[1]B. H. Flowers, Proc. Roy. Soc. (London) A212, 248 (1952).
[2]A. de Shalit and J. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963).
[3]R. D. Lawson and M. H. Macfarlane, Nucl. Phys. 66, 80 (1965).
[4]M. Ichimura, Progr. Theoret. Phys. (Kyoto) 33, 215 (1965).
[5]K. Helmers, Nucl. Phys. 23, 594 (1961).
[6]M. Baranger, Phys. Rev. 122, 992 (1961).
[7]A. M. Navon and A. K. Bose, Phys. Rev. 177, 1514 (1969).
[8]A. Navon, J. Math. Phys. 10, 821 (1969).
[9]A. K. Bose and A. Navon, Phys. Letters 17, 112 (1965).
[10]N. Jacobson, *Lie Algebras* (Interscience Publishers, Inc., New York, 1962).
[11]W. W. Morozov, Dokl. Akad. Nauk SSSR, 36, 259 (1942).
[12]E. B. Dynkin, Am. Math. Soc., Transl. Ser. 2, 6 (1957), p. 245.
[13]K. Helmers, Nucl. Phys. 69, 593 (1965).
[14]B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. (London) 84, 673 (1964).
[15]K. T. Hecht, Phys. Rev. 139, B794 (1965).
[16]J. C. Parikh, Nucl. Phys. 63, 214 (1965).

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Boundary-Condition-Model T Matrix

Michael G. Fuda
Department of Physics and Astronomy, State University of New York at Buffalo, Buffalo, New York 14214
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A number of results on the boundary-condition-model (BCM) T matrix are developed. It is shown that the half-off shell T matrix is unique, but the fully off-shell T matrix is not. Further, it is shown that the ambiguity in the T matrix resides in that part of the complete T matrix which is identifiable as the T matrix for the pure BCM, where by pure BCM is meant no forces outside the boundary-condition radius. Three different formulas for the pure BCM T matrix are presented. The first is derived by using the relations that exist between the half-off-shell T matrix and the fully off-shell T matrix for well-behaved potentials, and is found to be separable. The second is taken from the work of Kim and Tubis. The third is derived from a pseudopotential constructed by Hoenig and Lomon. All three agree exactly half off shell, and satisfy the off-shell unitarity relation. Numerical comparisons are given which show that significant differences can occur in the fully off-shell T matrices. An integral- as well as a differential-equation approach are given for finding the contribution to the BCM T matrix from the forces outside the boundary-condition radius. Separable representations for the BCM T matrix are developed, and their usefulness in carrying out calculations on the three-nucleon system is discussed.

I. INTRODUCTION

In a boundary-condition model (BCM), part or all of the force between a pair of particles is represented by a logarithmic boundary condition on the Schrödinger wave function. The boundary condition may or may not be energy dependent. It appears that the BCM was first used to describe the two-nucleon interaction by Breit and Bouricius,[1] who showed that the low-energy $^1S_0$ scattering data could be fitted by a pure BCM (no outside forces). They considered both energy-independent and energy-dependent logarithmic derivatives. The pure BCM was extended to higher energies and to tensor forces by Feshbach and Lomon,[2] who found that they could obtain a reasonable fit to the scattering data up to 274 MeV if they allowed the $^1S_0$ core radius to change with energy.

A good fit to pp data was later obtained by using an energy-independent boundary condition in conjunction with a local potential outside the core. The local potentials were of two types: a purely phenomenological exponential potential,[3] and a meson-theoretic potential[4] which included one- and two-pion-exchange contributions. In recent years, the fits to the nucleon-nucleon scattering data have been improved and the effects of mesons other than π mesons have been incorporated into the model.[5] The analytic properties of the scattering amplitudes arising in the BCM have been studied and found to be similar to those of the more conventional potential models.[6] It is now clear that the BCM with outside forces taken from meson field theories leads to as reasonable a description of the two-nucleon system as conventional potential models, in that it fits the elastic scattering data and gives rise to scattering amplitudes with acceptable analytic properties. The situation with respect to the many-nucleon problem is not so clear.

The application of the pure BCM (no outside forces) to nuclear matter was treated by Lomon and McMillan,[7] who found a pseudopotential which could be used to replace the boundary condition. They also investigated the two-body half-off-shell reaction or K matrix. An important conclusion of their work is that an ambiguity arises in applying the BCM to the many-body problem. This ambiguity comes about because it is not necessary to specify the potential inside the core radius for the pure two-body problem, but it is necessary to do so for the many-body problem. A calculation of the properties of nuclear matter based on a BCM with a square well representing the forces outside the core was carried out by Razavy and Sprung.[8] They represented the force which gives rise to the energy-independent boundary condition by a hard-core potential with a very narrow but very deep attractive well just outside the hard core but just inside the distance at which the logarithmic derivative is to be obtained. The calculated binding energy per nucleon came out very small (2 MeV per nucleon). It is not clear whether this was due to the force model inside the boundary-condition radius, the force model outside, or both. The BCM with outside forces was also applied to nuclear matter by Hoenig and Lomon.[9] They further clarified the ambiguity in the BCM. In particular, they showed that, if the boundary condition is energy independent and the potential is local outside the core, then the two-body *Schrödinger* wave function must vanish inside the core. They found, however, that this was not true for the Bethe-Goldstone (BG)[10] wave function. The solution of the BG equation depends on an arbitrary parameter, which is the logarithmic derivative of the BG wave function just inside the core radius. In this connection Hoenig and Lomon[9] have constructed a pseudopotential which, besides producing the correct logarithmic derivative just outside the core radius, also produces an arbitrary logarithmic derivative just inside the core radius. One cannot obtain this parameter from the pure two-body problem.

The T matrix for the BCM has been studied by Kim and Tubis.[11] They have shown that one can produce an energy-independent logarithmic derivative at some distance c from the origin by applying a limiting procedure to a square repulsive potential inside c with a δ-function interaction at c. Their idea is similar to that used by Razavy and Sprung[8] in their nuclear-matter calculation. Using this limiting procedure Kim and Tubis have obtained an expression for the T matrix arising from a pure BCM, as well as an integral equation for the contribution to the T matrix from the forces outside the core. Their procedure is by no means unique, since one could, for example, use the Hoenig-Lomon pseudopotential[9] to produce the energy-independent boundary condition.

It appears clear that further study of the ambiguity in the BCM is necessary. In this paper a number of results on the BCM T matrix are obtained. The ambiguity in the BCM T matrix is isolated, and some methods are presented for applying the BCM to the three-nucleon system.

