Direct Reduction to Talmi Integrals Without Use of Transformation Brackets*

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(Received 6 January 1970; revised manuscript received 24 August 1970)

The two-body shell-model matrix elements for central and tensor forces can be expanded as linear combinations of Talmi integrals. A closed form, which does not involve vector-coupling coefficients or transformation brackets, is obtained for the coefficients appearing in this expansion. For the case that all the oscillator constants are the same, the result is simple enough to allow tabulation. A table of these coefficients for matrix elements involving the shell-model states 0s, 0p, and 0d is included. The main advantage of this method over the Talmi-Brody-Moshinsky method is discussed.

I. INTRODUCTION

Investigations of nuclear properties in terms of both spherical and deformed shell models involve either the reduction or the reductions as well as the evaluation of the matrix elements of an appropriate two-nucleon interaction $V(r_{ij})$. Only the reduction of the spin and angular momentum part of this interaction is necessary if one wishes to determine the matrix elements over the radial part by fitting observed spectra. Both the reduction and the evaluation of these matrix elements are desired if one wants either to predict a nuclear spectrum for a given nucleon-nucleon interaction, or to compare matrix elements obtained by fitting a given set of spectra with those obtained using a two-nucleon interaction which is compatible with deuteron properties and the two-nucleon scattering data (at least at low energies).

A very useful method of reduction of these matrix elements was first suggested by Talmi[1] and later investigated by Moshinsky[2,3] and Brody. In order to reduce a typical shell-model matrix element

$$I=\int d\vec{r}_1\int d\vec{r}_2\phi^*_{n_1l_1m_1}(\vec{r}_1)\phi^*_{n_2l_2m_2}(\vec{r}_2)\times V(r_{12})\phi_{n_1'l_1'm_1'}(\vec{r}_1)\phi_{n_2'l_2'm_2'}(\vec{r}_2), \tag{1}$$

they noted that it is useful to transform the integrand to relative and center-of-mass coordinates. They accomplished this by coupling the product state wave functions in Eq. (1) to states with angular momenta $\vec{\lambda}$ and $\vec{\lambda}'$ defined by

$$\begin{aligned}\vec{\lambda}&=\vec{l}+\vec{L}=\vec{l}_1+\vec{l}_2,\\ \vec{\lambda}'&=\vec{l}'+\vec{L}'=\vec{l}_1+\vec{l}_2,\end{aligned} \tag{2}$$

where $\vec{l}$ and $\vec{L}$ (and $\vec{l}'$ and $\vec{L}'$) are, respectively, the relative angular momentum and the angular momentum associated with the center-of-mass motion. The desired integral (1) is then transformed to a sum of integrals over center-of-mass and relative cooordinates

$$I=\sum_{\lambda,\,\lambda'}\langle l_1l_2m_1m_2|\lambda\mu\rangle\langle l_1'l_2'm_1'm_2'|\lambda'\mu'\rangle I_{\lambda\lambda'} \tag{3}$$

with

$$\begin{aligned}I_{\lambda\lambda'}&\equiv\langle n_1l_1n_2l_2;\lambda\mu|V_{12}|n_1'l_1'n_2'l_2';\lambda'\mu'\rangle\\ &=\sum_{\substack{NLnl\\N'L'n'l'}}\langle n_1l_1n_2l_2,\lambda|nlNL,\lambda\rangle\\ &\quad\times\langle n'l'N'L',\lambda|n_1'l_1'n_2'l_2',\lambda'\rangle\\ &\quad\times\langle nlNL;\lambda\mu|V|n'l'N'L';\lambda'\mu'\rangle.\end{aligned} \tag{4}$$

In (4), the first two terms on the right-hand side are the Talmi-Brody-Moshinsky transformation brackets, and the last term can be expressed in terms of the Talmi integrals.

The double sum over $\vec{\lambda}$ and $\vec{\lambda}'$ in (3) is reduced to essentially a single sum by the restriction that $\vec{\lambda}=\vec{\lambda}'$ if $V(r)$ is a central-force potential or $\vec{\lambda}=\vec{\lambda}'$, $\vec{\lambda}'\pm2$ if V contains a tensor-force component. In evaluating the $I_{\lambda\lambda'}$ for each permissible λ,λ' pair, one finds that the number of terms in the multiple summations in (4) is considerably reduced by parity, orthogonality, and conservation-of-energy restrictions. Nevertheless, it requires a good deal of computational labor to evaluate all the required transformation brackets, the $I_{\lambda\lambda'}$ for each λ,λ' pair, and finally the desired integral I itself, using (3).

In the above approach, it should be noted that it is more difficult to handle tensor forces than central forces. Secondly, in the original work of Moshinsky, the transformation brackets were defined only for the case that all the oscillator constants were the same. Recently, Gal[4] has alleviated one of these difficulties by extending the definition of the brackets to the case of different oscillator constants for the various wave functions $\phi_{nlm}(r)$ in (1). The current work also allows one to treat the case of different oscillator constants, and in addition also allows one to treat tensor forces on an equal footing with central forces.

The purpose of this paper is to point out that the interaction matrix (1) for

$$V(r_{12}) = V_C(r_{12}) + V_T(r_{12}) \tag{5}$$

can be reduced directly to Talmi integrals I_l

$$\langle V_{12}\rangle = \sum_l (c_l I_l^C + t_l I_l^T) \tag{6}$$

without any recourse to an intermediate step involving the coupling scheme (2). In (6), the superscripts C and T indicate that the Talmi integrals are to be evaluated using $V_C(r)$ and $V_T(r)$, respectively, and c_l and t_l are the desired expansion coefficients.

