b d\Omega_b = -(2\pi/\hbar v_a) d(E_b) \mathrm{Im} \left[\langle \Psi^{(+)} \,|\, V_b \,|\, \chi^{(-)}_b \phi_b \rangle \frac{1}{E_x^+ - H_{x,A}} (\phi_b \chi^{(-)}_b \,|\, V_b \,|\, \Psi^{(+)} \rangle / \pi \right] . \tag{A5}$$

The final step taken by AV was to approximate $|\Psi^{(+)}\rangle$ by $|\chi^{(-)}_a \phi_a \phi_A\rangle$, and also $H_{x,A}$ by the optical model Hamiltonian H_x. (Because of these two approximations, the AV formalism employs the two-step approximation discussed in the text.) Note that the latter approximation is to replace the Green's function $(E_x^+ - H_{x,A})^{-1}$ in (A5) by $|\phi_x \phi_A\rangle(E_x^+ - H_x)\langle\phi_A \phi_x|$. We thus find that (A5) is replaced by

$$d^2\sigma/dE_b d\Omega_b = (2\pi/\hbar v_a) d(E_b) \times [\langle \rho^{(b)} | -\mathrm{Im}(E_x^+ - H_x)^{-1} | \rho^{(b)} \rangle / \pi] , \tag{A6}$$

where we have used $\rho^{(b)}$ that was defined in Eq. (10a) of the text.

The derivation of (A6) more or less completes the review of the operator derivation of AV. We find it convenient, however, to go one step further, by remarking that an identity exists, which reads[8,15]

$$-\mathrm{Im}[(E_x^+ - H_x)^{-1}] \equiv -\mathrm{Im} G_x^{(+)} = \pi | \chi_x^{(-)} \rangle \langle \chi_x^{(-)} | + G_x^{(+)+} W_x G_x^{(+)} . \tag{A7}$$

If we insert (A7) into (A6), it is easy to see that the first and second terms of the rhs of (A7) give rise, respectively, to the EB and BF cross sections. We thus have, in particular, the BF cross section of AV as

$$\begin{aligned} d^2\sigma/dE_b d\Omega_b &= (2\pi/\hbar v_a) d(E_b) \times (\langle \rho^{(b)} G_x^{(+)} | W_x | G_x^{(+)} \rho^{(b)} \rangle / \pi) \\ &= (2\pi/\hbar v_a) d(E_b) \times (\langle \Psi_{x,b}^{(+)} | W_x | \Psi_{x,b}^{(+)} \rangle / \pi) . \end{aligned} \tag{A8}$$

We now show that (A8) is not the unique result one gets from the starting equation (A1). In order to see this, we first note that the following identity holds:

$$\langle \Phi_{x,A}^{(-)} \phi_b \chi_b^{(-)} | V_b | \Psi^{(+)} \rangle = \langle \Phi_{x,A}^{(-)} \phi_b \chi_b^{(-)} | [V_b + \alpha(H_{x,A} - E_x)] | \Psi^{(+)} \rangle . \tag{A9}$$

Here, α is a constant. Eq. (A9) is true, because $\langle \Phi_{x,A}^{(-)} | (H_{x,A} - E_x) = 0$.

We may use in (A4) the rhs, instead of the lhs, of (A9), and then follow the procedure we did in going from (A4) to (A5). We then find that the factor $(\phi_b \chi_b^{(-)} | V_b | \Psi^{(+)} \rangle$ in (A5) is replaced by

$$(\phi_b \chi_b^{(-)} | V_b | \Psi^{(+)} \rangle + \alpha(H_{x,A} - E_x)(\phi_b \chi_b^{(-)} | \Psi^{(+)} \rangle . \tag{A10}$$

Note that the $(H_{x,A} - E_x)$ factor does not vanish (identically) any more.

Let us use (A10), in place of $(\phi_b \chi_b^{(-)} | V_b | \Psi^{(+)} \rangle$, in (A5), and again go through the procedure by which we obtained (A8) from (A5). We then have

$$d^2\sigma/dE_b d\Omega_b = (2\pi/\hbar v_a) d(E_b) \times [\langle (\Psi_{x,b}^{(+)} - \alpha n) | W_x | (\Psi_{x,b}^{(+)} - \alpha n) \rangle / \pi] , \tag{A11}$$

where n stands for the nonorthogonality term $n(\vec{r})$. We may now choose $\alpha = 1$. We then see that (A11) reduces to our formula, given in Eq. (2) of the text.

