similar form of statistical relation as the probability of occupation of states, with appropriate parameters to suit the respective boundary conditions. This gives us the statistical representation of Eqs. (5) to (10), without the systematic and irregular deviations:

$$Z^0(A) = 2c(A)[e^{(\varphi - \mathcal{E}_n^0)/\epsilon} - e^{(\varphi_Z - \mathcal{E}_n^0)/\epsilon_Z}] , \quad (14)$$

$$-E(A, Z) = -4\mathcal{E}_n^0 c(A)[e^{(\varphi - \mathcal{E}_n^0)/\epsilon} - e^{(\varphi^0 - \mathcal{E}_n^0)/\epsilon^0} - e^{(\varphi^+ - \mathcal{E}_n^0)/\epsilon^+}(Z^0 - Z)^2 - e^{(\varphi^* - \mathcal{E}_n^0)/\epsilon^*}(1 - e^{-\sigma^* n_i^*})] , \quad (15)$$

where

$$\varphi = 7.20 , \ \varphi_Z = 5.557 , \ \varphi^0 = 5.846 ,$$

$$\varphi^+(e, e; e, o; o, e) = 6.757 , \ \varphi^+(o, o) = 6.792 ,$$

$$\varphi^*(e) = 7.168 , \ \varphi^*(o) = 7.060 ,$$

$$\epsilon = 0.399 , \ \epsilon_Z = \epsilon^0 = 0.4642 ,$$

$$\epsilon^+(e, e; e, o; o, e) = 0.2687 , \ \epsilon^+(o, o) = 0.2701 ,$$

$$\epsilon^*(e \text{ or } o) = 0.2646 .$$

These parameters along with the values for σ^*, n_i^*, $c(A)$, and $\mathcal{E}_n^0$, as indicated previously, determine the energies in MeV in a statistical representation of the previous set of equations and actually gives us the underlying principle determining those equations.

[1]L. A. Konig, J. H. Mattauch, and A. H. Wapstra, Nucl. Phys. 31, 18 (1962).
[2]J. W. Dewdney, Nucl. Phys. 43, 303 (1963).
[3]A. K. Dutta, Progr. Theoret. Phys. (Kyoto) 39, 1069 (1968).
[4]*Nuclear Data Sheets*, compiled by K. Way *et al.* (Printing and Publishing Office, National Academy of Sciences – National Research Council, Washington, D. C.).
[5]Ph. A. Seeger, Nucl. Phys. 25, 1 (1964); J. Wing and P. Fong, Phys. Rev. 136, B923 (1964); W. D. Meyers and W. J. Swiatecki, Nucl. Phys. 81, 1 (1966); H. Kummel, J. H. Mattauch, W. Thiele, and A. H. Wapstra, Nucl. Phys. 81, 129 (1966).
[6]A. E. S. Green, Rev. Mod. Phys. 39, 569 (1958).
[7]A. K. Dutta, B. Pal, P. Ganguly, and D. Banerjee, Progr. Theoret. Phys. (Kyoto) 33, 1129 (1965).
[8]A. K. Dutta, Indian J. Phys. 40, 181 (1966).
[9]M. Harada, R. Tamagaki, and H. Tanaka, Progr. Theoret. Phys. (Kyoto) 29, 933 (1963); O. Endo, I. Shimodya, and J. Hiuva, *ibid.* 31, 1157 (1964); J. W. Clark and T. P. Wang, Ann. Phys. (N.Y.) 40, 127 (1966).

PHYSICAL REVIEW C VOLUME 3, NUMBER 2 FEBRUARY 1971

Separable-Expansion Method for the Three-Nucleon System

Michael G. Fuda
Department of Physics and Astronomy, State University of New York, Buffalo, New York 14214
(Received 15 October 1970)

A previously developed separable expansion for the two-particle T matrix is applied to calculating the properties of the three-nucleon system, using spin-dependent static central potentials. The force model is identical to one that was previously employed by Brayshaw and Buck. Results are presented for the triton binding energy, the doublet and quartet scattering lengths, and for the doublet and quartet s-wave phase shifts below and above the three-body breakup threshold. It is found that two or three terms of the expansion in each two-particle spin state give satisfactory convergence in the doublet state of the three-nucleon system. Only one term is needed for the quartet state. The over-all agreement with experiment is good, but some possibly significant discrepancies are found between the theoretical and experimental quartet phase shifts above the breakup threshold. Also, it is shown that a two-particle T matrix or an approximation to one which satisfies the off-shell unitarity relation and has the correct behavior at the bound-state poles leads to three-particle scattering amplitudes which satisfy the three-particle off-shell unitarity relations. The proof does not depend on the existence of a two-particle energy-independent Hermitian potential.

I. INTRODUCTION

In recent years several new techniques have been developed for solving the nonrelativistic three-body problem with local potentials. Some of these are based on the Schrödinger equation, while others take the Faddeev[1] equations as their starting point.