The advantage of our method is its simplicity, in that it directly evaluates the interaction matrix. Most physical problems leading to (1) involve product states, rather than vector-coupled states of angular momentum $\vec{\lambda} = \vec{l}_1 + \vec{l}_2$ and $\vec{\lambda}' = \vec{l}_1 + \vec{l}_2$, as in the conventional method of Talmi-Brody-Moshinsky (as for example, in the case of the interaction between Nilsson orbitals). Moreover, the expansion coefficients obtained here can be used *directly* in evaluating these matrix elements without any additional 3-j or 9-j symbols, and independently of any coupling scheme.

On the other hand, even with the use of the published table of the transformation brackets and the auxiliary table of B coefficients, the Talmi-Brody-Moshinsky method requires lengthy calculations. One must first vector-couple states, which replaces (1) by the sum (3), subject to the previously noted restrictions on $\vec{\lambda}$ and $\vec{\lambda}'$. Each of the integrals $I_{\lambda,\lambda'}$ is then transformed to Talmi integrals using Brody and Moshinsky's table of coefficients $B(nl, n'l'; p)$. Finally all the contributions are summed [the summations indicated in (3) and (4)] to achieve the final result. The amount of work involved, particularly in the case of a tensor force, was a principal motivation in developing a more direct means of evaluating (1). Finally, we note that in the event that one wishes to use deformed orbitals of the Nilsson type, the conventional method becomes even more involved, while no particular difficulty is encountered using the method developed here. The exact relationship between the expansion coefficients c_l and t_l and the various coefficients in the Brody-Moshinsky tables is discussed in Sec. VI.

The principal work in this paper is the derivation of closed-form expressions for the c_l and t_l in the expansion of (1) in terms of Talmi integrals, as in (6). These expressions, which may be programmed for a computer, allow all four oscillator constants ν_1, ν_2, ν_1', and ν_2' to be different. This leads to the possibility of multiwell calculations, such as using different oscillator constants for neutron and proton states, or for states with different l values. (As has been noted, this is now also possible using the transformation-bracket approach, due to the work of Gal.[4])

For the usual case of equal oscillator constants ($\nu_1 = \nu_2 = \nu_1' = \nu_2' = \nu$), the coefficients c_l and t_l are independent of ν, so that tabulation becomes feasible. Furthermore, if the integrals are written without complex conjugates of wave functions, as in (1), a large number of symmetries exist which considerably reduce the size of the tables.

The method could easily be extended to include a spin-orbit force. This has not been done explicitly, because the integrals in this case can be reduced to those for central forces by letting one of the operators L_+, L_-, or L_0 act on either of the product states in (1). In this way one can express the coefficients for a spin-orbit force in terms of the c_l.

A brief report of the method for equal oscillator constants has been presented previously.[5,6] Apart from presenting proofs and details, the present article extends the results to the case of unequal oscillator constants. For illustrative purposes, a table of c_l and t_l has been included. This table gives the reduction to Talmi integrals of matrix elements containing the $\mu = 0$ component of the potential V_{12}^μ when the particles are restricted to be in the states $0s$, $0p$, or $0d$, and have equal oscillator constants. Both central- and tensor-force coefficients have been included.

II. DEFINITIONS AND NOTATIONS

For our purpose, it will be sufficient to deal with the μth spherical component, V_{12}^μ of the interaction $V(r_{12})$, i.e.,[7]

$$V_{12}^\mu = V_C(r)\delta_{\mu,0} + V_T(r)T_2^\mu \tag{7}$$

with $\mu = 0, \pm1, \pm2$. T_2^μ is the μth spatial component of the tensor operator, and will be defined explicitly later.

Because of the usual properties of the spherical harmonics, we have

$$\int \phi_{n_1 l_1 m_1}(1)\phi_{n_2 l_2 m_2}(2) V_{12}^\mu \phi_{n_1' l_1' m_1'}(1)\phi_{n_2' l_2' m_2'}(2)\, d\tau_1 d\tau_2$$
$$= (-)^{m_1+m_2}\int \phi^*_{n_1 l_1 -m_1}(1)\phi^*_{n_2 l_2 -m_2}(2) \times V_{12}^\mu \phi_{n_1' l_1' m_1'}(1)\phi_{n_2' l_2' m_2'}(2)\, d\tau_1 d\tau_2\,, \tag{8}$$

and this integral (8) vanishes unless

$$\mu + m_1 + m_2 + m_1' + m_2' = 0 . \tag{9}$$

We want to expand the integral (8) in Talmi integrals to obtain

$$\langle V_{12}^{\mu}\rangle = \sum_l (c_l I_l^C + t_l I_l^T) . \tag{10}$$

We shall use spherical coordinates

$$\begin{aligned} r^{\pm} &= x \pm iy = r \sin\theta\, e^{\pm i\phi} , \\ r^0 &= z = r\cos\theta . \end{aligned} \tag{11}$$

Note the difference in normalization from the usual definition.

