Coulomb scattering of wave packets

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(Received 13 September 1983; revised manuscript received 13 August 1984)

Advantages are pointed out in studying S-matrix elements between wave packets when long range Coulomb forces are involved. A straightforward method is presented to calculate such matrix elements in few-body systems. It is based on a strong approximation of the Møller wave operators. For the two-body system the method is tested for (a) short range, (b) Coulomb, and (c) short range plus Coulomb potentials, and is found in all cases to converge to the reference solution. As an application, d + p breakup cross sections are calculated close to the threshold, including Coulomb forces properly.

I. INTRODUCTION

In nuclear and atomic physics one often encounters a few-body scattering process among which are charged particles, which involves the long range Coulomb force. For some processes, like charged nucleon scattering at high energies, the Coulomb force can be viewed as a weak perturbation, and adding a Rutherford-type correction to a neutral amplitude gives a good approximation. However, for such a process at low energies or in atomic collisions the Coulomb force is dominant. E.g., major differences in absolute magnitude between p + d and n + d breakup cross sections very close to the threshold have been pointed out in Ref. 1. In such cases a systematic approach for long range forces is a necessity.

There are a variety of rigorous and applicable methods to calculate a scattering amplitude for a charged two-particle system—the pure Coulomb amplitude, e.g., is known analytically. Going over to three- or more-particle processes with at least two charged components, no such well settled approach like the Faddeev equations for neutral particles exists. There has been much effort to establish a stationary multichannel scattering theory involving Coulomblike potentials.[2–11] This is not a trivial problem since already a two-particle stationary scattering theory based on the Lippmann-Schwinger equation exhibits certain difficulties.[12–19] E.g., the transition amplitude does not exist for forward scattering[20] and hence the conventional partial wave expansion is not possible.[21] Coulomb scattering wave functions, being explicitly known, are rapidly oscillating functions in momentum representation with nasty on shell singularities.[22] This behavior also shows up in Faddeev-type integral equations where the charged subsystem transition amplitudes carry those singularities. One way of treating this problem is the screening approach. Using this approach Alt *et al.* have calculated p + d elastic scattering in the few MeV region.[9] But most of these attempts[2–11] have not been completely satisfactory since either not sufficient attention has been paid to practical applicability or the range of application was restricted.

In this paper we want to describe a new method of calculating quantum mechanical few-body S-matrix elements for short range and Coulomblike interactions. Its main motivation is to avoid the singularities of Faddeev-type integral equations when Coulomb forces are involved. It is based on the observation that the singularities of Coulomblike potentials, the corresponding scattering wave functions, and S-matrix elements in momentum representation are softened or removed when being integrated over with square integrable functions, i.e., wave packets. Also, an S-matrix element between wave packets has the property to be decomposable in partial waves. On the other hand, partially averaged quantities are often measured in few-body reactions with charged particles.[23] The new method will enable us to calculate S-matrix elements between wave packets. The question arises: How are the physically interesting quantities like the S- and T-matrix elements on the energy shell related and how are they obtained from the corresponding matrix elements between wave packets? The initial and final wave packets have to be chosen in momentum space with peaks around the on shell values with a small width. In this work the wave packets are built from some expansion functions and all calculations are done in the expansion function basis. These functions are chosen in momentum space ($\vec{q}$) as step functions covering a finite interval to describe the $|\vec{q}|$ dependence, and spherical harmonics are used for the $\hat{q}$ dependence. The step functions $e_\nu(q)$ are constant in a small subinterval $[q_{\nu-1},q_\nu]$ and 0 elsewhere. Thus we approximate a sharp momentum continuum state $|qlm\rangle$ by a Hilbert state $|e_\nu,lm\rangle$. Using these expansion functions, it is easy to construct wave packets concentrated at a certain momentum with a small width. That leads to an approximation of the physical on shell S-matrix elements by matrix elements in terms of the step functions. Let us briefly describe the method. The S matrix is composed from wave operators, which contain all the scattering information. The wave operators are approximated by exponentials of an approximated finite rank Hamilton operator using a large but finite time. That applies to the standard Møller wave operator as well as to the Dollard modified wave operator if Coulomb forces are involved. The finite rank Hamiltonian is obtained from the original Hamiltonian by projecting onto the space of expansion functions. That enables one to cal-

culate the exponential in the eigenrepresentation of the finite rank Hamilton matrix. One advantage of choosing step functions as expansion functions is that the asymptotic channel Hamiltonian is diagonal in that basis. Moreover the matrix elements of the two-body Coulomb potential and the kinetic energy can be calculated analytically. It can be proven rigorously that the method yields S-matrix elements which approximate the original ones. Proofs and mathematical details are given in Refs. 24 and 25. In this paper we want to describe the method in more detail. In Sec. II the method is explained for the two-body system. In Sec. III numerical tests are given for the two-body system for short range potentials, the Coulomb potential, and a sum of both types. Section IV gives a generalization to the N-body case. In Sec. V the method is applied to a three-body reaction with two charged particles, namely to the d + p breakup process, including Coulomb forces properly. That extends the results of Ref. 26 and represents the first reasonably rigorous calculation of the d + p breakup cross section close to the threshold.

II. TWO-BODY SYSTEM

The units $\hbar=c=1$ are used in the following. Let V denote a two-body pair potential. The problem is treated in the center of mass system. Let H^0 denote the kinetic energy of the free relative motion, m the reduced mass, and $H=H^0+V$ the total Hamiltonian. For pedagogical reasons let us consider firstly V to be of short range. The Møller wave operator is given by the strong limit

$$\Omega^{(\pm)}=\text{s-}\lim_{t\to\mp\infty} e^{iHt}e^{-iH^0t} . \tag{2.1}$$

The operators H^0,H are self-adjoint unbounded operators on the Hilbert space $\mathscr{H}$ (actually they can be defined at most on a dense subspace $\mathscr{D}$ of $\mathscr{H}$, as a consequence of the Hellinger-Toeplitz theorem). In a first step H^0,H are approximated by self-adjoint bounded operators (defined on all of $\mathscr{H}$) $H^0(u),H(u)$ such that on $\mathscr{D}$

$$H^0=\text{s-}\lim_{u\to 0}H^0(u),\quad H=\text{s-}\lim_{u\to 0}H(u) . \tag{2.2}$$