In particular, by combining the Hoenig-Lomon theorem[9] on the two-body Schrödinger wave function with a result of Noyes,[12] it is shown that the *half-off-shell* T matrix can be obtained directly from a knowledge of the two-body Schrödinger wave function; the potential need not be specified. It is shown, however, that by applying three different procedures to the pure BCM (no outside forces), one obtains three different off-shell T matrices. The half-off-shell T matrices are the same. The first procedure is based on the relationship that exists between the half-off-shell T matrix and the fully off-shell T matrix for *well-behaved* potentials.[12,13] These relations lead to a one-term separable T matrix for the pure BCM. The second procedure is the Kim-Tubis[11] prescription mentioned above. The third T matrix is obtained from the Hoenig-Lomon[9] pseudopotential, and contains an arbitrary parameter (see above) which can be varied without effecting the half-off-shell T matrix. Furthermore, it is shown that all three T matrices satisfy the off-shell unitarity relation,[14] so that this general relation cannot be used to discriminate between the various T matrices.

Fortunately, it is found that adding in the contribution to the T matrix from the forces outside the core introduces no additional ambiguities. All of the ambiguity in the BCM T matrix resides in that part of it which is the T matrix for the pure BCM. In particular, it is shown that the integral equation for the contribution to the T matrix from the outside forces derived by Kim and Tubis[11] holds in general and does not depend on their prescription for the potential inside the boundary condition. A differential-equation approach is also developed for obtaining the BCM T matrix when forces outside the core are present. This is an extension of the method developed by other authors[15] for treating hard-core potentials and should be of use in studying analytic models.

Finally, some separable representations of the BCM T matrix are presented. Two of these are extensions of this author's[16] previous work on hard-core potentials; the other is related to the so-called unitary pole expansion of Harms.[17] These expansions have proven useful in carrying out calculations on the three-nucleon system. Calculations on this system should be useful in assessing the effect of the ambiguity in the fully off-shell BCM T matrix, since this is what enters into calculations based on the Faddeev[18] equations.

In Sec. II the various pure BCM T matrices mentioned above are developed and their properties are dicussed. A numerical comparison of the T matrices is given so as to demonstrate that significant differences among them do exist. In Sec. III the contribution to the BCM T matrix from the forces outside the core is investigated. The integral equation for this contribution is developed in part A of this section and is shown not to depend on the prescription which is used for the pure BCM. A differential-equation approach is also presented. In part B of Sec. III various separable representations for the contribution to the BCM T matrix from the outside forces are developed. A summary and some discussion are given in Sec. IV.

II. T MATRIX FOR THE CORE REGION

In this section we will treat the contribution to the two-body T matrix arising from the force inside the boundary-condition radius c. We begin by establishing notation and normalization. In the BCM one uses the Schrödinger equation for $r>c$ and applies a logarithmic boundary condition at $r=c$; i.e., we solve

$$\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}+V(r)\right]u_l(k,r)=k^2u_l(k,r)\,, \quad r>c \tag{2.1}$$

with the boundary condition

$$\frac{du_l}{dr}(k,r)\bigg|_{r=c}=\frac{f_l}{c}u_l(k,c)\,. \tag{2.2}$$

We will assume that f_l is independent of energy. Our free-particle eigenstates are normalized so that in the coordinate representation they are given by

$$\langle\vec{r}\,|plm\rangle=(2\pi^2)^{-1/2}j_l(pr)Y_{lm}(\hat{r})\,. \tag{2.3}$$

j_l is the usual spherical Bessel function, Y_{lm} is a spherical harmonic, and $\hat{r}$ is a unit vector in the direction of $\vec{r}$. With this normalization, the completeness relation is

$$\sum_{l,m}\int_0^\infty |plm\rangle\, 4\pi p^2dp\langle plm|=1\,. \tag{2.4}$$

We let $t(s)$ stand for the T matrix arising from the core region. The normalization (2.4) implies that the on-shell T matrix is related to the phase shift by

$$\langle klm\,|\,t(k^2+i\epsilon)klm\rangle=-(2\pi^2k)^{-1}e^{i\delta_l^c(k)}\sin\delta_l^c(k)\,. \tag{2.5}$$

Here δ_l^c is the phase shift arising from the pure BCM; i.e., no outside forces, and is given by

$$\cot\delta_l^c(k)=-\frac{kcn_l'(kc)+(1-f_l)n_l(kc)}{g_l(kc;f_l)}\,, \tag{2.6}$$

where

$$g_l(x;f_l)=xj_l'(x)+(1-f_l)j_l(x)\,. \tag{2.7}$$

Following Noyes,[12] we introduce the half-off-shell extension function $F_l(p,k)$ by the relation

$$F_l(p,k)=\frac{\langle plm\,|\,t(k^2+i\epsilon)\,|\,klm\rangle}{\langle klm\,|\,t(k^2+i\epsilon)\,|\,klm\rangle}\,. \tag{2.8}$$

It is shown in Ref. 12 that this function can be obtained from a knowledge of just the two-body wave function by means of the relation

$$F_l(p,k)=\left(\frac{p}{k}\right)^l+(p^2-k^2)\int_0^\infty dr\,kr^2j_l(pr)\times[\psi_l(k,r)-n_l(kr)-\cot\delta_l(k)\,j_l(kr)]\,, \tag{2.9}$$

with ψ_l normalized so that

$$\psi_l(k,r)\underset{r\to\infty}{\simeq}n_l(kr)+\cot\delta_l(k)j_l(kr)\,. \tag{2.10}$$

Hoenig and Lomon[9] have shown that if f_l [see (2.2)] is independent of energy and the potential is local for $r>c$, then the two-body wave function must vanish for $0<r<c$. Using this fact and (2.9), it follows immediately that for the pure BCM

$$F_l(p,k)=g_l(pc;f_l)/g_l(kc;f_l)\,. \tag{2.11}$$

Note that this function is separable in p and k.

We now consider going from half off shell to fully off shell. In Ref. 12 it is shown that for a *well-behaved* potential, the half-off-shell T matrix and the fully off-shell T matrix are related by

$$\begin{aligned}t_l(p,q;k^2+i\epsilon)&=t_l(p,q;q^2+i\epsilon)+\int_0^\infty W_l(p,q;x^2)dx\left(\frac{1}{k^2+i\epsilon-x^2}-\frac{1}{q^2+i\epsilon-x^2}\right)\\&=t_l(p,q;p^2+i\epsilon)+\int_0^\infty W_l(p,q;x^2)dx\left(\frac{1}{k^2+i\epsilon-x^2}-\frac{1}{p^2+i\epsilon-x^2}\right),\end{aligned} \tag{2.12}$$

where

$$t_l(p,q;k^2+i\epsilon)=\langle plm|t(k^2+i\epsilon)|qlm\rangle, \tag{2.13}$$

and

$$W_l(p,q;k^2)=F_l(p,k)\pi^{-3}\sin^2\delta_l(k)F_l(q,k). \tag{2.14}$$

Using these relations and the fact that F_l is separable [see(2.11)], it is easy to show that for the pure BCM

$$t_l^N(p,q;s)=F_l(p,k)t_l(k,k;k^2+i\epsilon)F_l(q,k), \tag{2.15}$$

where F_l is given by (2.11) and the on-shell T matrix is given by

$$t_l(k,k;k^2+i\epsilon)=(2\pi^2k)^{-1}g_l(kc;f_l)/D_l(kc), \tag{2.16}$$

$$D_l(x)=xh_l^{(+)\prime}(x)+(1-f_l)h_l^{(+)}(x). \tag{2.17}$$

The spherical Hankel function is normalized as in Messiah.[19] The T matrix (2.15) has the important property that it is separable in p and q.