For the general case of unequal oscillator constants, we introduce the "center-of-mass" transformation

$$\begin{aligned} \vec{R} &= (1-\lambda)\vec{r}_1 + \lambda\vec{r}_2 , \\ \vec{r} &= \vec{r}_1 - \vec{r}_2 , \end{aligned} \tag{12}$$

with

$$\lambda = \frac{\nu_2 + \nu_2'}{\nu_1 + \nu_2 + \nu_1' + \nu_2'} . \tag{13}$$

For the spherical coordinates of the two particles, this gives the transformation properties

$$\begin{aligned} r_1^{\mu} &= R^{\mu} + \lambda r^{\mu} \equiv R^{\mu} + \rho^{\mu} , \\ r_2^{\mu} &= R^{\mu} - (1-\lambda) r^{\mu} \equiv R^{\mu} - \gamma\rho^{\mu} \end{aligned} \tag{14}$$

for $\mu = 0, \pm 1$, and where

$$\begin{aligned} \rho^{\mu} &= \lambda r^{\mu} , \\ \gamma &= (1-\lambda)/\lambda . \end{aligned} \tag{15}$$

Using (14) and the binomial theorem, one obtains, for a and b integers

$$(r_1^{\mu})^a (r_2^{\mu})^b = \sum_{\beta=0}^{a+b} (M_{\gamma})_{\beta}^{ab} (R^{\mu})^{a+b-\beta} (\rho^{\mu})^{\beta} , \tag{16}$$

with

$$(M_{\gamma})_{\beta}^{ab} = \sum_j (-\gamma)^j \binom{b}{j} \binom{a}{\beta - j} . \tag{17}$$

The M_{γ} are most easily calculated by recurrence relations.

For the spherical spatial components T_2^{μ} of the tensor operator S_{12}, we use

$$\begin{aligned} T_2^{\pm 2} &= 3 \sin^2\theta\, e^{\pm 2i\phi} , \\ T_2^{\pm 1} &= 3\cos\theta \sin\theta\, e^{\pm i\phi} , \\ T_2^0 &= 2\cos^2\theta - \sin^2\theta , \end{aligned} \tag{18}$$

which differ in normalization from the usual spherical tensor operator.

Finally, we introduce the notation

$$\begin{aligned} \Omega(n, m) &= \int \sin^{2n}\theta \cos^{2m}\theta\, d\Omega , \\ &= 4\pi \frac{2^n n!\,(2m-1)!!}{[2(n+m)+1]!!} . \end{aligned} \tag{19}$$

III. EVALUATION OF c_l AND t_l

One may write the single-particle harmonic-oscillator wave functions as

$$\phi_{nlm}(r) = \left[\frac{(2\nu)^{3/2}[2(n+l)+1]!!}{\pi^{1/2} 2^{n-2} n!}\right]^{1/2} \sum_{q=0}^{n} D_q^{nl} (4\nu)^{q+l/2} r^{2q} (r^l Y_l^m)\, e^{-\nu r^2} , \tag{20}$$

where

$$D_q^{nl} = (-)^q \binom{n}{q} \Big/ [2(q+l)+1]!! . \tag{21}$$

The spherical harmonic $r^l Y_l^m$ can be expanded as

$$r^l Y_l^m = N_{lm} \sum_{k=0}^{(l-p)/2} G_k^{lp} r^{2k} (r^+)^{(p+m)/2} (r^-)^{(p-m)/2} (r^0)^{l-p-2k} \tag{22}$$

with $p = |m|$,

$$G_k^{lp} = (-)^k \binom{l}{k} \binom{2(l-k)}{l+p} , \tag{23}$$

and

$$N_{lm} = \frac{(-)^{(p+m)/2}}{2^l l!} \left[\frac{(2l+1)(l+p)!\,(l-p)!}{4\pi}\right]^{1/2} . \tag{24}$$

Since

$$(r^2)^n = [(r^0)^2 + r^+ r^-]^n = \sum_{i=0}^{n} \binom{n}{i} (r^0)^{2(n-i)} (r^+ r^-)^i , \tag{25}$$

one finds for $\phi_{nlm}(r)$,

$$\phi_{nlm}(r)=(-)^{(p+m)/2}\left(\frac{2\nu}{\pi}\right)^{3/4}N_{nlp}\sum_{s=l/2}^{n+l/2}\sum_{t=p/2}^{s}(4\nu)^s K^{nlp}_{s-l/2,\,t-p/2}(r^+)^{t+m/2}(r^-)^{t-m/2}(r^0)^{2(s-t)}e^{-\nu r^2} \tag{26}$$

with

$$N_{nlp}=\frac{1}{2^l l!}\left[\frac{(2l+1)(l-p)!(l+p)![2(n+l)+1]!!}{2^n n!}\right]^{1/2}, \tag{27}$$

and

$$K^{nlp}_{q,c}=D^{nl}_q\sum_{k=0}^{(l-p)/2}\binom{q+k}{c}G^{lp}_k. \tag{28}$$

Then using (26) and (28), one may write for the product of two oscillator wave functions of the same variable (with $\nu'=\delta\nu$)

$$\phi_{nlm}(r)\phi_{n'l'm'}(r)=(-)^{(P+M)/2}(\delta)^{(l'+3/2)/2}(2\nu/\pi)^{3/2}N_{nlp}N_{n'l'p'}\sum_{S=L/2}^{N+L/2}\sum_{T=P/2}^{S}(4\nu)^S K^{nlp,\,n'l'p'}_{S-L/2,\,T-P/2}$$
$$\times(r^+)^{T+M/2}(r^-)^{T-M/2}(r^0)^{2(S-T)}e^{-(\nu+\nu')r^2} \tag{29}$$

with

$$P=p+p',\quad L=l+l',\quad M=m+m', \tag{30}$$

and

$$K^{nlp,\,n'l'p'}_{Q,C}=\sum_{q'}(\delta)^{q'}D^{n'l'}_{q'}D^{nl}_{Q-q'}\left[\sum_{K,k'}\binom{Q+K}{C}G^{l'p'}_{k'}G^{lp}_{K-k'}\right]. \tag{31}$$

