Reply to "Volume conservation versus boson-number conservation"

R. Gilmore, D. H. Feng,* and M. Vallieres
Department of Physics and Atmospheric Science, Drexel University, Philadelphia, Pennsylvania 19104

John L. Wood
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

(Received 6 February 1984)

We show that second order conservation of the time-averaged volume in a two-fluid modification of the liquid drop model gives rise to a constraint analogous to boson-number conservation in the interacting-boson model.

Following the procedure outlined by Dirac for quantizing the electromagnetic field,[1] we expand the classical nuclear surface shape in normal mode amplitudes as follows:

$$R(\theta,\phi)=R_c+\Delta R(\theta,\phi) \ , \tag{1a}$$

$$\Delta R(\theta,\phi)=\sqrt{4\pi}R_c\sum_{l,m}(\alpha_{lm}Y_m^{l*}e^{i\omega_{lm}t}+\alpha_{lm}^*Y_m^le^{-i\omega_{lm}t}) \ . \tag{1b}$$

The nuclear volume is given by the integral $V=\int\frac{1}{3}R(\theta,\phi)^3d\Omega$. Using (1) in this expression for V and expanding, we obtain four terms, the first three of which are

$$\frac{1}{3}\int R_c^3d\Omega=\frac{4\pi}{3}R_c^3=V_c \ , \tag{2a}$$

$$\int R_c^2\Delta R d\Omega=3V_c(\alpha_{00}e^{i\omega_{00}t}+\alpha_{00}^*e^{-i\omega_{00}t}) \ , \tag{2b}$$

$$\int R_c\Delta R^2d\Omega=3V_c\sum_{l,m}[\alpha_{lm}e^{i\omega_{lm}t}+(-)^{l-m}\alpha_{l-m}^*e^{-i\omega_{l-m}t}]\times[\alpha_{lm}^*e^{-i\omega_{lm}t}+(-)^{l-m}\alpha_{l-m}e^{i\omega_{l-m}t}] \tag{2c}$$

The expression for the volume is

$$V(t;\alpha,\alpha^*)-V_c=(2b)+(2c)+(\text{third order terms}) \ . \tag{3}$$

This difference is a function of both time and the deformation parameters α_{lm}, α_{lm}^*. The time averaged volume is given in terms of the multipole amplitudes, to second order, by

$$\bar{V}-V_c=3V_c\sum_{l,m}(\alpha_{lm}\alpha_{lm}^*+\alpha_{lm}^*\alpha_{lm}) \ . \tag{4}$$

If we require the time averaged nuclear volume to be conserved under deformation to second order in the multipole amplitudes, the quadratic form on the right hand side of (4) must be invariant.

Under the two fluid interpretation of the nucleus previously described,[2] with R_c the radius of the inert spherical core, $\bar{V}-V_c$ represents the volume of the nuclear fluid outside the closed core. This is proportional to the number of valence nucleons, N (shell model input).[2] The canonical quantization conditions,[1]

$$\alpha_{lm}\to\hat{\alpha}_{lm}, \quad \alpha_{lm}^*\to\hat{\alpha}_{lm}^\dagger, \quad [\hat{\alpha}_{lm},\hat{\alpha}_{l'm'}^\dagger]=\kappa\delta_{ll'}\delta_{mm'} \ ,$$

lead to the conserved quantity

$$\sum_{l,m}\{\hat{\alpha}_{lm},\hat{\alpha}_{lm}^\dagger\}\sim(\bar{V}-V_c)/3V_c\sim N \ . \tag{5}$$

If only the monopole ($\alpha_{00}=s\to\hat{s}$) and quadrupole ($\alpha_{lm}=d_{lm}\to\hat{d}_{lm}$) modes are important, these classical and quantum second order volume conservation conditions reduce to

$$|s|^2+\sum_m|d_m|^2=(\bar{V}-V_c)/6V_c \ , \tag{6a}$$

$$\{\hat{s},\hat{s}^\dagger\}+\sum_m\{\hat{d}_m,\hat{d}_m^\dagger\}\sim N \ . \tag{6b}$$

If the monopole amplitude is secular rather than harmonic, only minor modifications are encountered in the treatment above. One again finds that second order volume conservation requires the invariance of a quadratic form.[3]