Unfortunately the prescription which led to (2.15) is not unique. The difficulty lies in the use of (2.12), a relation whose validity depends on the potential being well behaved. Some of the pseudopotentials which are used to produce the energy-independent boundary condition at $r=c$ have a hard core for $0<r<c$, as well as δ functions and derivatives of δ functions.[7-9,11] For a hard-core potential, (2.12) is not valid, although (2.9) is. In particular, Kim and Tubis[11] use the following potential to produce the pure BCM,

$$U(r)=U_0-U_1c\,\delta(r-c),\quad 0\le r\le c, \tag{2.18}$$

where U_0 and U_1 are made to approach infinity so that

$$f_1=(\sqrt{U_0}-U_1c)c. \tag{2.19}$$

The potential (2.18) leads to the T matrix

$$\begin{aligned}t_l^{\mathrm{KT}}&(p,q;k^2+i\epsilon)\\ &=F_l(p,k)t_l(k,k;k^2+i\epsilon)F_l(q,k)\\ &+\frac{c}{2\pi^2(p^2-q^2)g_l(kc;f_l)}[g_l(pc;f_l)w_l(qc,kc)(k^2-p^2)\\ &-g_l(qc;f_l)w_l(pc,kc)(k^2-q^2)],\end{aligned} \tag{2.20}$$

$$w_l(x,y)=yj_l(x)j_l{}'(y)-xj_l(y)j_l{}'(x). \tag{2.21}$$

It is easy to check that the two T matrices, (2.15) and (2.20), agree half off shell; i.e., when $p=k$ or $q=k$. In the two-body problem only the half-off-shell T matrix arises; however, in a three-body calculation, for example, one needs the fully off-shell T matrix. Thus, as pointed out by Hoenig and Lomon[9] in connection with nuclear-matter theory, there is an ambiguity in the BCM.

Another potential which gives rise to (2.2) is the pseudopotential of Hoenig and Lomon.[9] Theirs is a separable potential which is defined by the relations

$$\begin{aligned}\langle\vec{r}|U|\vec{r}'\rangle&=\sum_{l,m}Y_{lm}(\hat{r})\frac{U_l(r,r')}{rr'}Y_{lm}(\hat{r}'),\\ U_l(r,r')&=\left[\frac{f_l}{c}\delta(r-c^+)+\delta'(r-c^-)\right]\frac{c}{f_l}\\ &\times\left[\frac{f_l}{c}\delta(r'-c^+)+\delta'(r-c^-)\right]\\ &+\delta'(r-c^-)\left[\frac{c}{b_l}-\frac{c}{f_l}\right]\delta'(r'-c^-).\end{aligned} \tag{2.22}$$

c^+ (c^-) stands for a distance slightly greater (smaller) than c, and the primes indicate differentiation with respect to the argument of the δ function. In order to derive the T matrix which arises from (2.22), we follow the method developed by Van Leeuwen and Reiner[15] for square wells. The T matrix is the solution of the equation

$$t(s)=U+UG_0(s)t(s), \tag{2.23}$$

where

$$G_0(s)=(s-H_0)^{-1}.$$

H_0 is the kinetic energy operator. This can be converted to a differential equation by introducing

$$\Omega(s)=1+G_0(s)t(s). \tag{2.24}$$

From (2.23) we have

$$t(s)=U\Omega(s), \tag{2.25}$$

so (2.24) becomes

$$(s-H_0-U)\Omega(s)=(s-H_0). \tag{2.26}$$

In a mixed representation we obtain

$$(s+\nabla^2)\langle\vec{r}|\Omega(s)|qlm\rangle-\int\langle\vec{r}|U|\vec{r}'\rangle d\vec{r}'\langle\vec{r}'|\Omega(s)|qlm\rangle=(s-q^2)\langle\vec{r}|qlm\rangle. \tag{2.27}$$

If we let

$$\langle\vec{r}|\Omega(s)|qlm\rangle=\frac{G_l(r,q;s)}{(2\pi^2)^{1/2}r}Y_{lm}(\hat{r}), \tag{2.28}$$

and use (2.3) and (2.22), then (2.27) becomes

$$\left[s+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right]G_l(r,q;s)-\int_0^\infty dr'U_l(r,r')G_l(r',q;s)=(s-q^2)rj_l(qr). \tag{2.29}$$

The free wave rj_l is a solution of

$$\left[p^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right]rj_l(pr)=0\,. \tag{2.30}$$

If we multiply (2.29) by $rj_l(pr)$ and (2.30) by $G_l(r,q;s)$, subtract, integrate from $c-\epsilon$ to $c+\epsilon$, and then let $\epsilon\to 0$, we find

$$\left[G_l'(c^+,q;s)-\frac{f_l}{c}G_l(c^+,q;s)\right]cj_l(pc) +\left[G_l(c^-,q;s)-\frac{c}{b_l}G_l'(c^-,q;s)\right]\frac{d}{dc}cj_l(pc)=0\,. \tag{2.31}$$

Since $cj_l(pc)$ and $d\,cj_l(pc)/dc$ are linearly independent, we must have

$$G_l'(c^+,q;s)=\frac{f_l}{c}G_l(c^+,q;s)\,,\qquad G_l'(c^-,q;s)=\frac{b_l}{c}G_l(c^-,q;s)\,. \tag{2.32}$$

Thus, the pseudopotential (2.22) imposes two logarithmic boundary conditions; one just outside c and one just inside c. It is completely trivial to solve (2.29) with the boundary conditions (2.32). The result is

$$G_l(r,q;s)=-\frac{g_l(qc;f_l)}{D_l(kc)}rh_l^{(+)}(kr)+rj_l(qr)\,,\quad r>c\,;$$
$$G_l(r,q;s)=-\frac{g_l(qc;b_l)}{g_l(kc;b_l)}rj_l(kr)+rj_l(qr)\,,\quad 0\le r<c\,. \tag{2.33}$$

We have assumed $s=k^2+i\epsilon$. This dictates using $h_l^{(+)}$ in the outer region. Note that

$$G_l(r,k;k^2+i\epsilon)=0\,,\quad 0\le r<c\,. \tag{2.34}$$

This was to be expected, since (2.29) is the Schrödinger equation when $q=k$. From (2.25) and (2.28), it follows that

$$t_l(p,q;s)=(2\pi^2)^{-1}\int_0^\infty dr\,rj_l(pr)\int_0^\infty dr'U_l(r,r')G_l(r',q;s)\,. \tag{2.35}$$

Combining (2.22), (2.33), and (2.35), the Hoenig-Lomon T matrix is found to be

$$t_l^{\rm HL}(p,q;k^2+i\epsilon)=F_l(p,k)t_l(k,k;k^2+i\epsilon)F_l(q,k)+\frac{c(f_l-b_l)}{2\pi^2}\frac{w_l(pc,kc)w_l(qc,kc)}{g_l(kc;f_l)g_l(kc;b_l)}\,. \tag{2.36}$$

From (2.21) it follows that (2.36) agrees with (2.20) and (2.15) when $p=k$ or $q=k$. It is interesting to note that (2.36) becomes the same as (2.15) if b_l is set equal to f_l; i.e., if the logarithmic derivative of the wave function is continuous at $r=c$, the pseudopotential (2.22) gives the same T matrix as one would get from a well-behaved potential. By well-behaved potential we mean one for which (2.12) is valid.