Using (29), the integral of (8) (hereafter designated by $\langle V^{\mu}_{12}\rangle$) may be written

$$\langle V^{\mu}_{12}\rangle=\int\phi_{n_1l_1m_1}(1)\phi_{n_2l_2m_2}(2)V^{\mu}_{12}\phi_{n'_1l'_1m'_1}(1)\phi_{n'_2l'_2m'_2}(2)d\tau_1 d\tau_2$$
$$=(-)^{(P_1+P_2+M_1+M_2)/2}(\delta_1)^{(l'_1+3/2)/2}(\delta_2)^{(l'_2+3/2)/2}(\delta_{12})^{3/2}(2\nu_1/\pi)^{3/2}\prod_i(N_{n_il_ip_i}N_{n'_il'_ip'_i})$$
$$\times\sum_{S_1=L_1/2}^{N_1+L_1/2}\sum_{S_2=L_2/2}^{N_2+L_2/2}\sum_{T_1=P_1/2}^{S_1}\sum_{T_2=P_2/2}^{S_2}(\delta_{12})^{S_2}(4\nu_1)^{S_1+S_2}K^{n_1l_1p_1,\,n'_1l'_1p'_1}_{S_1-L_1/2,\,T_1-P_1/2}K^{n_2l_2p_2,\,n'_2l'_2p'_2}_{S_2-L_2/2,\,T_2-P_2/2}I(1,2) \tag{32}$$

with

$$I(1,2)=\iint(r_1^+)^{T_1+M_1/2}(r_2^+)^{T_2+M_2/2}(r_1^0)^{2(S_1-T_1)}(r_2^0)^{2(S_2-T_2)}(r_1^-)^{T_1-M_1/2}(r_2^-)^{T_2-M_2/2}e^{-(\nu_1+\nu'_1)r_1^2-(\nu_2+\nu'_2)r_2^2}V_{12}d\tau_1d\tau_2, \tag{33}$$

and where δ_i and δ_{12} are defined by

$$\nu'_i=\delta_i\nu_i,$$
$$\nu_2=\delta_{12}\nu_1. \tag{34}$$

Using (16) to transform to center-of-mass and relative coordinates, one has

$$I(1,2)=\sum_{a,b,c}[(M_\gamma)^{T_1+M_1/2,\,T_2+M_2/2}_a(M_\gamma)^{2(S_1-T_1),\,2(S_2-T_2)}_b(M_\gamma)^{T_1-M_1/2,\,T_2-M_2/2}_c]$$
$$\times\int(R^+)^{T+M/2-a}(R^-)^{T-M/2-c}(R^0)^{2(S-T)-b}e^{-\bar{\nu}R^2}d\tau_R(\lambda)^{a+b+c}\int(r^+)^a(r^-)^c(r^0)^b e^{-\nu r^2}[V_C(r)+V_T(r)T_2^{-M}]d\tau_r \tag{35}$$

with

$$S=S_1+S_2,\quad T=T_1+T_2,\quad M=M_1+M_2, \tag{36}$$

and

$$\bar{\nu}=\sum_i(\nu_i+\nu_i'),$$
$$\nu=\prod_i(\nu_i+\nu_i')/\sum_i(\nu_i+\nu_i'). \tag{37}$$

The angular integrals vanish unless

$$T+M/2-a=T-M/2-c, \tag{38}$$

i.e.,

$$c=a-M,$$

so that one may define parameters k and l by

$$2k=2a-M,$$
$$2l=2a-M+b, \tag{39}$$

giving

$$a=k+M/2,$$
$$c=k-M/2, \tag{40}$$
$$b=2(l-k).$$

The center-of-mass integral then becomes

$$\int R^{2(S-l)+2}e^{-\nu R^2}dR\iint(\sin^2\theta_R)^{T-k}(\cos^2\theta_R)^{S-l-T+k}d\Omega_R=(\pi)^{1/2}\frac{[2(S-l)+1]!!}{2^{S-l+2}\bar{\nu}^{S-l+3/2}}\Omega(T-k,S-l-T+k)$$
$$=(\pi)^{3/2}\frac{(T-k)![2(S-l-T+k)-1]!!}{\bar{\nu}^{S-l+3/2}2^{S-l-T+k}}. \tag{41}$$

Using the definition of the T_2^μ and (19), one may write for the integrals over Ω_r:
(a) For central forces,

$$\iint(\sin^2\theta_r)^k(\cos^2\theta_r)^{l-k}d\Omega_r=D_0^0\left[\frac{(4\pi)2^k k![2(l-k)+1]!!}{(2l+1)!!}\right]. \tag{42}$$

(b) For tensor forces

$$\iint[\sin^2\theta_r]^k[\cos^2\theta_r]^{l-k}e^{iM\phi}T_2^{-M}d\Omega_r=D_2^M\left[(4\pi)2^{k+|M|/2}\left(k+\frac{|M|}{2}\right)!\frac{[2(l-k)-|M|+1]!!}{(2l+1)!!}\right], \tag{43}$$

with

$$D_0^0=1/[2(l-k)+1],$$
$$D_2^0=\frac{1}{[2(l-k)+1]}\left[\frac{4l-6k}{2l+3}\right], \tag{44}$$
$$D_2^{\pm1}=D_2^{\pm2}=3/[2l+3].$$