An example for $H^0(u)$ would be

$$H^0(u)=\frac{E}{u}\arctan\left[\frac{uH^0}{E}\right] , \tag{2.3}$$

where E is a positive scaling factor of dimension energy. If V is bounded one can put

$$H(u)=H^0(u)+V . \tag{2.4}$$

Let

$$\Omega(u,t)=e^{iH(u)t}e^{-iH^0(u)t} . \tag{2.5}$$

It has been shown in Ref. 24 that expression (2.5) yields a strong approximation of the Møller wave operator, i.e., for every $\Phi\in\mathscr{H}$ one can find $u,t_\mp\epsilon||R$ such that

$$||\Omega^{(\pm)}\Phi-\Omega(u,t_\mp)\Phi|| \tag{2.6}$$

becomes arbitrarily small. The exponentials in expression (2.5) are built from bounded operators which allows us to calculate e^B, e.g., by an expansion in a power series

$$e^B=\sum_{\nu=0}^{\infty}\frac{B^\nu}{\nu!} \tag{2.7}$$

or by the expression

$$e^B=\lim_{\nu\to\infty}\left[1+\frac{B}{\nu}\right]^\nu . \tag{2.8}$$

Although expressions (2.7) and (2.8) converge, they may turn out to be numerically inconvenient because of a large B, which arises from large times t or small u. Thus an additional approximation is introduced which allows us to calculate exponential operators of large arguments. In a second step $H^0(u),H(u)$ are approximated by self-adjoint finite rank operators (which are of course bounded and defined on all of $\mathscr{H}$) $H^0(u,n),H(u,n)$ such that

$$\begin{aligned} H^0(u)&=\text{s-}\lim_{n\to\infty}H^0(u,n) ,\\ H(u)&=\text{s-}\lim_{n\to\infty}H(u,n) . \end{aligned} \tag{2.9}$$

An example would be

$$\begin{aligned} H^0(u,n)&=P_nH^0(u)P_n ,\\ H(u,n)&=P_nH(u)P_n , \end{aligned} \tag{2.10}$$

where P_n is the orthogonal projector on the subspace spanned by $\phi_1,\ldots,\phi_n$, where $\{\phi_\nu\}_{\nu\in||N}$ is a complete set in $\mathscr{H}$. Let

$$\Omega(u,n,t)=e^{+iH(u,n)t}e^{-iH^0(u,n)t} . \tag{2.11}$$

It has been shown in Ref. 24 that for every $\Phi\in\mathscr{H}$ one can find $u,n,t_\mp$ such that

$$||\Omega^{(\pm)}\Phi-\Omega(u,n,t_\mp)\Phi|| \tag{2.12}$$

becomes arbitrarily small. The choice of u,n depends on the choice of $t_\mp$ (see also Sec. III). The calculation of (2.11) means some finite dimensional matrix algebra. Various methods to calculate the exponential of a matrix have been reviewed by Moler and Loan.[27] In this case the most promising way is to diagonalize $H^0(u,n),H(u,n)$

$$\begin{aligned} H^0(u,n)&=\sum_{\nu=1}^{n}|e_\nu^0\rangle\epsilon_\nu^0\langle e_\nu^0| ,\\ H(u,n)&=\sum_{\nu=1}^{n}|e_\nu\rangle\epsilon_\nu\langle e_\nu| . \end{aligned} \tag{2.13}$$

Thus $\Omega(u,n,t)$ takes the form

$$\Omega(u,n,t)=\sum_{\nu,\mu=1}^{n}|e_\nu\rangle e^{+i\epsilon_\nu t}\langle e_\nu|e_\mu^0\rangle e^{-i\epsilon_\mu^0 t}\langle e_\mu^0| . \tag{2.14}$$

Now let us consider the modifications when a long range Coulomb potential is involved. In particular let

$$V=V^s+V^c , \tag{2.15}$$

where V^s is of short range and V^c is the long range Coulomb potential, given in coordinate space by

$$V^c(r)=\frac{e_1e_2}{r} . \tag{2.16}$$

An extension of the Møller wave operator taking into account the long range Coulomb potential has been given by Dollard[28]

$$\Omega^{c(\pm)}=\underset{t\to\mp\infty}{\text{s-lim}}\, e^{iHt}e^{-iH^{0c}(t)}\,, \tag{2.17}$$

where

$$H^{0c}(t)=H^0t+\text{sign}(t)\left[\frac{m}{2H^0}\right]^{1/2}e_1e_2\ln(4H^0\,|\,t\,|\,)\,. \tag{2.18}$$

Analogously to the short range case already outlined one can introduce self-adjoint bounded operators $H^{0c}(u,t),H(u)$ such that on a suitable dense subspace $\mathscr{D}$

$$\begin{aligned}&H^{0c}(t)=\underset{u\to0}{\text{s-lim}}H^{0c}(u,t)\,,\\ &H=\underset{u\to0}{\text{s-lim}}H(u)\end{aligned} \tag{2.19}$$

and self-adjoint finite rank operators $H^{0c}(u,n,t),H(u,n)$ such that

$$\begin{aligned}&H^{0c}(u,t)=\underset{n\to\infty}{\text{s-lim}}H^{0c}(u,n,t)\,,\\ &H(u)=\underset{n\to\infty}{\text{s-lim}}H(u,n)\,.\end{aligned} \tag{2.20}$$

Examples for $H^{0c}(u,t)$, $H^{0c}(u,n,t)$, $H(u)$, and $H(u,n)$ are given in Ref. 24. Defining

$$\Omega^c(u,n,t)=e^{iH(u,n)t}e^{-iH^{0c}(u,n,t)}\,, \tag{2.21}$$

then for every $\Phi\in\mathscr{H}$ one can find $u,n,t_\mp$ such that

$$||\Omega^{c(\pm)}\Phi-\Omega^c(u,n,t_\mp)\Phi|| \tag{2.22}$$

becomes arbitrarily small.[24] Again the choice of u,n depends on the choice of $t_\mp$. We want to remark that $\Omega^c(u,n,t)$ is unitary, while the original wave operator is isometric. Finally the approximations of the wave operators yield for the S matrix

$$S^c=\Omega^{c(-)^\dagger}\Omega^{c(+)}\,, \tag{2.23}$$

$$S^c(u,n,t_+,t_-)=\Omega^{c^\dagger}(u,n,t_+)\Omega^c(u,n,t_-)\,, \tag{2.24}$$

and for every Φ,Ψ,

$$|\langle\Phi\,|\,S^c\,|\,\Psi\rangle-\langle\Phi\,|\,S^c(u,n,t_+,t_-)\,|\,\Psi\rangle\,| \tag{2.25}$$

can be made arbitrarily small.

We want to conclude this section with a remark on the partial wave decomposition. The (full) S-matrix is unitary, mapping the Hilbert space onto itself. Considering, e.g., a rotationally symmetric potential (short range or/and Coulomb), then

$$\langle\Phi\,|\,S\,|\,\psi\rangle=\sum_{l=0}^{\infty}\langle\Phi\,|\,l0\rangle\langle l0\,|\,S\,|\,l0\rangle\langle l0\,|\,\psi\rangle\,,$$

which decouples different angular momenta. Thus it is sufficient to choose wave packets Φ,ψ as angular momentum eigenfunctions.

