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Convergence of the Sasakawa Expansion for the Scattering Amplitude*

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Sasakawa has rewritten Schrödinger's integral equation in such a manner that the inhomogeneous term has the asymptotic behavior of the exact scattering wave function. This paper gives a proof that the iterative solution for the scattering amplitude converges for all local potentials for which the function $rV(r)$ is absolutely integrable. Under very general conditions the kernel of the Sasakawa equation is a Hilbert-Schmidt kernel. The integral equation is convenient for numerical solution in both the radial representation and the momentum representation.

I. INTRODUCTION

A scattering wave function $\varphi(r)$ may be written as a sum of that solution of the unperturbed Schrödinger equation, $\varphi^0(r)$, that is asymptotically equal to $\varphi(r)$, plus a function $\chi(r)$ that vanishes for large r. The function $\varphi^0(r)$ depends on the unknown scattering amplitude. Schrödinger's integral equation yields a coupled set of linear equations for $\chi(r)$ and the scattering amplitude. The Sasakawa[1] expansion is the iterative solution of these equations. It appears to have the remarkable property that it converges for arbitrary potential strength. Sasakawa has shown this for the square-well, the exponential, and the Yukawa potential.[1] Recently, Austern[2] has advocated the use of this approximation for coupled-channel reaction problems.

It seems worthwhile to gain better insight into the convergence properties by a more general proof. Section IV gives such a proof for all local potentials that satisfy the condition

$$\int_0^\infty dr\, r|V(r)| < \infty . \tag{1}$$

Under this condition the series can be majorized by an exponential series. The locality of the potential is essential for the proof. The series certainly does not converge for all interesting potentials of arbitrary strength. Any potential of rank 1 is a counterexample that disproves the conjecture that it does.

The Sasakawa expansion allows an obvious generalization. It is possible to modify the function $\varphi^0(r)$ at small distances while retaining its asymptotic form. There is a corresponding change in the kernel of the integral equation.

Section II will serve to introduce Sasakawa's form of Schrödinger's integral equation, including its generalizations, and to establish the notation. Sasakawa formulated the scattering problem in the radial representation. Better insight can be obtained by using both the radial and the momentum representation; they complement each other. Qualitative features that are obvious in one representation may not be apparent in the other. The transformation from one representation to the other is straightforward. In the momentum representation it is easy to see that Sasakawa's equations are closely related to those of Noyes[3] and Kowalski.[4]

In Sec. III it is proved under appropriate assumptions that the Sasakawa kernel is a Hilbert-Schmidt kernel. Its properties are important for noniterative numerical solutions.

II. VARIOUS FORMS OF THE SCHRÖDINGER EQUATION

Let the Hamiltonian for the Lth partial wave be represented by the kernel

$$(r'|H|r) = (r'|H_0|r) + \delta(r'-r)V(r) , \tag{2}$$

where

$$(r'|H_0|r) = \delta(r'-r)\left(-\frac{d^2}{dr^2} + \frac{L(L+1)}{r^2} + V_0(r)\right) . \tag{3}$$

The functions $f_L(r,k)$ shall be the regular solutions of the differential equation

$$\left(-\frac{d^2}{dr^2} + \frac{L(L+1)}{r^2} + V_0(r) - k^2\right) f_L(r,k) = 0 , \tag{4}$$

normalized such that

$$\int_0^\infty dr f_L(r,k') f_L(r,k) = \delta(k'-k) , \tag{5}$$

and

$$\int_0^\infty dk f_L(r',k) f_L(r,k) = \delta(r'-r) . \tag{6}$$

The completeness relation (6) implies the assumption that the Hamiltonian H_0 has no bound states. For $V_0(r)=0$, the function $f_L(r,k)$ is related to the spherical Bessel function $j_L(kr)$ by

$$f_L(r,k) = (2/\pi)^{1/2} rk\, j_L(kr)\,. \tag{7}$$

If $V_0(r) \neq 0$, then the f_L are the so-called distorted waves. We are interested in the scattering produced by the potential $V(r)$. The functions f_L, and hence the phase shifts produced by V_0, are considered known.