It is also interesting to note that for p, q, and k real, all three T matrices have the same imaginary part, since they differ by terms which are purely real. Furthermore, since they agree half off shell ($p=k$ or $q=k$), they all satisfy the off-shell unitarity relation[14]

$$\begin{aligned}{\rm Im}\,t_l(p,q;k^2+i\epsilon)&=-2\pi^2kt_l(p,k;k^2+i\epsilon)t_l^*(k,q;k^2+i\epsilon)\\&=-2\pi^2kt_l^*(p,k;k^2+i\epsilon)t_l(k,q;k^2+i\epsilon)\,.\end{aligned} \tag{2.37}$$

In order to see if the differences between the T matrices given by (2.15), (2.20), and (2.36) are of practical significance, values for s-wave T matrices have been calculated using $f=2.1$ and $c=0.71$ F for a variety of momenta and energies. The values f and c are appropriate for the $^1S_0$ state of the two-nucleon system.[20] Samplings of the numerical results are given in Tables I–III. We see from Table I that for $k^2=4$ F^{-2} (a lab energy of about 330 MeV) there are significant differences between the various T matrices. Of course, they agree if $p=k$ or $q=k$, but as soon as one is fully off shell, the disagreement becomes noticeable. Large differences between T-matrix elements are also found at large negative energies (lab energies of about -300 MeV). We see from Table II ($k^2=0$) that at low energies the differences are less dramatic but still significant. It is of interest to note that the Kim-Tubis T matrix (column 4) and the Hoenig-Lomon T matrix with $b=3f$ (column 5) are very similar. This is found to be true, in general, for c.m. energies in the range from 75 down to -75 MeV for most of the momenta given in the tables. Thus, by making an appropriate choice of b, one can use the Hoenig-Lomon T matrix as a good separable approximation to the Kim-Tubis T matrix. This will be of practical value in carrying out three-body calculations in the Faddeev approach. Table III gives some values of the various T matrices for $k^2=-1$ F^{-2}. From these tables, as well as from results which have not been shown, it can be concluded that there is a practical difficulty in going from the half-off-shell pure BCM T matrix to the fully off-shell T matrix. One way to study this difficulty is to carry out three-body calculations. In such calculations it should be sufficient to use the Hoenig-Lomon T matrix, since by simply varying the arbitrary parameter b_l one can produce significant differences in the off-shell T matrix. Of course, in such calculations one

TABLE I. Values of T matrix $t_0(p,q;s)$ for $s=4$ F^{-2}. t^N is given by (2.11), (2.15), and (2.16). t^{KT} is given by (2.11), (2.16), and (2.20). t^{HL} is given by (2.11), (2.16), and (2.36). p and q are in F^{-1}. All values for t_0 have been multiplied by 10^2. $f=2.1$, $c=0.71$ F. We give only the real parts here for t^{KT}, t^{HL} ($b=3f$), and t^{HL} ($b=\infty$), since all three T matrices have exactly the same imaginary part.

p	q	t^N	Re(t^{KT})	Re(t^{HL}), $b=3f$	Re(t^{HL}), $b=\infty$
0.0	0.0	$0.888-i\,0.962$	0.117	0.077	−0.286
0.0	0.5	$0.903-i\,0.978$	0.190	0.152	−0.185
0.0	1.0	$0.944-i\,1.023$	0.397	0.365	0.105
0.0	1.5	$1.001-i\,1.084$	0.704	0.683	0.541
0.0	2.0	$1.059-i\,1.147$	1.059	1.059	1.059
0.5	0.5	$0.918-i\,0.994$	0.259	0.222	−0.089
0.5	1.0	$0.960-i\,1.040$	0.454	0.423	0.183
0.5	1.5	$1.018-i\,1.103$	0.743	0.724	0.592
0.5	2.0	$1.076-i\,1.166$	1.076	1.076	1.076
1.0	1.0	$1.003-i\,1.087$	0.616	0.590	0.404
1.0	1.5	$1.064-i\,1.153$	0.853	0.837	0.736
1.0	2.0	$1.126-i\,1.219$	1.126	1.126	1.126
1.5	1.5	$1.129-i\,1.223$	1.014	1.004	0.948
1.5	2.0	$1.194-i\,1.293$	1.194	1.194	1.194
2.0	2.0	$1.262-i\,1.368$	1.262	1.262	1.262

should include the contribution to the T matrix which arises from the forces outside the boundary-condition radius. We turn our attention to this in the next section.

III. CONTRIBUTION TO THE BCM T MATRIX FROM FORCES OUTSIDE THE CORE

This section consists of two parts. In the first part we derive the exact equations, integral as well as differential, for the contribution to the total T matrix which arises from the forces outside the core. The second part presents three schemes for constructing separable approximations to the T matrix. Such approximations are of use in three-body calculations.

A. Exact T Matrix

We begin by dividing the complete potential into two parts; i.e., we write

$$V=U+W, \tag{3.1}$$

where[21] U stands for the potential or pseudopotential that gives rise to the pure BCM (no forces outside), and W stands for the potential outside the core. We assume W is a local potential, which means it can be written in the form

$$\langle\vec{r}\,|\,W\,|\,\vec{r}\,'\rangle=W(r)\,\delta(\vec{r}-\vec{r}\,'). \tag{3.2}$$

Of course

$$W(r)=0, \quad 0\leqslant r<c. \tag{3.3}$$

TABLE II. T matrices $t_0(p,q;0)$. $f=2.1$, $c=0.71$ F. See caption to Table I for explanation of notation.

p	q	t^N	t^{KT}	t^{HL}, $b=3f$	t^{HL}, $b=\infty$
0.0	0.0	1.884	1.884	1.884	1.884
0.0	0.5	1.916	1.916	1.916	1.916
0.0	1.0	2.003	2.003	2.003	2.003
0.0	1.5	2.125	2.125	2.125	2.125
0.0	2.0	2.247	2.247	2.247	2.247
0.5	0.5	1.948	1.944	1.944	1.942
0.5	1.0	2.037	2.020	2.020	2.015
0.5	1.5	2.160	2.125	2.124	2.115
0.5	2.0	2.285	2.228	2.226	2.211
1.0	1.0	2.130	2.066	2.064	2.047
1.0	1.5	2.259	2.124	2.120	2.083
1.0	2.0	2.389	2.172	2.163	2.104
1.5	1.5	2.396	2.112	2.102	2.025
1.5	2.0	2.534	2.078	2.057	1.932
2.0	2.0	2.680	1.948	1.907	1.705