It is thus seen that the AV formalism contains a serious ambiguity. It can derive either the AV formula or our formula (or anything else) by starting from the same (exact) expression given by (A1). And the above argument shows that the ambiguity was brought into the theory in the step of going from (A4) to (A5). Note that this step was to make a transformation of an expression, which contains exclusively on-the-energy shell quantities, into another, which begins to use off-the-energy shell quantities for the first time. The trouble encountered by the AV formalism, as shown above, means that such a transformation should not be attempted. Note that to attempt this is essentially the same as to attempt to obtain a wave function in the nonasymptotic region, when its behavior is known only in the asymptotic region. One easily sees why one encounters an ambiguity.

In the text, we have stated repeatedly that one should use the prior form if one wants to avoid the appearance of the nonorthogonality term. And in justifying this statement, we have referred to the arguments given in Refs. 13 and 14. Since this is the crucial point of the present paper, we shall review here briefly the arguments in these two references. As seen, we work in coordinate space, and work out equations which are valid both in nonasymptotic and asymptotic regions. It is thus very unlikely that our formalism would suffer from such an ambiguity.

We want to make our presentation as simple as possible, and make the two-step approximation of the text from the beginning. We thus start by writing the total wave function, which is an eigenstate of $H_a + V_a$, as

$$\Psi_a^{(+)} = | \chi_a^{(+)} \phi_a \phi_A \rangle + \Psi_x^{(+)} | \phi_x \phi_B \rangle . \tag{A12}$$

Here, for simplicity, we assume that the particle b has been captured by A so as to form a nucleus B in a state denoted by ϕ_B.

The most important point of the arguments given in Refs. 13 and 14 (which the reader will follow rather easily), is that $\Psi_x^{(+)}$ satisfies an equation written as

$$(E_x - H_x)\Psi_x^{(+)} = (\phi_b \phi_x | V_a | \chi_a^{(+)} \phi_a \phi_A \rangle . \tag{A13}$$

Namely, $\Psi_x^{(+)}$ satisfies an inhomogeneous equation with a prior form source term, without being corrected by the nonorthogonality term.

We can now use the relation in Eq. (8a) in the text to show that (A13) is replaced by

$$(E_x - H_x)\Psi_x^{(+)} = (\phi_B \phi_x | V_b | \chi_a^{(+)} \phi_a \phi_A \rangle - (E_x - H_x)(\phi_B \phi_x | \chi_a^{(+)} \phi_a \phi_A \rangle , \tag{A14}$$

showing the clear onset of the nonorthogonality correction, with which the post form source term must be accompanied.

It is interesting to show that (A14) can be rewritten as

$$(E_x - H_x)[\Psi_x^{(+)} + (\phi_B \phi_x | \chi_a^{(+)} \phi_a \phi_A \rangle] = (\phi_B \phi_x | V_b | \chi_a^{(+)} \phi_a \phi_A \rangle , \tag{A15}$$

which shows that the solution of an inhomogeneous equation, having only the post form source term, is not $\Psi_x^{(+)}$, but

$$\Psi_x^{(+)'} = \Psi_x^{(+)} + (\phi_B \phi_x | \chi_a^{(+)} \phi_a \phi_A \rangle . \tag{A16}$$

Clearly, we can express this function $\Psi_x^{(+)'}$ as

$$\Psi_x^{(+)'}=(\phi_B\phi_x \mid \Psi_a^{(+)}\rangle \ , \tag{A17}$$

which shows that it is a projection of the total wave function of (A12), including the incident wave component, upon the state $|\phi_B\phi_x)$. Since the latter component should never contribute to the x-channel wave function, it is quite legitimate to say that $\Psi_x^{(+)'}$ contains an unphysical component.

All the above arguments remain valid even after replacing the wave function ϕ_B by $\chi_b^{(-)}\phi_b$. The above arguments can then be used to justify the arguments we presented in the text. Namely, the post form BF formula must contain nonorthogonality corrections, if one wants to avoid unphysical contributions.

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