The remaining radial integral has the form

$$\int r^{2l+2}e^{-\nu r^2}V(r)dr=\frac{(\pi)^{1/2}(2l+1)!!}{2^{l+2}\nu^{l+3/2}}I_l, \tag{45}$$

where

$$I_l=\frac{2}{\Gamma(l+\frac{3}{2})}\int x^{2l+2}V(\nu^{-1/2}x)e^{-x^2}dx. \tag{46}$$

The total contribution from the integrals over $\vec{R}$ and $\vec{r}$ is therefore

$$\frac{\pi^3}{(\nu\bar{\nu})^{3/2}}\left[\frac{2|M|/2}{2^{S-T}\nu^l\bar{\nu}^{S-l}}\right]N_{S,T}(l,k)(\delta_{M,0}D_0^0I_l^C+D_2^MI_l^T), \tag{47}$$

where I_l^C and I_l^T refer to Talmi integrals calculated with $V_C(r)$ and $V_T(r)$, respectively, and

$$N_{S,T}(l,k)=(T-k)!\left(k+\frac{|M|}{2}\right)![2(l-k)-|M|+1]!![2(S-l-T+k)-1]!!. \tag{48}$$

Combining all these results, one obtains for the integral of (8)

$$\langle V_{12}^{\mu}\rangle=(-)^{(P+M)/2}(\delta_1)^{(l_1'+3/2)/2}(\delta_2)^{(l_2'+3/2)/2}(\delta_{12})^{3/2}N\left[\frac{(2\nu_1)^3}{(\nu\bar{\nu})^{3/2}}\right]$$
$$\times\sum_{S_1,T_1}\sum_{S_2,T_2}\sum_{k=|M|/2}^{T}\sum_{l=0}^{S}(\delta_{12})^S K_{S_1-L_1/2,\,T_1-P_1/2}^{n_1l_1p_1,\,n_1'l_1'p_1'}K_{S_2-L_2/2,\,T_2-P_2/2}^{n_2l_2p_2,\,n_2'l_2'p_2'}$$
$$\times[(M_\gamma)_{k+M/2}^{T_1+M_1/2,\,T_2+M_2/2}(M_\gamma)_{k-M/2}^{T_1-M_1/2,\,T_2-M_2/2}(M_\gamma)_{2(l-k)}^{2(S_1-T_1),\,2(S_2-T_2)}]\frac{1}{2^{S-T}}\left[\frac{(4\nu_1)^S\lambda^{2l}}{\nu^l\bar{\nu}^{S-l}}\right]N_{S,T}(l,k)[\delta_{M,0}D_0^0I_l^C+D_2^MI_l^T], \tag{49}$$

with

$$N=\frac{2^{|M|/2}}{2^{\frac{1}{2}\sum_i(l_i+l_i'+n_i+n_i')}}\prod_i\left[\frac{(2l_i+1)!!(2l_i'+1)!![2(n_i+l_i)+1]!![2(n_i'+l_i')+1]!!}{n_i!\,n_i'!\binom{2l_i}{l_i-p_i}\binom{2l_i'}{l_i'-p'}}\right]. \tag{50}$$

Considerable simplification takes place when one specializes to the most frequently used case that all the ν_i are the same. One then has

$$\delta_1=\delta_2=\delta_{12}=\gamma=1,\quad \nu=\nu_1,\quad \bar{\nu}=4\nu_1,\quad \lambda=\tfrac{1}{2}. \tag{51}$$

Denoting $(M_\gamma)_\beta^{a,b}$ by $\mathsf{M}_\beta^{a,b}$ for convenience, the expression for the matrix element $\langle V_{12}^{\mu}\rangle$ in this case becomes

$$\langle V_{12}^{\mu}\rangle=(-)^{(P+M)/2}N\sum_{S_1,T_1}\sum_{S_2,T_2}\sum_{k,l}K_{S_1-L_1/2,\,T_1-P_1/2}^{n_1l_1p_1,\,n_1'l_1'p_1'}K_{S_2-L_2/2,\,T_2-P_2/2}^{n_2l_2p_2,\,n_2'l_2'p_2'}$$
$$\times[\mathsf{M}_{k+M/2}^{T_1+M_1/2,\,T_2+M_2/2}\mathsf{M}_{k-M/2}^{T_1-M_1/2,\,T_2-M_2/2}\mathsf{M}_{2(l-k)}^{2(S_1-T_1),\,2(S_2-T_2)}]\frac{1}{2^{S-T}}N_{S,T}(l,k)[\delta_{M,0}D_0^0I_l^C+D_2^MI_l^T], \tag{52}$$

from which the coefficients c_l and t_l are easily extracted. It is perhaps well to mention that a convenient check sum exists for the c_l. Letting $V_C(r)=1$, one finds using orthogonality that

$$\sum_l c_l=(-)^{m_1+m_2} \tag{53}$$

if $(n_i,l_i,m_i)=(n_i',l_i',m_i')$, and 0 otherwise.

IV. AN EXAMPLE

To clarify the general procedure used above to derive expressions for c_l and t_l, we note the essential steps for the simple case

$$\int\phi_{011}(1)\phi_{010}(2)[V_C(r)+V_T(r)T_2^0]\phi_{010}(1)\phi_{01\,-1}(2)\,d\tau_1d\tau_2, \tag{54}$$

where all the oscillator constants are the same.