TABLE III. T matrices $t_0(p,q;-1)$. $f=2.1$, $c=0.71$ F. See caption to Table I for explanation of notation.

p	q	t^N	t^{KT}	t^{HL}, $b=3f$	t^{HL}, $b=\infty$
0.0	0.0	3.094	3.012	3.011	2.993
0.0	0.5	3.146	3.045	3.044	3.021
0.0	1.0	3.290	3.135	3.132	3.097
0.0	1.5	3.489	3.254	3.249	3.196
0.0	2.0	3.690	3.362	3.354	3.279
0.5	0.5	3.199	3.074	3.073	3.045
0.5	1.0	3.345	3.154	3.151	3.107
0.5	1.5	3.548	3.258	3.252	3.186
0.5	2.0	3.752	3.348	3.337	3.245
1.0	1.0	3.498	3.204	3.199	3.132
1.0	1.5	3.710	3.264	3.254	3.152
1.0	2.0	3.924	3.303	3.285	3.142
1.5	1.5	3.935	3.259	3.240	3.085
1.5	2.0	4.161	3.222	3.188	2.971
2.0	2.0	4.401	3.095	3.038	2.733

The complete T matrix is given by

$$T(s) = V + VG(s)V, \quad (3.4)$$

where

$$G(s) = (s - H_0 - V)^{-1}. \quad (3.5)$$

H_0 is the kinetic energy operator. The pure BCM T matrix is given by

$$t(s) = U + UR(s)U, \quad (3.6)$$

where

$$R(s) = (s - H_0 - U)^{-1}. \quad (3.7)$$

We also introduce the free Green's function

$$G_0(s) = (s - H_0)^{-1}. \quad (3.8)$$

The various resolvents or Green's functions are related by the well-known operator indenties

$$G(s) = R(s) + R(s)WG(s),$$
$$G(s) = R(s) + G(s)WR(s), \quad (3.9)$$

and

$$R(s) = G_0(s) + G_0(s)UR(s),$$
$$R(s) = G_0(s) + R(s)UG_0(s). \quad (3.10)$$

We define

$$\Omega(s) = 1 + G_0(s)t(s),$$
$$= 1 + R(s)U, \quad (3.11)$$

and

$$\Sigma(s) = 1 + t(s)G_0(s),$$
$$= 1 + UR(s). \quad (3.12)$$

The two forms of Ω and Σ can be shown to be identical by using (3.6) and (3.10). By combining (2.28) and (2.33), we can write out Ω for the Hoenig-Lomon pseudopotential

$$\langle \vec{r} | \Omega(s) | qlm \rangle = (2\pi^2)^{-1/2}\left[j_l(qr) - \frac{g_l(qc;f_l)}{D_l(kc)} h_l^{(+)}(kr)\right] Y_{lm}(\hat{r}), \quad r \geq c. \quad (3.13)$$

g_l and D_l are given by (2.7) and (2.17), respectively. As will be seen shortly, we do not need Ω inside the core in order to calculate the contribution to T from the outside forces. Furthermore, it is easy to show that the other two prescriptions for the pure BCM lead to the same result for Ω when $r \geq c$. In order to find an expression for Σ, we can use the identity

$$\Sigma(s) = \Omega^{\dagger}(s^*), \quad (3.14)$$

which follows from (3.11) and (3.12). From this identity one can show that

$$\langle qlm | \Sigma(s) | \vec{r} \rangle = \langle \vec{r} | \Omega(s^*) | qlm \rangle^*. \quad (3.15)$$

Calculating $\langle \vec{r} | \Omega(s^*) | qlm \rangle$ simply amounts to replacing $h_l^{(+)}$ by $h_l^{(-)}$.

The relation between the full T matrix and the pure BCM T matrix can be obtained by using (3.4), (3.6), (3.9), (3.11), and (3.12). The result is given by the relations

$$T(s) = t(s) + t^{(1)}(s), \quad (3.16)$$

$$t^{(1)}(s) = \Sigma(s)\tau(s)\Omega(s), \quad (3.17)$$

$$\tau(s) = W + WG(s)W. \quad (3.18)$$

From (3.9) it follows that the T-matrix-like operator $\tau(s)$ satisfies

$$\tau(s) = W + WR(s)\tau(s). \quad (3.19)$$

Using the identity

$$[1 - UG_0(s)][1 + UR(s)] = 1, \quad (3.20)$$

which can be derived from (3.10), one easily demonstrates that the contribution to the T matrix from the outside forces is the solution of

$$t^{(1)}(s) = \Sigma(s)W\Omega(s) + \Sigma(s)WG_0(s)t^{(1)}(s). \quad (3.21)$$

This equation can easily be written out in momentum space by using

$$\langle plm | \Sigma(s)W\Omega(s) | qlm \rangle = \int \langle plm | \Sigma(s) | \vec{r} \rangle d\vec{r}\, W(r) \langle \vec{r} | \Omega(s) | qlm \rangle, \quad (3.22)$$

$$\langle plm | \Sigma(s)W | qlm \rangle = \int \langle plm | \Sigma(s) | \vec{r} \rangle d\vec{r}\, W(r) \langle \vec{r} | qlm \rangle. \quad (3.23)$$

As promised above we see from (3.2) and (3.21)–(3.23) that $t^{(1)}(s)$ does not depend on the form of $\Sigma(s)$ and $\Omega(s)$ inside the core. Since all of the prescriptions give the same result for $\Sigma(s)$ and $\Omega(s)$ for $r \geq c$, there is no ambiguity in the contribution to the T matrix from the outside forces. Equation (3.21) agrees with that of Kim and Tubis[11] if one makes allowances for differences in normalization. The derivation presented here, while not as rigorous as theirs, is certainly more transparent.

The contribution to the T matrix from the forces outside the core can also be obtained by solving a differential, rather than an integral equation. In order to formulate this approach, we introduce an operator $\Gamma(s)$ defined by

$$\Gamma(s) = \Omega(s) + G_0(s)t^{(1)}(s). \quad (3.24)$$

$\Omega(s)$ is given by (3.11). From (3.21) it follows that

$$t^{(1)}(s) = \Sigma(s)W\Gamma(s). \quad (3.25)$$

Using (3.10), (3.12), (3.24), and (3.25), we obtain

$$\Gamma(s) = \Omega(s) + R(s)W\Gamma(s). \quad (3.26)$$

It is most convenient to study (3.26) in a mixed representation.

$$\langle \vec{r}\,|\,\Gamma(s)\,|\,qlm\rangle = \langle \vec{r}\,|\,\Omega(s)\,|\,qlm\rangle + \int \langle \vec{r}\,|R(s)|\,\vec{r}'\rangle\, d\vec{r}'\, W(r')\langle \vec{r}\,'\,|\,\Gamma(s)\,|\,qlm\rangle . \tag{3.27}$$