The momentum representation of any operator Θ is related to the radial representation by

$$(k'|\Theta|k) = \int_0^\infty dr' \int_0^\infty dr\, f_L(r',k)(r'|\Theta|r) f_L(r,k)\,. \tag{8}$$

We thus have

$$(k'|H_0|k) = \delta(k'-k)k^2\,, \tag{9}$$

and

$$(k'|V|k) = \int_0^\infty dr\, f_L(r,k') V(r) f_L(r,k)\,. \tag{10}$$

The scattering wave function $\varphi_L(r,k)$ is the mixed radial representation of the Møller[5] operator Ω, i.e.,

$$\varphi_L(r,k) = (r|\Omega|k) = \int_0^\infty dk' f_L(r,k')(k'|\Omega|k)\,. \tag{11}$$

The Møller matrix $(k'|\Omega|k)$ satisfies the Schrödinger integral equation

$$(k'|\Omega|k) = \delta(k'-k) + \lim_{\epsilon\to 0}\int_0^\infty dk''(k^2 - k'^2 + i\epsilon)^{-1} \times (k'|V|k'')(k''|\Omega|k)\,. \tag{12}$$

The corresponding radial integral equation is

$$\varphi_L(r,k) = f_L(r,k) + \int_0^\infty dr' G(r,r') V(r') \varphi_L(r',k)\,, \tag{13}$$

where the Green's function $G(r,r')$ is defined by

$$G(r,r') = \lim_{\epsilon\to 0}\int_0^\infty dk' \frac{f_L(r,k')f_L(r',k')}{k^2 - k'^2 + i\epsilon} = -(\pi/2k)[h_L(r,k)f_L(r',k)\theta(r-r') + f_L(r,k)h_L(r',k)\theta(r'-r)]\,, \tag{14}$$

and the function $h_L(r,k)$ is the solution of the differential equation (4) that satisfies the outgoing-wave boundary condition. The integral over k' in Eq. (14) can be evaluated by using the analytic properties of the functions $f_L(r,k)$.[6] The step function θ is

$$\theta(x) = \begin{cases} 1, & \text{if } x>0\,; \\ 0, & \text{if } x<0\,. \end{cases} \tag{15}$$

The scattering amplitude is obtained from the kernel of the operator T defined by

$$T \equiv V\Omega\,. \tag{16}$$

The phase shift δ_L is related to T by

$$e^{2i\delta_L} - 1 = -2\pi i (k|T|k)/2k\,. \tag{17}$$

Equation (12) is quite formal, and a side remark about the limits involved is in order. The kernel $(k'|\Omega|k)$ is supposed to represent the operator Ω defined by the strong operator limit

$$\operatorname*{s-lim}_{t\to-\infty} e^{iHt}e^{-iH_0t} = \Omega\,. \tag{18}$$

It can be shown that if the limit (18) exists, then the strong vector limit

$$\operatorname*{s-lim}_{\epsilon\to 0}\int dk \frac{(k'|T|k)\varphi(k)}{k^2 - k'^2 + i\epsilon} \tag{19}$$

also exists for square-integrable $\varphi(k)$. However, this condition is not sufficient for the existence of cross sections. For that purpose it is necessary that $(k|T|k)$ be a continuous function of k. From Eqs. (12) and (16) it then follows that the expression

$$\lim_{\epsilon\to 0}\int dk \frac{(k|V|k')(k'|T|k)}{k^2 - k'^2 + i\epsilon} \tag{20}$$

must converge to a continuous function of k. The properties of the potential should guarantee that these conditions are satisfied.

Let us return to the main topic. Sasakawa's expansion is based on rewriting Eq. (13) or (12) in the general form

$$\Omega = \Omega_0 + A\Omega\,, \tag{21}$$

where $(r|\Omega_0|k)$ has the same asymptotic form as $(r|\Omega|k)$; i.e., the function $(r|A\Omega|k)$ vanishes asymptotically. It follows that Ω_0 is defined such that the kernel $(k'|A|k'')$ does not have a singularity for $k' = k$. These requirements are satisfied if

$$(k'|\Omega_0|k) = \delta(k'-k) + (k' - k'^2 + i\epsilon)^{-1}(k'|\gamma|k)t(k)\,, \tag{22}$$

where

$$t(k) \equiv (k|T|k)\,, \tag{23}$$

and $(k'|\gamma|k)$ is a smooth function restricted by the conditions that $(1+|k'|)^{-1}(k'|\gamma|k)$ is bounded and

$$(k|\gamma|k) = 1\,. \tag{24}$$

The kernel A is then given by

$$(k'|A|k'') = (k^2 - k'^2)^{-1}[(k'|V|k'') - (k'|\gamma|k)(k|V|k'')]\,. \tag{25}$$