(1) Each ϕ_{nlm} is expressed in terms of the spherical coordinates $r_2^{\pm},r_i^0$ defined by (11). The subscript i stands for particle 1 or 2. For this example, one obtains

$$\text{constant}\times\int r_1^+r_1^0r_2^0r_2^-e^{-2\nu(r_1^2+r_2^2)}[V_C(r)+V_T(r)T_2^0]d\tau_1d\tau_2. \tag{55}$$

(2) The integral is transformed to center-of-mass and relative coordinates using (16). For this case $\lambda = \frac{1}{2}$, since all the ν_i are equal. One obtains

$$\text{constant} \times \int (R^+ + \tfrac{1}{2}r^+)(R^0 + \tfrac{1}{2}r^0)(R^0 - \tfrac{1}{2}r^0)(R^- - \tfrac{1}{2}r^-)\, e^{-4\nu R^2} e^{-\nu r^2}[V_C(r) + V_T(r)T_2^0]\, d\tau_r\, d\tau_R$$

$$= \text{constant} \times \int \{R^+R^-(R^0)^2 - \tfrac{1}{4}[R^+R^-(r^0)^2 + (R^0)^2 r^+ r^-] + \tfrac{1}{16} r^+ r^- (r^0)^2\} e^{-4\nu R^2} e^{-\nu r^2}[V_C(r) + V_T(r)T_2^0]\, d\tau_r\, d\tau_R\,. \quad (56)$$

In the last line we have used the fact that the center-of-mass angular integral vanishes unless the powers of R^+ and R^- are equal.

(3) Each term is integrated over the center-of-mass coordinates and the angular part of the relative coordinates to gave a constant times a Talmi integral [I^C or I^T depending upon whether one is dealing with the central force $V_C(r)$ or the tensor force $V_T(r)$]. These integrations are elementary, even with the tensor force operator T_2^μ of (18).

V. TABLES

For the special case of all ν_i equal, a FORTRAN program has been written which allows rapid calculation on the c_l and t_l. Some tables have been prepared, using the Fordham University IBM 360/40 system. Short excerpts are included here in the table, allowing the reduction to Talmi integrals under the following conditions:

(1) All oscillator constants are equal;

(2) central forces or tensor forces with $\mu = 0$;

(3) all particles are in the 0s, 0p, or 0d states.

These tables are laid out as follows: In the first line of each table are the values of l_1, l_2, l_1', l_2', in that order (written as either s, p, or d), and labels for the coefficients in the different columns, c_l denoting central-force coefficients, and t_l denoting those for the tensor-force case. If c_k is the highest nonvanishing central-force coefficient, it can be shown that

$$c_{k-i} = (-)^{l_2 + l_2'} c_i\,. \quad (57)$$

This symmetry has been used to reduce the number of columns and is indicated in the headings.

Each remaining line in the table consists of a set m_1, m_2, m_1', m_2', in that order and the corresponding values of the coefficients c_l and t_l. Not all sets of m values have been included, because the integrals

$$\langle l_1 m_1 l_2 m_2 | V | l_1' m_1' l_2' m_2' \rangle = \int \phi_{0 l_1 m_1}(1) \phi_{0 l_2 m_2}(2) V_{12} \phi_{0 l_1' m_1'}(1) \phi_{0 l_2' m_2'}(2) d\tau_1 d\tau_2 \quad (58)$$

have the following symmetry properties (1, 2, 1′, 2′ refer to the four wave functions):

(1) The integrals are invariant under any of the exchanges (a) 1 to 1′, (b) 2 to 2′, and (c) the pairs (1, 1′) to (2, 2′).

(2) The integrals are invariant if all m's are replaced by their negatives.

(3) If $m_1 + m_1' = 0$, the integral is invariant under the exchange m_1 to m_1', and similarly for m_2 and m_2'.

All nonvanishing integrals (i.e., $m_1 + m_2 + m_1' + m_2' = 0$) for our cases can be reduced to those in the tables by use of the symmetry properties. Examples:

$$\begin{aligned}\int \phi_{022}^*(1)\phi_{021}^*(2) V_C(r) \phi_{021}(1)\phi_{022}(2) d\tau_1 d\tau_2 &= -\langle 2\,-2; 2\,-1 | V_C | 21; 22\rangle \\ &= -\langle 22; 21 | V_C | 2\,-1; 2\,-2\rangle \\ &= 0.187(I_0^C + I_4^C) - 0.250(I_1^C + I_3^C) + 0.125 I_2^C\,,\end{aligned} \quad (59)$$

$$\begin{aligned}\int \phi_{010}^*(1)\phi_{01\,-1}^*(2) V_T T_2^0 \phi_{021}(1)\phi_{02\,-2}(2) d\tau_1 d\tau_2 &= -\langle 10; 11 | V_T T_2^0 | 21; 2\,-2\rangle \\ &= -\langle 21; 2\,-2 | V_T T_2^0 | 10; 11\rangle \\ &= -\langle 2\,-2; 21 | V_T T_2^0 | 11; 10\rangle \\ &= -\langle 22; 2\,-1 | V_T T_2^0 | 1\,-1; 10\rangle \\ &= -0.141 I_1^T - 0.101 I_2^T\,.\end{aligned} \quad (60)$$