From (3.13) we have, assuming $s = k^2 + i\epsilon$,

$$(s+\nabla^2)\langle \vec{r}\,|\,\Omega(s)\,|\,qlm\rangle = (s-q^2)\langle r\,|\,qlm\rangle , \quad r > c . \tag{3.28}$$

Since $R(s)$ is the Green's function for the pure BCM, it is obvious that

$$(s+\nabla^2)\langle \vec{r}\,|R(s)|\,\vec{r}'\rangle = \delta(\vec{r}-\vec{r}') , \quad r \text{ and } r' > c . \tag{3.29}$$

It is easy to show that this Green's function is given by the relations

$$\langle \vec{r}\,|R(s)|\,\vec{r}'\rangle = \sum_{l,m} \frac{R_l(r,r';s)}{rr'}\, Y_{lm}(\hat{r})Y_{lm}(\hat{r}') , \tag{3.30}$$

$$R_l(r,r';s) = -krr'\left[j_l(kr_<)h_l^{(+)}(kr_>) - \frac{g_l(kc;f_l)}{D_l(kc)}\, h_l^{(+)}(kr)h_l^{(+)}(kr')\right] , \tag{3.31}$$

where g_l and D_l are given by (2.7) and (2.17). Combining (3.27)–(3.29) we obtain the differential equation for $\Gamma(s)$,

$$[s+\nabla^2 - W(r)]\langle \vec{r}|\,\Gamma(s)\,|qlm\rangle = (s-q^2)\langle \vec{r}|qlm\rangle , \quad r > c . \tag{3.32}$$

If we use (3.13), (3.27), (3.30), (3.31), and let

$$\langle \vec{r}\,|\,\Gamma(s)\,|\,qlm\rangle = \frac{\Gamma_l(r,q;s)}{(2\pi^2)^{1/2} r}\, Y_{lm}(\hat{r}) , \tag{3.33}$$

it is easy to show that

$$\left.\frac{d\Gamma_l(r,q;s)}{dr}\right|_{r=c} = \frac{f_l}{c}\,\Gamma_l(c,q;s) . \tag{3.34}$$

The boundary condition at infinity is obviously (assuming $s = k^2 + i\epsilon$, $\epsilon > 0$)

$$\Gamma_l(r,q;s) \underset{r\to\infty}{\simeq} r j_l(qr) + r c_l h_l^{(+)}(kr) , \tag{3.35}$$

where c_l is some constant. After one solves (3.32) subject to the boundary conditions (3.34) and (3.35), one can find $t^{(1)}(s)$ in momentum space by using (3.25) in the form

$$\langle plm\,|\,t^{(1)}(s)\,|\,qlm\rangle = \int \langle plm\,|\,\Sigma(s)\,|\,\vec{r}\rangle\, d\vec{r}\, W(r)\langle \vec{r}\,|\,\Gamma(s)\,|\,qlm\rangle . \tag{3.36}$$

This differential-equation approach should be of use in obtaining closed-form results for simple models such as a square well outside the core. We will present results for the square-well model in the near future. We now turn our attention to separable representations of $t^{(1)}(s)$.

B. Separable Representations of the T Matrix

In this section we will derive three different schemes for producing separable expansions for the contribution to the T matrix from the forces outside the core. Two of these schemes[16] are generalizations of those obtained for hard-core potentials and could be obtained by extending the methods used in the references just given; however, we will derive the results here in a simpler fashion. The expansions for potentials with hard cores are special cases of those we are about to give, and can be recovered by letting f_l [see (2.2)] become infinite.

We begin by considering the differential equation

$$\left[s + \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right]\phi_{\nu l}(r;s) = \lambda_\nu(s)W(r)\phi_{\nu l}(r;s), \quad r > c . \tag{3.37}$$

The solutions of this are to satisfy the boundary conditions

$$\left.\frac{d}{dr}\phi_{\nu l}(r;s)\right|_{r=c} = \frac{f_l}{c}\,\phi_{\nu l}(r;s) , \tag{3.38}$$

and

$$\phi_{\nu l}(r;s) \sim e^{ikr} , \quad s = k^2 + i\epsilon , \quad \epsilon > 0 , \qquad \phi_{\nu l}(r;s^*) \underset{r\to\infty}{\simeq} e^{-ikr} . \tag{3.39}$$

From the nature of the boundary conditions it is clear that we can only obtain solutions of (3.37) for a discrete set of eigenvalues $\lambda_\nu(s)$. From (3.37) we have

$$\left[s + \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2}\right]\phi_{\nu l}{}^*(r;s^*) = \lambda_\nu{}^*(s^*)W\phi_{\nu l}{}^*(r;s^*) . \tag{3.40}$$

Since $\phi_{\nu l}{}^*(r;s^*)$ satisfies the same differential equation and the same boundary conditions [see (3.38) and (3.39)] as $\phi_{\nu l}(r;s)$, it follows that

$$\phi_{\nu l}(r;s^*) = \phi_{\nu l}{}^*(r;s) , \tag{3.41}$$

$$\lambda_\nu(s^*) = \lambda_\nu{}^*(s) . \tag{3.42}$$

It is trivial to show, using (3.37)–(3.39), that

$$(\lambda_\nu - \lambda_\mu)\int_c^\infty \phi_{\mu l}{}^*(r;s^*)W(r)\phi_{\nu l}(r;s)dr = 0 . \tag{3.43}$$

Thus for complex s we have a set of biorthogonal eigenfunctions,[22] with the orthogonality relation

$$\langle \Phi_\mu(s^*)|W|\Phi_\nu(s)\rangle = 0, \quad \mu \neq \nu, \tag{3.44}$$

and the completeness relation

$$W = \sum_\nu \frac{W|\Phi_\nu(s)\rangle\langle\Phi_\nu(s^*)|W}{\langle\Phi_\nu(s^*)|W|\Phi_\nu(s)\rangle}. \tag{3.45}$$

The completeness relation gives us a separable representation for the potential W. It is easy to convert the differential equation (3.37) with the boundary conditions (3.38) and (3.39) into an integral equation. One easily checks by using (3.29) –(3.31) that the integral equation is

$$R(s)W|\Phi_\nu(s)\rangle = |\Phi_\nu(s)\rangle\,\zeta_\nu(s), \tag{3.46}$$

where

$$\zeta_\nu(s) = \lambda_\nu^{-1}(s). \tag{3.47}$$

Combining (3.17), (3.19), (3.45), and (3.46), we obtain

$$t^{(1)}(s) = \sum_\nu \frac{\Sigma(s)W|\Phi_\nu(s)\rangle\langle\Phi_\nu(s^*)|W\Omega(s)}{\langle\Phi_\nu(s^*)|W|\Phi_\nu(s)\rangle}\,\frac{1}{1-\zeta_\nu(s)}. \tag{3.48}$$