III. NUMERICAL TESTS FOR THE TWO-BODY SYSTEM

In this section numerical examples shall demonstrate the feasibility of the approach. In all numerical calculations we have skipped the bounded approximation (corresponding to the parameter u), but we have directly applied the finite rank approximation (corresponding to the parameter n). It has been shown in Ref. 25 for a particular set of expansion functions (later denoted as step functions) that both the wave operator and the S-matrix converge to the same limits (and in an analogous way) as in the preceding. Thus one has as approximation parameters only the time $T=t_+=-t_-$ and the number n of expansion functions. Some care is necessary concerning convergence. Keeping T fixed and letting n increase the series tends to a limit. However, keeping n fixed and letting T increase the series starts to oscillate after a while. The correct way is to pick a T_1, let n increase until the series tends to a limit $S(T_1)$, then pick a larger T_2, find $S(T_2)$, and so on until $S(T_1),S(T_2),\ldots$, tend to a limit S.

(a) Firstly we consider a short range potential

$$\begin{aligned}&V^s=|\chi\rangle\lambda\langle\chi|\,,\\ &\langle\vec{q}\,|\,\chi\rangle=\begin{cases}\kappa(q_{\text{cut}}-q) & 0<q<q_{\text{cut}}\\ 0 & \text{elsewhere}\,,\end{cases}\end{aligned} \tag{3.1}$$

which is of separable type, rank 1 with an s-wave form factor. The parameters were chosen to be

$$\lambda=-10^{-2}\ \text{fm}^{-1},\quad q_{\text{cut}}=1\ \text{fm}^{-1},\quad \kappa:||\chi||=1\,.$$

We calculate the S matrix element $\langle\chi\,|\,S\,|\,\chi\rangle$, where for the sake of simplicity the wave packet has been chosen to be identical to the potential form factor. The reference solution is given by

$$\begin{aligned}&\langle\chi\,|\,S\,|\,\chi\rangle=1-2\pi i\int d\vec{q}\int d\vec{q}\,'\,\delta(E_q-E_{q'})\chi(\vec{q})\langle\vec{q}\,|\,T(E_{q'}+i0)\,|\,\vec{q}\,'\rangle\chi(\vec{q}\,')\,,\\ &T(z)=\frac{|\chi\rangle\lambda\langle\chi|}{1-\lambda\Delta(z)},\quad \Delta(z)=\int\frac{d\vec{q}\,|\,\chi(\vec{q})\,|^2}{z-E_q}\,,\end{aligned} \tag{3.2}$$

and yields the number $0.846\,718+0.482\,735\,i$.

The finite rank approximations $H^0(n),H(n)$ have been obtained by defining the orthogonal projections P_n with the following choice of expansion functions:

(i) $e_\nu(q)=\begin{cases}\kappa_\nu & q_{\nu-1}<q<q_\nu\\ 0 & \text{elsewhere}\end{cases}$ step function,

where the q_ν subdivide the integral $[0,q_{\text{cut}}]$ in subintervals of equal length, and κ_ν is chosen to normalize e_ν to 1.

(ii) e_ν is the νth orthogonal Jacobi polynomial on the interval $[0,q_{\text{cut}}]$ corresponding to the weight function $4\pi q^2$. Note that it is not necessary in this case to introduce firstly a bounded approximation because V is bounded and the cutoff in the potential means that H^0 is effec-

TABLE I. The approximate matrix element $\langle \chi | S | \chi \rangle$ dependent on the parameters time T and the number N of expansion functions (step functions). The potential is short range, separable, rank 1 with an s-wave form factor χ given by Eq. (3.1). The wave packet is chosen identical to the form factor.

T (fm/c) \ N	10	20	30	40
100	0.854 088 0.459 915 i	0.857 905 0.464 903 i	0.858 621 0.465 837 i	0.858 872 0.466 164 i
200	0.840 440 0.488 535 i	0.849 241 0.475 326 i	0.849 920 0.476 282 i	0.850 159 0.476 618 i
300		0.847 204 0.478 208 i	0.847 898 0.479 020 i	0.848 132 0.479 359 i
400			0.847 139 0.480 168 i	0.847 372 0.480 511 i
500			0.846 726 0.481 118 i	0.847 013 0.481 109 i
600				0.846 821 0.481 458 i
700				0.846 604 0.482 444 i
	Reference	0.846 718	$+0.482\,735\,i$	

tively bounded by $||H^0|| \leq q_{\text{cut}}^2/2m$. The results are given in Table I for the choice (i) of expansion functions and in Table II for the choice (ii).

Next we consider a more realistic separable potential given by Eq. (3.1) with $\lambda = -1$, but now with a Yamaguchi form factor

$$\langle \vec{q} | \chi \rangle = \frac{A}{q^2+\beta^2}\,, \qquad A = 1.9 \text{ fm}^{-1},\ \beta = 1.786 \text{ fm}^{-1}\,. \tag{3.3}$$

We have calculated the S-matrix element $\langle \psi | S | \psi \rangle$ for a

TABLE II. Same as Table I but with Jacobi polynomials as expansion functions.

T (fm/c) \ N	10	20	30	40
100	0.859 292 0.466 583 i	0.859 195 0.466 585 i	0.859 195 0.466 585 i	0.859 195 0.466 585 i
200		0.850 453 0.477 060 i	0.850 466 0.477 049 i	0.850 455 0.477 048 i
300		0.847 985 0.481 648 i	0.848 436 0.479 789 i	0.848 434 0.479 793 i
400			0.847 694 0.480 926 i	0.847 672 0.480 949 i
500				0.847 313 0.481 547 i
600				0.846 660 0.483 460 i
	Reference	0.846 718	$+0.482\,735\,i$	

TABLE III. Dependence of the matrix element $\langle\psi|S|\psi\rangle$ on the wave packet ψ. The potential is separable with a Yamaguchi form factor χ given by Eq. (3.3). Wave packet I is identical to the Yamaguchi form factor [Eq. (3.3)], but normalized to 1, wave packet II is Gaussian, given by the antibound state channel in Eq. (3.5), but normalized to 1. Wave packets IIIa–f are given by Eq. (3.4), but differ in the choice of the parameters $q_{\rm low}, q_{\rm up}$. They all have a peak at $q_M=(q_{\rm up}+q_{\rm low})/2=0.173\,555$ fm^{-1}, but differ in their half-width $w=(q_{\rm up}-q_{\rm low})/2$. The on shell matrix element (limes $w\to 0$) is given also.