TABLE I. The coefficients c_l and t_l in the expansion of the integrals

$$\int \phi_{0l_1m_1}(1)\phi_{0l_2m_2}(2)V_{12}^0\,\phi_{0l_1'm_1'}(1)\phi_{0l_2'm_2'}(2)d r_1 d r_2=\sum_l [c_l I_l^C+t_l I_l^T]$$

in terms of Talmi integrals. All oscillator constants are equal and all particles are in $0s$, $0p$, or $0d$ states. To the left of each table are, in the first line, the values of l_1, l_2, l_1', and l_2', in that order, written as s, p, or d. Each succeeding line contains a set of m_1, m_2, m_1', and m_2'. Other integrals with s, p, and d particles may be reduced to those in the tables by using the symmetry properties listed in the text.

s	*s*	*s*	*s*	c_0				
0	0	0	0	1.000				
p	*p*	*s*	*s*	$c_0=-c_1$		t_1		
1	−1	0	0	−0.500		−0.200		
0	0	0	0	0.500		−0.400		
p	*s*	*p*	*s*	$c_0=c_1$		t_1		
1	0	−1	0	−0.500		0.200		
0	0	0	0	0.500		0.400		
d	*s*	*s*	*s*			t_1		
0	0	0	0			0.346		
d	*d*	*s*	*s*	$c_0=c_2$	c_1	t_1	t_2	
2	−2	0	0	0.250	−0.500	0.200	−0.143	
1	−1	0	0	−0.250	0.500	0.100	−0.071	
0	0	0	0	0.205	−0.500	−0.200	0.143	
d	*s*	*d*	*s*	$c_0=c_2$	c_1	t_1	t_2	
2	0	−2	0	0.250	0.500	−0.200	0.143	
1	0	−1	0	−0.250	−0.500	−0.100	−0.071	
0	0	0	0	0.250	0.500	0.200	0.143	
p	*p*	*p*	*p*	$c_0=c_2$	c_1	t_1	t_2	
1	1	−1	−1	0.500	0	0	−0.286	
1	0	0	−1	−0.250	0.500	0.100	−0.071	
1	0	−1	0	−0.250	−0.500	−0.100	−0.071	
1	−1	1	−1	0.500	−1.000	0.400	−0.286	
0	0	0	0	0.750	−0.500	−0.400	0.857	
d	*p*	*p*	*s*	$c_0=-c_2$	c_1	t_1	t_2	
2	−1	−1	0	0.354	0	0	0.202	
1	0	−1	0	−0.250	0	+0.300	0.071	
1	−1	0	0	−0.250	0	−0.300	0.071	
0	1	−1	0	0.144	0	−0.173	0.206	
0	0	0	0	0.289	0	0.173	−0.454	
d	*p*	*s*	*p*	$c_0=c_2$	c_1	t_1	t_2	
2	−1	0	−1	0.354	−0.707	0.283	−0.202	
1	0	0	−1	−0.250	0.500	0.100	−0.071	
0	1	0	−1	0.144	−0.289	−0.058	−0.206	
0	0	0	0	0.289	−0.577	−0.289	0.454	
d	*d*	*d*	*s*	$c_0=c_3$	$c_1=c_2$	t_1	t_2	t_3
2	0	−2	0	−0.144	0.144	0.029	0.041	0.144
2	−1	−1	0	0.177	−0.177	−0.141	0.101	0
2	−2	0	0	−0.144	0.144	0.029	−0.206	0.144
1	0	−1	0	−0.072	0.072	0.318	−0.289	−0.144
1	−1	0	0	−0.072	0.072	−0.202	0.330	−0.144
1	−2	1	0	0.177	−0.177	0.283	−0.202	0
0	0	0	0	0.144	−0.144	0.087	−0.371	0.433
d	*d*	*p*	*p*	$c_0=-c_3$	$c_1=-c_2$	t_1	t_2	t_3
2	0	−1	−1	−0.204	0.204	0.041	0.117	−0.204
2	−1	0	−1	0.177	−0.530	0	0	0
2	−1	−1	0	0.177	0.177	0.141	0.101	0
2	−2	1	−1	−0.375	1.125	−0.450	0.643	−0.250
2	−2	0	0	0.125	−0.375	0	0	0

TABLE I *(Continued)*

d	d	p	p	$c_0=-c_3$	$c_1=-c_2$	t_1	t_2	t_3
2	−2	−1	1	−0.375	0.125	−0.050	0.071	−0.250
1	1	−1	−1	0.250	−0.250	−0.200	−0.143	0
1	0	0	−1	−0.072	0.505	0.144	−0.206	0.144
1	0	−1	0	−0.072	−0.361	−0.029	0.165	0.144
1	−1	1	−1	0.250	−0.750	0	0	0
1	−1	0	0	−0.375	0.625	0.050	−0.500	0.250
0	0	1	−1	−0.208	0.458	0.117	−0.024	−0.083
0	0	0	0	0.458	−0.708	−0.267	0.667	−0.667

d	p	d	p	$c_0=c_2$	$c_1=c_3$	t_1	t_2	t_3
2	1	−2	−1	−0.375	−0.125	0.050	0.071	0.250
2	0	−1	−1	0.177	−0.177	−0.141	0.101	0
2	0	−2	0	0.125	0.375	0	0	0
2	−1	0	−1	−0.204	0.204	0.041	−0.292	0.204
1	1	−1	−1	0.250	0.250	0.200	−0.143	0
1	0	0	−1	−0.072	0.072	−0.202	0.330	−0.144
1	0	−1	0	−0.375	−0.125	0.350	−0.357	−0.250
1	−1	1	−1	0.250	−0.250	0.400	−0.286	0
0	1	0	−1	−0.208	−0.292	0.017	−0.262	0.083
0	0	0	0	0.458	0.042	0.133	−0.381	0.667