This is a separable representation for $t^{(1)}(s)$. If one obtains the functions Φ_ν by solving the differential equation (3.37), the factors in (3.48) can easily be obtained in momentum space by doing an integral; e.g.,

$$\langle plm|\Sigma(s)W|\Phi_\nu(s)\rangle = \int \langle plm|\Sigma(s)|\vec{r}\rangle\, d\vec{r}\, W(r)\langle\vec{r}|\Phi_\nu(s)\rangle. \tag{3.49}$$

From (3.10) and (3.12) it follows that

$$R(s) = G_0(s)\Sigma(s); \tag{3.50}$$

hence the integral equation (3.46) can easily be written in momentum space. It becomes

$$(s-p^2)^{-1}\int_0^\infty \langle plm|\Sigma(s)W|qlm\rangle\, 4\pi q^2 dq \langle qlm|\Phi_\nu(s)\rangle = \langle plm|\Phi_\nu(s)\rangle\,\zeta_\nu(s). \tag{3.51}$$

The factors in (3.48) are given by, e.g.,

$$\langle plm|\Sigma(s)W|\Phi_\nu(s)\rangle = (s-p^2)\langle plm|\Phi_\nu(s)\rangle\zeta_\nu(s), \tag{3.52}$$

and are therefore directly obtainable from (3.51). The results (3.51) and (3.52) give a more practical scheme for constructing the expansion for $t^{(1)}(s)$ than the method given previously for hard-core potentials,[16] although the formal equations are similar.

If the potential W is a negative definite operator, i.e., if

$$\langle\Psi|W|\Psi\rangle \leq 0, \tag{3.53}$$

where $|\Psi\rangle$ is an arbitrary vector, then it is obvious from (3.45) that for negative, real energies

$$\langle\Psi|\sum_{\nu=1}^{N} \frac{W|\Phi_\nu(s)\rangle\langle\Phi_\nu(s)|W|\Psi\rangle}{\langle\Phi_\nu(s)|W|\Phi_\nu(s)\rangle} \geq \langle\Psi|W|\Psi\rangle. \tag{3.54}$$

Thus, for negative, real energies, the separable potential obtained by truncating (3.45) is always less attractive than the original potential W, if W itself is a purely attractive potential. If one carries out a calculation of the triton binding energy by using one of the separable T matrices presented in Sec. II for the core region, combined with a truncation of (3.48) for $t^{(1)}(s)$, the result will be an upper bound on the three-body energy. An alternative would be to use (3.45) but keep the energy s fixed. One no longer has (3.48) but rather an expansion for $t^{(1)}(s)$ which is a generalization of the so-called unitary pole expansion of Harms.[17] The upper-bound property would still be true.

We now turn our attention to deriving another separable representation for that part of the T matrix which arises from the forces outside the core. This expansion is also based on the differential equation (3.37) with the boundary condition (3.38), however we no longer use (3.39) but rather

$$\langle\vec{r}|\Phi_\nu(s)\rangle \underset{r\to\infty}{\simeq} \langle\vec{r}|klm], \tag{3.55}$$

where

$$\langle\vec{r}|klm] = \langle\vec{r}|\Omega(k^2 \pm i\epsilon)|klm\rangle e^{\mp i\delta_l^c(k)}. \tag{3.56}$$

From (3.11) it follows that $|klm]$ is a solution of the Schrödinger equation for the pure BCM. δ_l^c is the phase shift for the pure BCM. It is easy to show, using (3.13), that putting in the phase factor in (3.56) makes the radial part of (3.13) real, for k real. The set of functions obtained by solving (3.37) with the boundary conditions (3.38) and (3.55) form an orthogonal rather than biorthogonal set for both positive and negative real energies. Under such circumstances one does need the star in (3.44) and (3.45). It is obvious that one solution of (3.37), (3.38), and (3.55) is the BCM wave function itself with an eigenvalue λ equal to zero. We label these with the subscript zero; i.e.,

$$\lambda_0 = 0, \qquad |\Phi_0(s)\rangle = |klm]. \tag{3.57}$$

The system (3.37), (3.38), and (3.55) can be converted to the integral equation

$$|\Phi_\nu(s)\rangle = |klm] + R(s)\lambda_\nu W|\Phi_\nu(s)\rangle, \tag{3.58}$$

where $R(s)$ is given by (3.30) and (3.31). From (3.57), we see that (3.58) is obviously true for

$\nu = 0$. It is a straightforward matter to check (3.58) for the other values of ν by using (3.30), (3.31), (3.44) without the star, and (3.57). In order to make (3.58) look more like an eigenvalue problem, we introduce

$$|\pi_\nu(s)\rangle = |klm] - |\Phi_\nu(s)\rangle . \qquad (3.59)$$

Combining (3.58) and (3.59) we can write

$$R(s)W_1(s)|\pi_\nu(s)\rangle = |\pi_\nu(s)\rangle \zeta_\nu(s) , \quad \nu \neq 0 , \qquad (3.60)$$

where

$$W_1(s) = W - \frac{W|klm][klm|W}{[klm|W|klm]} , \qquad (3.61)$$

and ζ_ν is given by (3.47). In deriving (3.60) we have used the fact that

$$[klm|W|\Phi_\nu(s)\rangle = 0 , \quad \nu \neq 0 . \qquad (3.62)$$

In order to derive a separable representation for $t^{(1)}(s)$, we first derive a separable representation for $\tau(s)$ [see (3.16)–(3.19)]. To do this we insert the completeness relation into (3.19) and obtain

$$\tau(s) = \sum_{\nu=0}^{\infty} \frac{W|\Phi_\nu(s)\rangle\langle A_\nu(s)|}{\langle\Phi_\nu(s)|W|\Phi_\nu(s)\rangle} , \qquad (3.63)$$

where

$$\langle A_\nu(s)| = \langle\Phi_\nu(s)|W[1 + R(s)\tau(s)] . \qquad (3.64)$$

Combining (3.63), (3.64), and (3.58), we find

$$\langle A_\nu(s)| = \frac{\langle\Phi_\nu(s)|W}{1-\zeta_\nu(s)} + \frac{1}{1-\lambda_\nu} \langle A_0(s)| , \quad \nu \neq 0 . \qquad (3.65)$$

Inserting (3.65) into (3.63) we have

$$\tau(s) = \sum_{\nu=0}^{\infty} \frac{W|\Phi_\nu(s)\rangle\langle A_0(s)|}{\langle\Phi_\nu(s)|W|\Phi_\nu(s)\rangle} \frac{1}{1-\lambda_\nu(s)} + \tau_1(s) , \qquad (3.66)$$

where

$$\tau_1(s) = \sum_{\nu=1}^{\infty} \frac{W|\Phi_\nu(s)\rangle\langle\Phi_\nu(s)|W}{\langle\Phi_\nu(s)|W|\Phi_\nu(s)\rangle} \frac{1}{1-\zeta_\nu(s)} . \qquad (3.67)$$