The radial representation of Ω_0 is

$$(r|\Omega_0|k) = f_L(r,k) - (\pi/2k)H_L(r,k)t(k)\,, \tag{26}$$

where

$$-\frac{\pi}{2k} H_L(r,k) = \lim_{\epsilon\to 0}\int_0^\infty dk' \frac{f_L(r,k')(k'|\gamma|k)}{k^2-k'^2+i\epsilon}. \quad (27)$$

The radial representation of the kernel A is

$$(r'|A|r'') = \bar{G}(r',r'')V(r''), \quad (28)$$

where

$$\bar{G}(r',r'') = (\pi/2k)\{[h_L(r',k)f_L(r'',k) - f_L(r',k)h_L(r'',h)]\theta(r''-r') + [H_L(r',k)-h_L(r',k)]f_L(r'',k)\}. \quad (29)$$

The exact solution of Eq. (21) is

$$\Omega = (1-A)^{-1}\Omega_0. \quad (30)$$

According to Eqs. (23), (16), and (30), the exact expression for the scattering amplitude $t(k)$ is

$$t(k) = (k|V(1-A)^{-1}|k)/[1-J(k)], \quad (31)$$

where

$$J(k) = \lim_{\epsilon\to 0}\int_0^\infty dk' \frac{(k|V(1-A)^{-1}|k')(k'|\gamma|k)}{k^2-k'^2+i\epsilon} = -(\pi/2k_0)\int_0^\infty dr(k|V(1-A)^{-1}|r)H_L(r,k). \quad (32)$$

The iterative solution is equivalent to the expansion

$$(1-A)^{-1} = \sum_{n=0}^{\infty} A^n. \quad (33)$$

The preceding equations reduce to those of Sasakawa for

$$(k'|\gamma|k) = (k'|\gamma_0|k) \equiv \begin{cases} k'/k, & \text{if } L \text{ is even,} \\ 1, & \text{if } L \text{ is odd.} \end{cases} \quad (34)$$

The equations of Kowalski[4] follow for

$$(k'|\gamma|k) = (k'|V|k)/(k|V|k). \quad (35)$$

In that case $(k'|A|k)=0$, and hence

$$(k'|T|k) = (k^2-k'^2)(k'|\Omega|k) = (k'|V|k)t(k)/(k|V|k) + \int_0^\infty dk''(k'|B|k'')(k''|T|k), \quad (36)$$

where

$$(k'|B|k'') = \left[(k'|V|k'') - \frac{(k'|V|k)(k|V|k'')}{(k|V|k)}\right](k^2-k''^2)^{-1}. \quad (37)$$

III. PROPERTIES OF THE SASAKAWA KERNEL

For the sake of convenience and precision, let $V_0=0$ in the following. In this case the functions $f_L(r,k)$ are functions of kr, i.e., $f_L(r,k)=f_L(kr)$. We do not attempt to prove results under the weakest possible assumptions for the potential. Instead the aim is to make assumptions that are strong enough to allow simple proofs while including all potentials that are of interest in practice except the Coulomb potential and potentials that are more singular than r^{-1} at the origin.

For local potentials the kernel $(k'|V|k'')$ and its derivatives satisfy the inequalities

$$k'^{-1}|(k'|V|k'')| = \left|\int_0^\infty dr f(k'r)(k'r)^{-1}f(k''r)rv(r)\right| \le bb_1\int_0^\infty dr\, r|V(r)|, \quad (38)$$

$$\left|\frac{\partial}{\partial k'}(k'|V|k'')\right| = \left|\int_0^\infty dr f'(k'r)f(k''r)rV(r)\right| \le bb'\int_0^\infty dr\, r|V(r)|, \quad (39)$$

$$\left|\frac{\partial}{\partial k'}\frac{\partial}{\partial k''}(k'|V|k'')\right| \le b'^2\int_0^\infty dr\, r^2|V(r)|, \quad (40)$$

$$k'^{-2}\int_0^\infty dk''|(k'|V|k'')|^2 = \int_0^\infty dr|rV(r)|^2 f(k'r)^2(k'r)^{-2} \le b_1^2\int_0^\infty dr|rV(r)|^2, \quad (41)$$

and

$$\int_0^\infty dk''\left|\frac{\partial}{\partial k''}(k'|V|k'')\right|^2 \le b'^2\int_0^\infty dr|rV(r)|^2, \quad (42)$$

where the constants b, b_1, and b' are defined by

$$b = \sup_x |f(x)|,$$

$$b_1 = \sup_x |f(x)/x|,$$

and

$$b' = \sup_x |f'(x)|.$$

Let us assume that the integrals on the right-hand side of Eqs. (38)–(42) are finite. In other words, the function $rV(r)$ shall be absolutely integrable and square-integrable. The volume integral $\int_0^\infty dr\, r^2|V(r)|$ shall also exist. Then it follows that the left-hand sides of these inequalities are continuous and bounded. For nonlocal potentials, we explicitly assume the same properties for the kernels $(k'|V|k'')$. The derivative $(\partial/\partial k') \times (k'|\gamma|k)$ shall also be continuous and bounded.