Wave packet	Width (fm^{-1})	S
I		$0.127\,759-0.367\,535\,i$
II		$-0.288\,895-0.723\,391\,i$
IIIa	0.173 555	$0.891\,391-0.441\,356\,i$
IIIb	$0.867\,777-1$	$0.913\,091-0.404\,012\,i$
IIIc	$0.433\,889-1$	$0.918\,759-0.393\,824\,i$
IIId	$0.216\,944-1$	$0.920\,190-0.391\,219\,i$
IIIe	$0.108\,472-1$	$0.920\,549-0.390\,564\,i$
IIIf	$0.542\,361-2$	$0.920\,639-0.390\,400\,i$
On shell limes		$0.920\,668-0.390\,345\,i$

set of different wave packets ψ. Firstly, we have taken the wave packet identical to the Yamaguchi form factor, given by Eq. (3.3), but normalized to 1. Secondly, we have taken as a wave packet a Gaussian form factor, given by the antibound state channel in Eq. (3.5), but normalized to 1. Finally we have taken wave packets, given by

$$\langle\vec{q}\,|\,\psi\rangle=\begin{cases}\kappa\left[1-\cos\left[2\pi\dfrac{q_{\rm up}-q}{q_{\rm up}-q_{\rm low}}\right]\right], & q_{\rm low}<q<q_{\rm up}\\ 0, \text{ elsewhere}\end{cases}, \qquad \kappa:||\psi||=1\ . \tag{3.4}$$

This wave packet is physically more meaningful, as it has a peak at $q_M=(q_{\rm up}+q_{\rm low})/2$ and half-width $w=(q_{\rm up}-q_{\rm low})/2$. Keeping q_M fixed and letting w tend to 0, the wave packet approximates a delta function, i.e., a sharp energy state. We have chosen several packets of the latter type, all with a peak at $q_M=0.173\,5555$ fm^{-1}, but different width w. This momentum q_M corresponds in NN scattering to $E_{\rm lab}=2.5$ MeV. The reference values are given in Table III. For the approximate solutions we have used as expansion functions the step functions with a quadratic distribution of nodes in $[0,q_{\rm cut}]$. For the Gaussian wave packet $q_{\rm cut}=5$ fm^{-1} has been chosen. The results are given in Table IV. For the wave packets given by Eq. (3.4), we have used $q_{\rm cut}=12$ fm^{-1}. The results are displayed in Tables V–VIII. Moreover we consider a separable NN potential which distinguishes between spin singlet and triplet states. It has been used in

TABLE IV. The approximate matrix element $\langle\psi|S|\psi\rangle$ dependent on the parameters time T and the number N of expansion functions (step function). The potential is separable with a Yamaguchi form factor χ given by Eq. (3.3). The wave packet ψ is Gaussian, given by the antibound state channel in Eq. (3.5), but normalized to 1 (wave packet II of Table III).

T (fm/c) \ N	30	40	60	100
10	$-0.296\,839$ $-0.655\,948\,i$	$-0.295\,398$ $-0.660\,259\,i$	$-0.293\,750$ $-0.663\,268\,i$	$-0.292\,710$ $-0.664\,584\,i$
20	$-0.303\,713$ $-0.689\,112\,i$	$-0.304\,173$ $-0.688\,708\,i$	$-0.303\,116$ $-0.689\,333\,i$	$-0.302\,764$ $-0.689\,042\,i$
30	$-0.303\,097$ $-0.704\,238\,i$	$-0.307\,222$ $-0.696\,145\,i$	$-0.307\,761$ $-0.695\,519\,i$	$-0.307\,635$ $-0.695\,446\,i$
40		$-0.306\,382$ $-0.703\,680\,i$	$-0.309\,412$ $-0.698\,678\,i$	$-0.309\,674$ $-0.698\,344\,i$
50		$-0.294\,164$ $-0.716\,005\,i$	$-0.310\,640$ $-0.700\,916\,i$	$-0.310\,640$ $-0.700\,009\,i$
60			$-0.309\,684$ $-0.705\,043\,i$	$-0.311\,109$ $-0.701\,038\,i$
70			$-0.305\,112$ $-0.714\,014\,i$	$-0.311\,397$ $-0.701\,753\,i$
80				$-0.311\,784$ $-0.702\,625\,i$
	Reference	$-0.288\,895$	$-0.723\,391\,i$	

TABLE V. Same as Table IV, but with a wave packet given by Eq. (3.4). The wave packet has a peak at $q_M = 0.173\,555\ \text{fm}^{-1}$ and a width $w = 0.173\,555\ \text{fm}^{-1}$ (wave packet IIIa in Table III).

T (fm/c) \ N	30	40	60	100
100	0.937 861 −0.289 166 i	0.940 457 −0.252 571 i	0.940 430 −0.251 025 i	0.939 535 −0.241 610 i
200	0.882 801 −0.394 748 i	0.913 942 −0.372 571 i	0.906 431 −0.382 453 i	0.905 022 −0.377 890 i
300	0.880 197 −0.398 859 i	0.897 726 −0.424 304 i	0.896 387 −0.423 627 i	0.894 048 −0.427 064 i
400		0.880 323 −0.450 597 i	0.893 384 −0.434 135 i	0.892 204 −0.436 585 i
500			0.891 375 −0.437 707 i	0.891 949 −0.438 616 i
600			0.890 084 −0.438 276 i	0.891 721 −0.439 742 i
700				0.891 671 −0.440 137 i
800				0.891 242 −0.440 702 i
	Reference	0.891 391	−0.441 356 i	

TABLE VI. Same as Table V but with the wave packet having a width $w = 0.086\,7777\ \text{fm}^{-1}$ (wave packet IIIb in Table III).

T (fm/c) \ N	30	40	60	100
100	0.972 445 −0.133 932 i	0.968 701 −0.151 419 i	0.972 982 −0.127 965 i	0.974 700 −0.119 073 i
200	0.959 503 −0.207 772 i	0.947 304 −0.264 654 i	0.950 736 −0.225 956 i	0.953 839 −0.215 543 i
300	0.955 966 −0.273 957 i	0.921 793 −0.327 075 i	0.935 700 −0.301 114 i	0.936 069 −0.292 231 i
500		0.920 245 −0.347 754 i	0.923 559 −0.369 548 i	0.918 725 −0.376 039 i
700			0.914 665 −0.400 097 i	0.913 728 −0.399 570 i
900			0.902 759 −0.419 411 i	0.913 423 −0.402 368 i
1100				0.913 187 −0.403 069 i
1300				0.912 022 −0.404 454 i
	Reference	0.913 091	−0.404 012 i	

TABLE VII. Same as Table V, but with the wave packet having a width $w=0.043\,388\,89$ fm^{-1} (wave packet IIIc in Table III).