Using (3.62) we find

$$\tau(s) = \frac{\tau(s)|klm][klm|\tau(s)}{[klm|\tau(s)|klm]} + \tau_1(s) , \qquad (3.68)$$

and

$$\frac{\tau(s)|klm]}{[klm|\tau(s)|klm]} = \sum_{\nu=0}^{\infty} \frac{W|\Phi_\nu(s)\rangle}{\langle\Phi_\nu(s)|W|\Phi_\nu(s)\rangle} \frac{1}{1-\lambda_\nu(s)} . \qquad (3.69)$$

Combining (3.11), (3.12), (3.14), (3.17), (3.57), and (3.68), we finally arrive at another separable representation for $t^{(1)}(s)$,

$$t^{(1)}(s) = \frac{t^{(1)}(s)|klm\rangle\langle klm|t^{(1)}(s)}{\langle klm|t^{(1)}(s)|klm\rangle} + \Sigma(s)\tau_1(s)\Omega(s) . \qquad (3.70)$$

It is obvious from (3.14), (3.57), (3.62), and (3.67) that any approximation to $t^{(1)}(s)$ obtained by truncating (3.70) (as long as one keeps the first term on the right-hand side) is exact half off the energy shell. Furthermore, one can show by using the arguments of Sec. IV (see second work Ref. 16) that the T matrix obtained by adding a truncated version of (3.70) to any of the T matrices obtained in Sec. II, exactly satisfies the off-shell unitarity relation (2.37).

The factors in the second term on the right-hand side of (3.70) can be obtained by using

$$\Sigma(s)W|\Phi_\nu(s)\rangle = -\Sigma(s)W_1(s)|\pi_\nu(s)\rangle , \qquad (3.71)$$

which follows from (3.59), (3.61), and (3.62), and by solving

$$(s-p^2)^{-1} \int_0^\infty \langle plm|\Sigma(s)W_1(s)|qlm\rangle 4\pi q^2 dq \langle qlm|\pi_\nu(s)\rangle = \langle plm|\pi_\nu(s)\rangle\zeta_\nu(s) , \qquad (3.72)$$

which we have obtained from (3.50) and (3.60). From (3.14), (3.56), and (3.61), we have

$$\langle klm|\Sigma(s)W_1(s)|qlm\rangle = 0 , \quad s = k^2 + i\epsilon ; \qquad (3.73)$$

hence, in general, one will not have a singularity in (3.72) when $p = k$. This is one practical advantage this expansion has over the one presented earlier in this section.

IV. SUMMARY AND DISCUSSION

A number of important results have been obtained for the BCM T matrix. In particular, it has been shown that the half-off-shell BCM T matrix is unique, but the fully off-shell T matrix is not unique. It has been demonstrated that the ambiguity occurs only in that part of the complete T matrix which is identifiable as the T matrix of the pure BCM (no outside forces). In other words, the Kim-Tubis[11] integral equation for the contribution to the T matrix from the outside forces has been shown not to depend on the potential inside the boundary-condition radius. A differential-equation approach has also been developed for finding the contribution to the T matrix from the outside forces.

Numerical results obtained for the $^1S_0$ pure BCM T matrix show that significant differences do occur when one uses various prescriptions for producing an energy-independent boundary condition at the core radius. It is of interest to note that the pure BCM T matrix, which was obtained by only assuming that the potential inside the core was well behaved, turned out to be exactly separable. This is of great calculational convenience, and it would be pleasant if this turned out to be the best prescription for going fully off shell. It is probably significant that the T matrix obtained from the Hoenig-Lomon pseudopotential[9] goes over into the T matrix just mentioned if one assumes that the logarithmic derivative of the Schrödinger wave function is continuous in the neighborhood of the core radius. Such continuity is certainly physically resonable. Of practical significance is the fact that by a proper choice of the parameter b_l (the logarithmic derivative just inside the core radius) in the Hoenig-Lomon pseudopotential[9] one can produce a two-term separable T matrix that is similar to that of Kim and Tubis.[11] Thus by using the Hoenig-Lomon T matrix in a three-body calculation, for example, and by varying b_l, one should be able to assess the effect of the ambiguity in the pure BCM T matrix. Of course, one must be able to include the effect of the outside forces in such a calculation. The separable representations developed in part B of Sec. III are one way of doing this. The practicality of using these expansions in three-body calculations with hard-core potentials has been demonstrated previously.[16]

A calculational program based on the results of this paper is now under way. In particular, the BCM T matrix for simple force models outside the core is being calculated in order to see if the ambiguities in the pure BCM are masked by the contribution to the T matrix from the outside forces. Also, calculations on the three-nucleon system using simple force models are under way to see if the nonuniqueness of the T matrix makes itself felt in a system in which the fully off-shell T matrix plays a role.

In order to carry out calculations with realistic forces the results of this paper must be extended to tensor forces. This is being done.

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[1]G. Breit and W. G. Bouricius, Phys. Rev. 75, 1029 (1949).
[2]H. Feshbach and E. L. Lomon, Phys. Rev. 102, 891 (1956).
[3]E. L. Lomon and M. Nauenberg, Nucl. Phys. 24, 474 (1961).
[4]H. Feshbach, E. L. Lomon, and A. Tubis, Phys. Rev. Letters 6, 635 (1961).
[5]E. L. Lomon and H. Feshbach, Rev. Mod. Phys. 39, 611 (1967); Ann. Phys. (N.Y.) 48, 94 (1968).
[6]H. Feshbach and E. L. Lomon, Ann. Phys. (N.Y.) 29, 19 (1964).
[7]E. L. Lomon and M. McMillan, Ann. Phys. (N.Y.) 23, 439 (1963).
[8]M. Razavy and D. W. L. Sprung, Phys. Rev. 133, B300 (1964).
[9]M. M. Hoenig and E. L. Lomon, Ann. Phys. (N.Y.) 36, 363 (1966).
[10]H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) A238, 551 (1957).
[11]Y. E. Kim and A. Tubis, Phys. Rev. C 1, 414 (1970).
[12]H. P. Noyes, Phys. Rev. Letters 15, 538 (1965).
[13]M. Baranger, G. Giraud, S. K. Mukopadhyay, and P. U. Sauer, Nucl. Phys. A138, 1 (1969).
[14]C. Lovelace, Phys. Rev. 135, B1225 (1964).
[15]J. M. J. Van Leeuwen and A. S. Reiner, Physica 27, 99 (1961); R. Lauglin and B. L. Scott, Phys. Rev. 171, 1196 (1968).
[16]M. G. Fuda, Phys. Rev. 178, 1682 (1969); 186, 1078 (1969).
[17]E. Harms, Phys. Rev. C 1, 1667 (1970).
[18]L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [transl.: Soviet Phys.–JETP 12, 1014 (1961)].
[19]A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1965).
[20]H. Enge, *Introduction to Nuclear Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1966).
[21]If U contains potentials which are infinite in strength, one should think of U as finite until the final equations are derived, at which point the strengths can be allowed to approach infinity.
[22]K. Meetz, J. Math. Phys. 3, 690 (1961); S. Weinberg, Phys. Rev. 131, 440 (1963); R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill Book Company, Inc., New York, 1966).