It follows from these conditions that the Sasakawa kernel $(k'|A|k'')$ defined in Eq. (25) is a Hilbert-Schmidt kernel, i.e.,

$$\int_0^\infty dk'\int_0^\infty dk''|(k'|A|k'')|^2 < \infty. \quad (43)$$

It follows further that the function $\chi(k')$ defined by

$$\chi(k') = \lim_{\epsilon\to 0}\int_0^\infty dk'' \frac{(k'|A|k'')(k''|\gamma|k)}{k^2 - k''^2 + i\epsilon} \qquad (44)$$

is square-integrable and that the functions $J(k)$ and $t(k)$ defined in Eqs. (32) and (31) are continuous functions of k.

IV. CONVERGENCE OF THE SASAKAWA SERIES

The series

$$(1-A)^{-1} = \sum_n A^n \qquad (45)$$

converges uniformly if and only if the largest eigenvalue of A is less than unity.[7,8] However, uniform convergence is not required in practice. It is sufficient that the series

$$\sum_n (k|VA^n|k) \qquad (46)$$

and

$$\sum_n \int dr (k|VA^n|r) H_L(r,k) \qquad (47)$$

converge. We therefore prove the following theorem.

Theorem: For $\gamma = \gamma_0$ the series (46) and (47) converge for all local potentials for which $rV(r)$ is absolutely integrable.

Proof: When $\gamma = \gamma_0$ and hence $H_L = h_L$, it follows from Eqs. (28) and (29) that

$$J_n(k) \equiv \int_0^\infty dr (k|VA^n|r) h_L(kr)$$
$$= \int_0^\infty dr \int_r^\infty dr_1 \int_{r_1}^\infty dr_2 \cdots \int_{r_{n-1}}^\infty dr_n rV(r) r_1 V(r_1)\cdots r_n V(r_n) f_L(kr) r^{-1}\bar{G}(r,r_1) r_1^{-1}\cdots \bar{G}(r_{n-1},r_n) r_n^{-1} h_L(kr_n) . \qquad (48)$$

Since the functions $f_L(kr)h_L(kr)r^{-1}$ and $\theta(r'-r)r' f_L(k')^2/f_L(kr')^2 r$ are bounded, there exists a constant C independent of r and r' such that

$$|r^{-1} f_L(kr)\bar{G}(r,r')| \le C|f(kr')| . \qquad (49)$$

Hence

$$|J_n| \le \text{const}\, C^n \int_0^\infty dr \int_r^\infty dr_1 \cdots \times \int_{r_{n-1}}^\infty dr_n |rV(r)\cdots r_n V(r_n)|$$
$$= \text{const}\frac{C^n}{(n+1)!}\left(\int_0^\infty dr\, r|V(r)|\right)^{n+1} . \qquad (50)$$

Thus, the series (47) can be majorized by an exponential series.

The convergence proof for the series (46) is exactly the same. It is only necessary to replace h_L by f_L in Eq. (48). The inequality (50) obviously remains valid.

The features that the Green's function $\bar{G}(r',r'')$ is proportional to $\theta(r''-r')$ and that the potential is local are essential to the preceding proof. It is well known[9] that the integral equation (50) for $F_L(r,k) \equiv \varphi_L(r,k)e^{-i\delta}$ – Eq. (51) below – has an iterative solution with the same convergence properties for the same reason. From Eq. (13) it follows that

$$F_L(r,k) = f_L(r,k) + \int_0^\infty dr' g(r,r')V(r')F(r',k) , \qquad (51)$$

where

$$g(r_1 r') = -(\pi/2k)[h_L(r,k)f_L(r',k) - f_L(r,k)h_L(r',k)]\,\theta(r-r') . \qquad (52)$$

That the series does not converge equally well for nonlocal potentials is easily seen in the simple case of a potential of rank 1. Let

$$(k'|V|k) = \lambda w(k')w(k) . \qquad (53)$$

Then

$$(k'|A|k'') = \lambda\frac{w(k') - w(k)}{k^2 - k'^2} w(k'') , \qquad (54)$$

and hence

$$(k|VA^n|k) = w(k)^2\lambda^{n+1}\left(\int dk' w(k')\frac{w(k') - w(k)}{k^2 - k'^2}\right)^n . \qquad (55)$$