T (fm/c) \ N	30	40	60	100
100	0.980 766 −0.093 372 i	0.982 309 −0.091 696 i	0.982 921 −0.083 293 i	0.987 083 −0.063 962 i
300	0.954 164 −0.267 251 i	0.957 023 −0.210 980 i	0.952 133 −0.224 337 i	0.963 635 −0.173 884 i
500	0.890 542 −0.427 984 i	0.955 700 −0.271 512 i	0.933 410 −0.335 452 i	0.945 732 −0.264 419 i
700		0.933 998 −0.340 998 i	0.903 566 −0.418 380 i	0.931 653 −0.327 997 i
900			0.872 572 −0.459 076 i	0.924 109 −0.371 350 i
1100				0.916 804 −0.392 499 i
1300				0.913 551 −0.399 213 i
1500				0.911 719 −0.400 994 i
	Reference	0.918 759	−0.393 824 i	

TABLE VIII. Same as Table V, but with the wave packet having a width $w=0.021\,6944$ fm^{-1} (wave packet IIId in Table III).

T (fm/c) \ N	30	40	60	100
200	0.972 385 −0.169 297 i	0.979 285 −0.113 404 i	0.971 332 −0.131 415 i	0.978 470 −0.107 698 i
400	0.917 913 −0.349 464 i	0.967 182 −0.209 863 i	0.947 335 −0.244 575 i	0.958 640 −0.203 615 i
600		0.938 351 −0.312 481 i	0.920 506 −0.361 777 i	0.941 803 −0.283 261 i
800		0.904 690 −0.401 631 i	0.879 353 −0.451 311 i	0.931 822 −0.338 656 i
1000			0.839 725 −0.518 450 i	0.920 504 −0.378 492 i
1200				0.914 649 −0.393 665 i
1400				0.912 950 −0.396 869 i
1600				0.910 434 −0.397 927 i
	Reference	0.920 190	−0.391 219 i	

the three-body calculation described in Sec. V.

$$V=\sum_{\omega=\phi,d}|\chi^{\omega}\rangle\lambda^{\omega}\langle\chi^{\omega}|p^{\omega}\,,$$
$$\langle\vec{q}\,|\chi^{\omega}\rangle=e^{-\mu^{\omega}q^2},\quad \omega=\phi,d\,,$$
$$\lambda^{\phi}=-0.02047\ \text{fm}^{-1},\quad \mu^{\phi}=0.7172\ \text{fm}^2\,, \tag{3.5}$$
$$\lambda^{d}=-0.02459\ \text{fm}^{-1},\quad \mu^{d}=0.4164\ \text{fm}^2\,.$$

p^{ω} projects on the spin states $\omega=\phi$ (antibound state, singlet) and $\omega=d$ (bound state, triplet). The parameters $\lambda^{\omega},\mu^{\omega}$ are adjusted to the low energy data deuteron binding energy, singlet and triplet scattering length, and singlet effective range. Again we calculate the matrix element $\langle\psi|S|\psi\rangle$ with ψ given by Eq. (3.4) in both the singlet and triplet channel. One obtains as reference

$$\langle\psi|S^{\phi}|\psi\rangle=-0.533572+0.844763\,i\,,$$
$$\langle\psi|S^{d}|\psi\rangle=-0.244219-0.915836\,i\,.$$

The step functions have been used covering the interval $[0,q_{\text{cut}}]$, $q_{\text{cut}}=5\ \text{fm}^{-1}$. The results are shown in Tables IX and X. Another test is the relation

$$\langle\psi|H^0|\psi\rangle={}^{(\pm)}\langle\psi|H|\psi\rangle^{(\pm)}\,, \tag{3.6}$$

where $|\psi\rangle^{(+)}$ denotes the scattered wave packet corresponding to the incoming packet $|\psi\rangle$. Equation (3.6) follows from the intertwining relation. One has for both spin channels

$$\langle\psi|H^0|\psi\rangle=0.878020\ 10^{-2}\ \text{fm}^{-2}\,.$$

Using $T=800$ fm/c and $N=40$ expansion functions, one obtains

$${}^{(+)}\langle\psi|H|\psi\rangle^{(+)}=0.886274\ 10^{-2}\ \text{fm}^{-2}\ \text{(singlet)}\,,$$
$$=0.873612\ 10^{-2}\ \text{fm}^{-2}\ \text{(triplet)}\,.$$

(b) Now we consider a long range Coulomb potential,

TABLE IX. The singlet channel matrix element $\langle\psi|S^{\phi}|\psi\rangle$. The potential is separable with Gaussian form factors χ^{ω}, given by Eq. (3.5). The wave packet is given by Eq. (3.4).

T (fm/c) \ N	40	60	80
200	−0.180627 0.864393 i	−0.187995 0.855594 i	−0.193773 0.853536 i
400	−0.456263 0.874496 i	−0.483258 0.860613 i	−0.492106 0.855857 i
600	−0.484207 0.862403 i	−0.506203 0.856867 i	−0.514489 0.852251 i
800	−0.496876 0.841284 i	−0.510965 0.856771 i	−0.519406 0.851681 i
900			−0.520637 0.851398 i
Reference	−0.533572	+0.844763 i	

TABLE X. Same as Table IX but in the triplet channel.

T (fm/c) \ N	40	60
200	−0.133454 −0.622121 i	−0.130547 −0.610353 i
400	−0.253278 −0.881276 i	−0.248867 −0.886531 i
600	−0.253806 −0.900901 i	−0.248684 −0.908682 i
800		−0.250676 −0.912073 i
900		−0.250576 −0.911252 i
Reference	−0.244219	−0.915836 i

$$V(r)=\frac{e^2}{r},\quad e^2=1/137\cdots \tag{3.7}$$

and a wave packet

$$\langle\vec{q}\,|\psi\rangle=\begin{cases}\kappa & q_{\text{low}}<q<q_{\text{up}}\\ 0 & \text{elsewhere}\end{cases} \tag{3.8}$$

which is also s wave. The parameters were chosen to be $q_{\text{low}}=0.5\ \text{fm}^{-1}$, $q_{\text{up}}=1\ \text{fm}^{-1}$, $\kappa:||\psi||=1$. Again we calculate the S-matrix element $\langle\psi|S|\psi\rangle$. The reference solution is given by

$$\langle\psi|S|\psi\rangle=\int d\vec{q}\,|\psi(\vec{q})|^2\frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)},\quad \gamma=\frac{e^2m}{q} \tag{3.9}$$

and yields the number $0.999614-0.027271\,i$. The finite rank approximations $H^{0c}(n,t),H(n)$ have been obtained by orthogonal projections P_n with the step functions [choice (i)] as expansion functions, but which now cover an interval $[0,q_{\text{cut}}]$, $q_{\text{cut}}=40\ \text{fm}^{-1}$ which is larger than the interval $[q_{\text{low}},q_{\text{up}}]$ of the wave packet. The results are shown in Table XI.