In the classical approach described by Lipas and Warner, the nuclear shape is given by Eq. (1) of Ref. 4. To second order the volume conservation equations are[3]

$$(\sqrt{\pi}+\alpha_{00})^2+\sum_m\alpha_{2m}^*\alpha_{2m}=\pi \ . \tag{7}$$

In the limit of small deformations α_{00} is well approximated by

$$\alpha_{00}=-\frac{1}{\sqrt{4\pi}}\sum_m\alpha_{2m}^*\alpha_{2m} \ , \tag{8}$$

which is Lipas and Warner's Eq. (2), to leading order.[4] This approximation breaks down as the deformation increases and fails completely for $\sum\alpha_{2m}^*\alpha_{2m}>\pi$. This is an important point, because the classical quantization procedure treats the amplitudes as if they were variables on the real line, e.g., $-\infty<\alpha_{2m}+\alpha_{2m}^*<+\infty$, placing no bounds on the number of quanta allowed.

Our approach differs from the standard description of the liquid drop model employed by Lipas and Warner[4] in two essential ways. These involve a "core plus mantle" decomposition of the nucleus and the introduction of a time-averaging procedure.

First, the length scale, R_c, appearing in our Eq. (1) describes the radius of the inert closed core about which the active fluid (valence nucleons) sloshes. The length scale, R_0, appearing in Eq. (1) of Ref. 4 describes the radius of the entire liquid drop in its undeformed shape. As a consequence, in our approach the undeformed nucleus has a large monopole amplitude [$\alpha_{00} \sim (\bar{V} - V_c/6V_c)^{1/2}$] while in the classical approach $\alpha_{00} = 0$. In our approach the monopole amplitude will decrease from this nonzero value as the quadrupole deformation increases in such a way that the quadratic constraint (6) is satisfied. In the classical approach the monopole amplitude decreases from zero as the quadrupole deformation increases. However, even in the classical approach the monopole and quadrupole amplitudes obey the quadratic constraint equation (7).

At the quantum-mechanical level in our approach the number of $l = 0$ excitations is largest in the spherically symmetric shape, and decreases as the number of quadrupole excitations increases. The number of $l = 0$ and $l = 2$ excitations is conserved. In the classical approach the number of monopole excitations is zero in the spherically symmetric state, and increases[4] as the number of quadrupole excitations increases. The total number of s and d excitations is neither conserved nor bounded. If the number of monopole excitations in the classical picture were proportional to $\sqrt{\pi} + \alpha_{00}$ rather than α_{00}, then the differences between our physical interpretations of excitation numbers would disappear. In fact, conservation of s- and d-mode excitation numbers would be a feature of the classical approach.

Second, we have explicitly introduced a time-averaging procedure in our approach. The multipole amplitudes α_{lm} are introduced to describe the normal modes of the fluid. These amplitudes have harmonic time dependence. A Hamiltonian constructed in the harmonic approximation is time independent, but the volume is necessarily time dependent. We feel the presence of these time-dependent surface fluctuations is reasonable, and are surprised they are not considered important.[4]

We have required that the time averaged value of the volume should be conserved under deformation. This volume conservation condition, when carried out to second order, leads to a quadratic constraint (4) on the normal mode amplitudes. The value of this conserved quantity is proportional to the amount of fluid outside the inert closed core. As such, it can be identified as proportional to the number of valence nucleons. Thus, both in structure ($s^\dagger s + d^\dagger d = \text{constant}$) and physical interpretation (constant $\propto$ number of valence nucleons) the quantum mechanical version of the second-order volume conservation condition (6b) is directly comparable with the boson number conservation condition in the Interacting Boson Model.

Our approach[2,3,5] is different from the approach taken by Lipas and Warner.[4] The group theoretical relations described in Ref. 5 remain correct and unchanged.

We wish to thank P. O. Lipas and D. D. Warner for useful discussions which led to the realization that a "core plus mantle" decomposition[2] was necessary to obtain the conservation condition expressed in Eq. (4). This work has been supported in part by NSF Grant No. PHY-830-4868.

---

*Present address: National Science Foundation, 1800 G Street, Northwest, Washington, D.C. 20550.

[1]P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed. (Clarendon, Oxford, 1958).
[2]R. Gilmore and D. H. Feng, Phys. Rev. C **26**, 766 (1982).
[3]R. Gilmore and D. H. Feng, Phys. Lett. **125B**, 99 (1983).
[4]P. O. Lipas and D. D. Warner, Phys. Rev. C **30**, 1097 (1984).
[5]R. Gilmore, D. H. Feng, M. Vallieres, and J. L. Wood. Phys. Rev. C **28**, 903 (1983).