Therefore the series must diverge for sufficiently large λ. On the other hand, for $(k'|\gamma|k) = w(k')/w(k)$ we have $A = 0$, and therefore $\Omega = \Omega_0$. In general we have $\Omega = \Omega_0$ if

$$(k'|\gamma|k) = (k'|T|k)/t(k) . \qquad (56)$$

It is always possible to chose γ such that Ω_0 is a good approximation to Ω. But such a choice does not imply convergence of the series (45).

Let A_0 be the original Sasakawa kernel $\gamma = \gamma_0$. The generalized kernel may then be written in the form

$$(k'|A|k'') = (k'|A_0|k'') + \frac{(k'|\gamma|k) - (k'|\gamma_0|k)}{k^2 - k'^2}(k|V|k'') . \qquad (57)$$

The kernel A differs from A_0 by a kernel of rank 1. If the largest eigenvalue of A_0 is nondegenerate, it can be removed by putting

$$(k'|\gamma|k) = (k'|\gamma_0|k) - \frac{\alpha_0 \chi(k')(k^2 - k'^2)}{\int_0^\infty dk''(k|V|k'')\chi(k'')}, \qquad (58)$$

where α_0 is the largest eigenvalue of A_0 and $\chi(k)$ is the corresponding eigenfunction, i.e.,

$$A_0\chi = \alpha_0\chi\,. \qquad (59)$$

This is the optimum choice for γ if uniform convergence of the series (45) is desired.

V. CONCLUSIONS

The preceding analysis can be trivially extended to a finite number of coupled channels.

The unmodified Sasakawa series converges for local potentials of arbitrary strength. This feature does not obtain for nonlocal potentials or the modified series. But under very general conditions the Sasakawa kernel A is a Hilbert-Schmidt kernel, and both $(r'|A|r'')$ and $(k'|A|k'')$ are continuous functions of their arguments. Thus, the integral equation

$$A\Omega = A\Omega_0 + A(A\Omega) \qquad (60)$$

can be solved numerically to arbitrary accuracy both in the radial representation and in the momentum representation.

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[1]T. Sasakawa, Progr. Theoret. Phys. (Kyoto) Suppl. 27, 1 (1963).
[2]N. Austern, Phys. Rev. 188, 1595 (1969).
[3]H. P. Noyes, Phys. Rev. Letters 15, 538 (1965).
[4]K. L. Kowalski, Phys. Rev. Letters 15, 798 (1965).
[5]C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 23, No. 1 (1945).
[6]R. G. Newton, J. Math. Phys. 1, 319 (1960).
[7]S. Weinberg, Phys. Rev. 131, 440 (1963).
[8]F. Coester, Phys. Rev. 133, 1516 (1964).
[9]See Ref. 6, Sec. 3.

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Neutron-Proton Interaction in Odd-Odd Deformed Nuclei*

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Calculations of the energy splittings between the parallel and antiparallel coupling states of odd neutron and odd proton in deformed nuclei have been made for both zero-range and finite-range nuclear forces. The energy splittings for both types of forces agree with the corresponding experimental energies. Calculations have also been made for the odd-even shift in $K=0$ bands. Finite-range interactions improve the agreement between experiment and theory for the odd-even shift, but the results are still unsatisfactory. It is shown that a tensor force is important in reproducing the experimental odd-even shift, especially in the case where $\Sigma_n + \Sigma_p = 1$.

INTRODUCTION

One of the most important problems in nuclear physics is the determination of the effective residual interaction. In general, the nuclear force has many facets and plays involved roles in the nucleus. Nevertheless, sometimes we can see certain characteristic parts of the nuclear force by using the appropriate phenomena. For example, the spectroscopy of odd-odd nuclei gives us detailed information about the neutron-proton interaction. Many authors have investigated the neutron-proton interaction in the framework of the spherical shell model.[1] The Nordheim rule,[2] which was proposed for the spherical odd-odd nuclei in order to predict ground-state spins, has been extended to deformed odd-odd nuclei by Gallagher and Moszkowski.[3] In the deformed odd-odd nucleus, there are twofold degenerate states in which the coupling of the spins of neutron and proton is either parallel or antiparallel along the symmetry axis. After rotational energies are subtracted, the lowest-order term in the splitting energy between these intrinsic states is caused by the re-