(c) Finally we consider a potential which is a sum of the above types

$$V=V^{s}+V^{C}\,, \tag{3.10}$$

where V^{s} is of the type given by (3.1) but now with a different form factor

$$\langle\vec{q}\,|\chi\rangle=\begin{cases}\kappa\left[1-\cos\left[2\pi\dfrac{q_{\text{up}}-q}{q_{\text{up}}-q_{\text{low}}}\right]\right] & q_{\text{low}}<q<q_{\text{up}}\\ 0 & \text{elsewhere}\end{cases} \tag{3.11}$$

with the parameters $\lambda=-10^{-2}\ \text{fm}^{-1}$, $q_{\text{low}}=0.5\ \text{fm}^{-1}$, $q_{\text{up}}=1\ \text{fm}^{-1}$, and $\kappa:||\chi||=1$. We have calculated the matrix element $\langle\chi|S-S^{c}|\chi\rangle$ choosing a wave packet identical to the form factor, where S denotes the full S matrix and S^{c} the pure Coulomb S matrix, the latter being analytically known in momentum-angular momentum representation.[29] The reference solution has been obtained using the screening and renormalization technique[10]

TABLE XI. The approximate matrix element $\langle\psi|S|\psi\rangle$ for the Coulomb potential given by Eq. (3.7). The wave packet is given by Eq. (3.8).

T (fm/c) \ N	20	40	60	80	100
100	0.999 556 −0.018 302 i	0.999 430 −0.019 892 i	0.999 422 −0.019 889 i	0.999 419 −0.019 943 i	0.999 414 −0.020 153 i
200	0.999 686 −0.020 284 i	0.999 563 −0.022 695 i	0.999 527 −0.023 359 i	0.999 516 −0.023 621 i	0.999 506 −0.023 926 i
300		0.999 620 −0.022 361 i	0.999 551 −0.023 992 i	0.999 527 −0.024 592 i	0.999 512 −0.025 050 i
400			0.999 578 −0.023 823 i	0.999 540 −0.024 852 i	0.999 518 −0.025 512 i
500				0.999 573 −0.023 682 i	0.999 543 −0.025 584 i
600					0.999 571 −0.025 401 i
	Reference	0.999 614	−0.027 271 i		

$$\langle\chi|S-S^c|\chi\rangle=\lim_{R\to\infty}-2\pi i\int d\vec{q}\int d\vec{q}\,'\,\delta(E_q-E_{q'})\frac{1}{Z_R(E_q)}\chi(\vec{q})\langle\vec{q}\,|\,T_R(E_q+i0)-T_R^C(E_q+i0)\,|\,\vec{q}\,'\rangle\chi(\vec{q}\,') , \tag{3.12}$$

with

$$\begin{aligned} &Z_R(E_q)=e^{2i\phi_R(E_q)} ,\\ &\phi_R(E_q)=-\frac{me^2}{q}[\ln(2qR)-C] ,\\ &C=0.577216\cdots \text{ (Euler number)} , \end{aligned} \tag{3.13}$$

where Z_R represents a renormalizing phase factor and T_R^C, T_R are defined as the solutions of the Lippmann-Schwinger equations, respectively,

$$\begin{aligned} &T_R^C(z)=V_R^C+V_R^C G_0(z)T_R^C(z) ,\\ &T_R(z)=V_R+V_R G_0(z)T_R(z) , \end{aligned} \tag{3.14}$$

with the screened short range potentials, respectively,

$$\begin{aligned} &V_R^C(r)=V^C(r)e^{-r/R} ,\\ &V_R=V^S+V_R^C . \end{aligned} \tag{3.15}$$

The Lippmann-Schwinger equation has been solved using a Padé technique. The solution was found to be stable with diagonal Padé approximants of order 5 and a screening radius $R=40$ fm. It yields the number $-0.320056+0.699390\,i$. The finite rank approximations of $H^{0c}(n,t), H(n)$ have been obtained by the same type of orthogonal projections as in the case with a pure Coulomb potential. The results are shown in Table XII.

In summary of this section we have tested the proposed method by calculating S-matrix elements $\langle\psi|S|\psi\rangle$ corresponding to different potentials between s-wave packets ψ of different shape. The approximate S-matrix elements tend to the reference solution in all cases. The convergence is fast in the short range case, rather slow in the pure Coulomb case, but for the Coulomb subtracted S-matrix element corresponding to a short range plus Coulomb potential it is comparable to the short range case. The physically most relevant wave packets are those concentrating at some momentum [given by Eq. (3.4)] which approximate a sharp energy state. The corresponding S-matrix elements were found to be relatively stable under variation of the width of those wave packets ($w\to 0$), which seems to indicate a smooth phase shift function in the corresponding energy region. Wave packets concentrated at some momentum with a small width have been chosen also in the d + p calculation (Sec. V). The other wave packets used have no direct physical meaning, but its corresponding S-matrix elements can be viewed as bilinear functionals of the S matrix, which can be used to recover the energy dependence of the S matrix. Denoting the S matrix in momentum-angular momentum representation by

$$\langle q'l0\,|\,S\,|\,ql0\rangle=S_l(q)\delta\left[\frac{q'^2}{2m}-\frac{q^2}{2m}\right] , \tag{3.16}$$

one can choose as wave packets a set $\{P_n(q)\}$ of orthogonal functions in Hilbert space and calculate for each partial wave l

$$\langle P_i l0\,|\,S\,|\,P_j l0\rangle=\int dq\frac{q}{m}P_i^*(q)S_l(q)P_j(q) , \qquad i,j=1,2,\ldots , \tag{3.17}$$

TABLE XII. The approximate matrix element $\langle\chi|S-S^C|\chi\rangle$. S corresponds to a short range plus Coulomb potential given by Eq. (3.10), S^C corresponds to the pure Coulomb potential given by Eq. (3.7). The short range potential is separable, rank 1 with an s-wave form factor χ, given by Eq. (3.11). The wave packet is chosen identical to the form factor.