d	d	d	d	$c_0=c_4$	$c_1=c_3$	c_2	t_1	t_2	t_3	t_4
2	2	−2	−2	0.375	0	0.250	0	−0.143	0	−0.273
2	1	−1	−2	−0.187	0.250	−0.125	0.125	0.018	−0.125	0.034
2	1	−2	−1	−0.187	−0.250	−0.125	−0.125	0.018	0.125	0.034
2	0	0	−2	0.187	−0.417	0.458	−0.033	0.024	0.167	−0.136
2	0	−1	−1	0	0.204	−0.408	0.102	−0.073	−0.102	0.084
2	0	−2	0	0.187	0.083	0.458	−0.033	0.024	0.167	−0.136
2	−1	1	−2	−0.187	0.750	−1.125	−0.075	0.161	−0.125	0.034
2	−1	0	−1	0	−0.408	0.816	−0.143	0.102	−0.102	0.084
2	−2	2	−2	0.375	−1.500	2.250	0.600	−1.286	1.000	−0.273
1	1	−1	−1	0.375	−0.250	0.750	−0.200	0.214	0	0.136
1	0	0	−1	−0.187	0.677	−0.958	0.108	−0.399	0.458	−0.170
1	0	−1	0	−0.187	−0.583	0.542	−0.042	0.244	−0.292	−0.170
1	−1	1	−1	0.375	−1.000	1.250	0.100	0.357	−0.500	0.136
0	0	0	0	0.562	−0.750	1.375	−0.233	0.738	−0.833	0.682

VI. RELATION BETWEEN c (AND t) AND TALMI - BRODY-MOSHINSKY BRACKET

To establish a relation between the expansion coefficients c_l and t_l and the transformation bracket tabulated by Brody and Moshinsky, we will consider only the central-force case. A similar relation can be established for the tensor-force case.

The integral considered by Moshinsky is not the integral of (1) involving the product state functions, but rather the integral where the wave functions are coupled to intermediate states of angular momenta $\vec{\lambda}$ and $\vec{\lambda}'$, i.e., the integrals

$$I_{\lambda\lambda'} = \langle n_1 l_1 n_2 l_2; \lambda\mu | V | n_1' l_1' n_2' l_2'; \lambda'\mu'\rangle . \tag{61}$$

Moshinsky showed that such integrals may be written (for central forces, so that $\lambda=\lambda'$) as

$$I_{\lambda\lambda} = \sum_p I_p \Big\{ \sum_{nlNLn'} [\langle n_1 l_1 n_2 l_2, \lambda | nlNL, \lambda\rangle \langle n'lNL, \lambda | n_1' l_1' n_2' l_2', \lambda\rangle] B(nl, n'l; p)\Big\} . \tag{62}$$

The terms inside the square brackets on the right-hand side of (62) are the transformation brackets, which, along with the coefficients $B(nl, n'l'; p)$, have been tabulated by Brody and Moshinsky.

The desired integral (8) is related to the integrals evaluated by Brody and Moshinsky ($I_{\lambda\lambda'}$) by

$$I = \langle V_{12}\rangle = \sum_{\lambda=|l_1-l_2|}^{|l_1+l_2|} \sum_{\lambda'=|l_1'-l_2'|}^{|l_1'+l_2'|} \langle l_1 l_2 m_1 m_2 | \lambda\mu\rangle \langle l_1' l_2' m_1' m_2' | \lambda'\mu'\rangle I_{\lambda\lambda'} . \tag{63}$$

Since, in terms of the coefficients c_l introduced here, $\langle V_{12}\rangle$ may be written

$$\langle V_{12}\rangle = \sum_p (c_p I_p^C + t_p I_p^T) \tag{64}$$

one can relate the c_l to the various quantities used in the Brody-Moshinsky method by

$$c_p = \sum_{\lambda=|l_1-l_2|}^{|l_1+l_2|} \langle l_1 l_2 m_1 m_2|\lambda\mu\rangle\langle l_1' l_2' m_1' m_2'|\lambda\mu\rangle \sum_{nlNLn'} \Big(\langle nlNL,\lambda|n_1 l_1 n_2 l_2,\lambda\rangle\langle n_1' l_1' n_2' l_2',\lambda|n'lNL,\lambda\rangle B(nl,n'l;p)\Big). \tag{65}$$

From the expression (65), the principal advantage of the present method is clear. We have replaced a very complicated summation by a simple coefficient. In the Talmi-Brody-Moshinsky method, one is to use the Brody-Moshinsky tables to get the transformation brackets and the B coefficients, and then perform multiple summations over products of these coefficients with various Clebsch-Gordan coefficients, even for the central-force case. Obviously the tensor-force case is more complex. By contrast, in using the present method, one simply takes the desired coefficient c_l (or t_l for a tensor force) directly from the table. In addition, as should be clear from the above discussion, the amount of work eliminated by our method increases sharply as one deals with higher-energy states, where the number of values of $\vec{\lambda}$ and $\vec{\lambda}'$, the number of Clebsch-Gordan coefficients, the number of transformation brackets, and the number of B coefficients involved in calculating a single integral over product state wave functions all increase. On the other hand, the c_l and t_l may be tabulated once and for all for any reasonable case using a simple FORTRAN program.

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*Work performed in partial fulfillment of the requirements for the Ph. D. degree by R. H. Richardson at Fordham University.

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‡Work supported in part by a National Science Foundation grant.

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