T (fm/c) \ N	20	30	40	50	60
30	−0.135 738 0.494 467 i	−0.137 034 0.497 403 i	−0.137 709 0.498 987 i	−0.137 946 0.499 544 i	−0.138 056 0.499 802 i
60	−0.270 427 0.673 759 i	−0.273 083 0.677 988 i	−0.274 472 0.680 276 i	−0.274 958 0.680 107 i	−0.275 183 0.681 452 i
90	−0.310 304 0.691 587 i	−0.313 203 0.696 043 i	−0.314 703 0.698 459 i	−0.315 225 0.699 308 i	−0.315 466 0.699 702 i
120	−0.315 938 0.690 760 i	−0.318 545 0.695 344 i	−0.319 855 0.697 839 i	−0.320 304 0.698 718 i	−0.320 509 0.699 126 i
150	−0.317 245 0.690 768 i	−0.319 457 0.695 550 i	−0.320 496 0.698 172 i	−0.320 838 0.699 100 i	−0.320 992 0.699 531 i
180	−0.318 172 0.690 500 i	−0.319 988 0.695 485 i	−0.320 716 0.698 252 i	−0.320 931 0.699 237 i	−0.321 022 0.699 696 i
210	−0.319 089 0.690 110 i	−0.320 570 0.695 282 i	−0.320 971 0.698 204 i	−0.321 043 0.699 253 i	−0.321 063 0.699 745 i
	Reference	−0.320 056	+ 0.699 390 i		

from which one can obtain the expansion coefficients $S_{l,k}$

$$S_l(q)=\sum_{k=1,2,\ldots} S_{l,k}P_k(q)\ . \tag{3.18}$$

IV. N-BODY SYSTEM

The notation is taken from Ref. 24. Let α describe the situation (channel) where some of the N elementary particles form one or more composite particles. The situation without composite particles is denoted by $\alpha=0$. Corresponding to each channel α there is a channel interaction V^α which is the sum of interactions between those elementary particles which are contained in a composite particle ($V^0=0$). The total interaction is V. The kinetic energy is H^0. There are channel Hamiltonians H^α and the full Hamiltonian H,

$$H^\alpha=H^0+V^\alpha,\quad H=H^0+V\ . \tag{4.1}$$

In the case of short range potentials the Møller operator corresponding to channel α is given by

$$\Omega^{\alpha(\pm)}=\underset{t\to\mp\infty}{\text{s-lim}}\, e^{iHt}e^{-iH^\alpha t}\ . \tag{4.2}$$

Let $H^\alpha(u),H(u)$ denote self-adjoint bounded operators, such that on a dense subspace $\mathscr{D}$ of $\mathscr{H}$

$$H^\alpha=\underset{u\to0}{\text{s-lim}}H^\alpha(u),\quad H=\underset{u\to0}{\text{s-lim}}H(u)\ . \tag{4.3}$$

Let $H^\alpha(u,n),H(u,n)$ denote self-adjoint, finite rank operators such that

$$H^\alpha(u)=\underset{n\to\infty}{\text{s-lim}}H^\alpha(u,n),\quad H(u)=\underset{n\to\infty}{\text{s-lim}}H(u,n)\ , \tag{4.4}$$

and let

$$\Omega^\alpha(u,n,t)=e^{iH(u,n)t}e^{-iH^\alpha(u,n)t}\ . \tag{4.5}$$

Then for every $\Phi\in\mathscr{H}$ one can find $u,n,t_\mp$ such that

$$||\Omega^{\alpha(\pm)}\Phi-\Omega^\alpha(u,n,t_\mp)\Phi||$$

becomes arbitrarily small. The choice of u,n depends on the choice of $t_\mp$.

Inclusion of Coulomb potentials requires a modification of the wave operator. We follow again the proposal of Dollard.[28] Assume in channel α there are M particles, each of which is either elementary or composite (if there is at least one composite particle $M<N$), labeled by I,J. Let e_I denote the charge of particle I, m_{IJ} the reduced mass of particle I, and J, H^0_{IJ} the kinetic energy of relative motion between particle I,J.

$$A^{\alpha c}(t)=\text{sign}(t)\sum_{1\le I<J\le M}e_Ie_J\left[\frac{m_{I,J}}{2H^0_{IJ}}\right]^{1/2}\ln(4\,|\,t\,|\,H^0_{IJ})\ , \tag{4.6}$$

$$H^{\alpha c}(t)=H^\alpha t+A^{\alpha c}(t)\ . \tag{4.7}$$

Coulomb modified wave operators are

$$\Omega^{\alpha c(\pm)}=\underset{t\to\mp\infty}{\text{s-lim}}\, e^{iHt}e^{-iH^{\alpha c}(t)}\ . \tag{4.8}$$

Again let $H^{\alpha c}(u,t)$ denote a self-adjoint bounded operator such that

$$H^{ac}(t)=\text{s-}\lim_{u\to 0} H^{ac}(u,t)\ , \tag{4.9}$$

let $H^{ac}(u,n,t)$ denote a self-adjoint finite rank operator such that

$$H^{ac}(u,t)=\text{s-}\lim_{n\to\infty} H^{ac}(u,n,t)\ , \tag{4.10}$$

and let

$$\Omega^{ac}(u,n,t)=e^{iH(u,n)t}e^{-iH^{ac}(u,n,t)}\ . \tag{4.11}$$

Then for every $\Phi\in\mathscr{H}$ one can find $u,n,t_{\mp}$ such that

$$||\Omega^{ac(\pm)}\Phi-\Omega^{ac}(u,n,t_{\mp})\Phi||$$

becomes arbitrarily small. Again the choice of u,n depends on the choice of $t_{\mp}$. The approximated channel S matrices can be constructed from the approximated channel wave operators in analogy to the two-body case. We want to conclude this section with a remark and a word of warning. We consider it as an advantage of this approach that one has not to deal with singularities of the Green's functions as they occur in integral equation approaches.[2–9] Especially for the scattering of oppositely charged particles, e.g., e + d, the infinitely many bound state poles of the Coulomb Green's function do not occur.

However, a note of warning should be given here. There is an ambiguity (phase factor) with Dollard's Coulomb modified wave operators in channels with *more* than two charged particles (elementary or composite) corresponding to the incomplete knowledge of the boundary conditions.[28] Also in the approaches of Refs. 2–9 this phase ambiguity for three charged particles is not resolved. However, Merkuriev[30] has given boundary conditions in coordinate space for three charged particles. That should, in principle, serve to fix the abovementioned ambiguity in case of three charged particles.

V. d + p BREAKUP PROCESS

In this section we want to describe an application to the d + p→p + p + n process very near the threshold, i.e., in the low energy region where the correct treatment of the Coulomb force is supposed to be important. We have considered the potential

$$V=V^S+V^C,\quad V^S=\sum_{\alpha} V^{S\alpha},\quad \alpha\in 12,23,31 \tag{5.1}$$

where for each channel α we have taken the strong two-body potential $V^{S\alpha}$ given by Eq. (3.5) and the Coulomb potential V^C, acting only between p-p, given by Eq. (3.7). The breakup S matrix

$$S^C_{0\alpha}=\Omega^{0c(-)\dagger}\Omega^{ac(+)} \tag{5.2}$$

is related to the breakup T matrix, which enters in the cross section via[31]

$$_0\langle E'\,|\,S^c_{0\alpha}\,|\,E\rangle_\alpha = -2\pi i\delta(E'-E)_0\langle E'\,|\,T^c_{0\alpha}(E+i0)\,|\,E\rangle_\alpha\ . \tag{5.3}$$

The differential cross section is obtained from the T-matrix element in the same way as for short range forces. The phase space factor is given, e.g., by Ebenhöh.[32] The potential conserves the quantum numbers total spin and total angular momentum, but not total isospin. We have used the following basis of functions:

$$|\,e_m,e_nLM(l,\lambda)\rangle,\quad m,n=1,2,\ldots,\quad l,\lambda=0.1,2\ldots,$$

where e_m are the step functions, as given in Sec. III, used here for the variables $|\,\vec{p}\,|$, $|\,\vec{q}\,|$ which describe relative and subsystem motion, respectively. The quantum numbers l,λ describe relative and subsystem angular momenta, respectively, and LM describes the total angular momentum. Also spin and isospin have been included. In the ingoing state the wave packet for the relative motion has been represented by the function $e_k(p)$, such that the on shell (os) momentum $p_{\rm os}$ lies in the subinterval k ($p_{k-1}<p_{\rm os}<p_k$). It has been summed in the outgoing channel over all S-matrix elements with those step functions $e_m(p)e_n(q)$ which contribute to the energy interval

$$\frac{3}{4m_{\rm N}}p_{k-1}^2-|\,B^d\,|\le\frac{3}{4m_N}p^2+\frac{q^2}{m_{\rm N}}\le\frac{3}{4m_{\rm N}}p_k^2-|\,B^d\,|\ ,$$

$m_{\rm N}$ = nucleon mass .

The following approximations have been made. The subsystem, the relative, and the total angular momentum have been restricted only to s and p waves (note that the breakup amplitude does not have the forward singularity due to the Coulomb potential as the elastic amplitude). The number of expansion functions has been limited to 400. Only total isospin conserving Hamilton matrix elements have been taken into account. The differential cross sections $d^3\sigma/d\Omega_1 d\Omega_2 dE_s$ projecting on the kinematical curve have been calculated,[26] and the absolute normalization is adjusted to the experiment.[33] Figure 1 shows the results for 7.5 MeV incident deuteron energy, which corresponds to an energy of 0.275 MeV for the outgoing three nucleons, while Fig. 2 gives results for 7.402 MeV deuteron energy.

We want to discuss qualitatively the structure of the curves, in particular the major bump. Although the phase space factor has a maximum near that bump, it does not

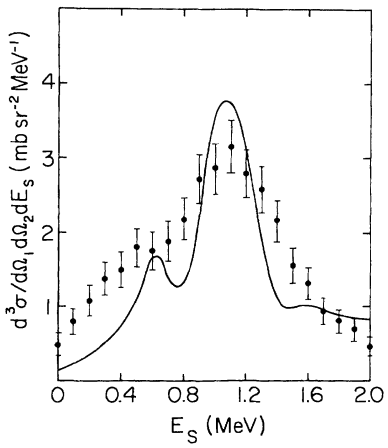

FIG. 1. The d + p breakup cross section at 7.5 MeV incident deuteron energy, 13°-13° proton-proton detection angle, along the kinematic curve; the data are taken from Ref. 33.

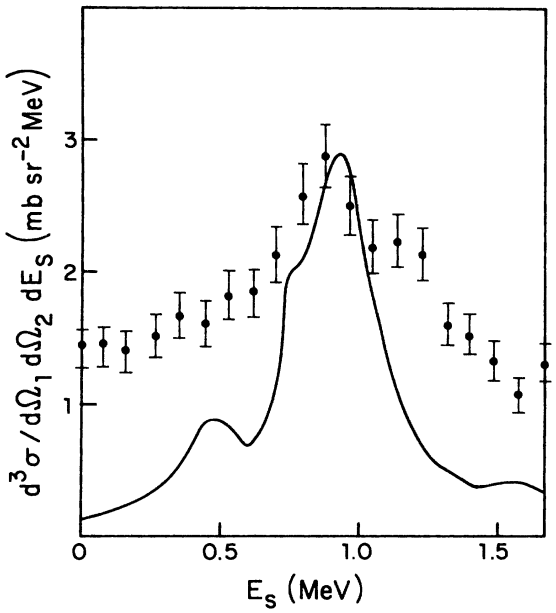

FIG. 2. Same as Fig. 1, but 7.402 MeV incident deuteron energy.

show such a pronounced peak (see also Ref. 23). On the other hand, the bump cannot be explained by a NN final state peak due to the strong force, because none of the NN pairs displays a minimum in its relative energy at the peak. However, the relative energy of the two protons has a maximum at the peak. With a total energy tending to the breakup threshold, i.e., the kinetic energy of the outgoing three nucleons tending to 0, the strong NN force will become negligible compared to the pp Coulomb force in the outgoing channel, as has been pointed out in Ref. 1. Thus we suggest to explain the bump mainly as a Coulomb effect in the outgoing channel, namely the Coulomb repulsion of the two protons, yields an enhanced probability to find the two protons widely separated which is most likely with a large pp relative momentum and that is exactly where the peak is found.

In the future, it is planned to investigate if the deviations in Figs. 1 and 2 can be ascribed (at least partly) to three-body forces. However, two-body forces are fairly dominant. The strategy is therefore to deal with the Coulomb problem in an "exact" way. The strength of the three-body force is believed to be comparable to the Coulomb force at short distances, but has partly an opposite sign and a different momentum dependence. Hence interference effects should be strongly energy dependent and produce deviations with respect to an exact two-body plus Coulomb treatment.

VI. CONCLUSION

We have proposed a method to calculate few-body S-matrix elements for Coulomblike interactions between wave packets using a strong approximation of the wave operators. The method has been tested in the two-body system and was found to give results converging to the reference solution. As an application we have calculated differential breakup cross sections for the d + p reaction close to the threshold. We consider as virtues of the method its technical simplicity and the absence of singularities, which appear in integral equations.

ACKNOWLEDGMENTS

The author is grateful to Q. Hokim for valuable discussions and a critical reading of the manuscript. This work has been supported by the National Science and Engineering Research Council of Canada.

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