In the method of six-dimensional or K harmonics[2] the orbital part of the three-body wave function is expanded as a sum of products of radial functions and a complete orthogonal set of harmonic functions depending on five angles. Substitution of the expansion into the Schrödinger equation leads to a system of coupled differential equations for the radial functions. This system of equations is truncated at various stages and solved numerically. The method has been applied to calculating the ground-state properties of the three-nucleon system under the assumption that the potential acting between each pair of particles is spin dependent, but central.[3-6] Of the various potential shapes considered (Yukawa, exponential, square well), the square well gives the best agreement with experiment. A variation of the K-harmonic method has been introduced by Erens, Visschers, and van Wageningen,[7] who introduce a complete orthogonal set of radial functions, thereby reducing the three-body problem to the diagonalization of a matrix. Their calculations indicate that the convergence of the K-harmonic method can be adversely effected by potentials with an attractive $1/r$ singularity at the origin (Yukawa or Hulthén) as well as by soft-core potentials. It is not yet known how satisfactory the method will be for scattering problems although there has been some effort in that direction.[8]

Most of the other methods recently developed for dealing with the three-body problem have been based on the Faddeev[1] equations. It is now well known that the Faddeev equations for nonseparable potentials can in general only be reduced to a system of coupled integral equations in two continuous variables.[9] One way to solve these equations is to use numerical quadrature rules for the two-dimensional integrals and then diagonalize or invert the resulting matrix. Methods of this type have been developed by Osborn[10] and Kim[11] and applied to the problem of calculating the bound-state energies of a system of three identical spinless particles interacting via simple potentials of the exponential or Yukawa type. Kim's method has also been applied to such a system assuming hard-core–square-well potentials acting between each pair of particles.[12] An iterative technique for solving the Faddeev equations has been developed by Malfliet and Tjon[13] and applied to the problem of finding the ground-state properties of the three-nucleon system. Their technique has proved to be quite successful in that they have been able to treat potentials with soft cores, as well as tensor coupling.

For separable T matrices the Faddeev equations can be reduced to a system of coupled integral equations in one continuous variable. The T matrix of a local potential is not separable; however, it can often be closely approximated by a sum of separable terms. One way[14] of doing this is to make a separable expansion of the two-particle T matrix in terms of the eigenfunctions of the kernel of the Lippmann-Schwinger equation. This expansion is often referred to as the Weinberg[15] series, since it was proposed by him in his study of the Born series. The method has been successfully applied to the bound state and bound-state scattering problem for a system of three identical spin-zero particles.[14] It has also been used to calculate the binding energy and scattering lengths (doublet and quartet) of the three-nucleon system assuming Hulthén[16] as well as square-well[17] potentials to act between each pair of particles. The square well gives good agreement with experiment. It has been shown[18,19] that the Weinberg series can be used even with potentials which have hard cores. A variation on the Weinberg series has been developed by Harms.[20] Its convergence rate has been studied by Harms[20] and by Kok[21] and found to be faster than that of the Weinberg series in certain cases. A rather different approach to separable approximations has been developed by Brayshaw,[22] who has given a two-term separable approximation which reproduces the exact right-hand cut of the off-shell T matrix as well as the exact behavior at each bound state or resonance pole.

In this paper we consider a separable expansion of the T matrix which has been developed by this author[19] and independently by Osborn.[23] The leading term of the expansion is the separable approximation suggested by Kowalski and Noyes.[24] Truncating the expansion gives an approximation for the T matrix which is exact half off the energy shell, exactly satisfies the off-shell unitary relation, and duplicates the exact T matrix in the neighborhood of bound-state and resonance poles. The unphysical poles that can occur in the Kowalski-Noyes approximation[24] are canceled by other terms in the expansion. This expansion is more convenient to use at positive energies than the Weinberg series, since the functions that occur in it can be obtained by solving nonsingular integral equations. Use of the Weinberg series at positive energies necessitates solving singular integral equations. The rate of convergence of the series has been investigated[19] by calculating the binding energy of a system of three identical spinless particles assuming a square-well potential to act between each pair of particles. The rate of convergence was found to be satisfactory.

In this paper we investigate the rate of convergence of the expansion for the three-nucleon system, assuming the two-body potentials to be spin dependent, but central. For the shape of the po-

tential we use square wells, since this allows us to obtain the expansion analytically. Furthermore, the square well appears to be a fairly "realistic" potential, since previous calculations[4, 6, 17] have shown that it gives reasonable results for the triton binding energy, as well as for the low-energy doublet and quartet n-d phase shifts below the three-body breakup threshold. Here we calculate the above properties, as well as the elastic amplitude for n-d scattering above the threshold for breakup.

In Sec. II, we summarize the basic equations that are used for constructing the separable expansion of the two-particle T matrix. The reduction of the Faddeev equations to coupled one-dimensional integral equations is presented in Sec. III. In Sec. IV we prove that a two-body T matrix, or an approximation to one, which has the correct behavior at the bound-state poles and which satisfies the two-body off-shell unitarity relations will lead to three-body transition operators which satisfy three-body off-shell unitarity. This proof is relevant to the expansion being studied, since the T matrix obtained by truncating the expansion does not correspond to an energy-independent Hermitian potential. The proof, while not rigorous, is extremely straightforward. In Sec. V the results of the calculations are given and compared with experiment. Section VI is devoted to a summary and discussion of the results.

II. SEPARABLE EXPANSION

In this section the basic equations for constructing the separable expansion of the T matrix are summarized. The two-body T matrix is defined by

$$T(s) = V + V(s - H)^{-1}V, \tag{2.1}$$

where V is the two-body potential, H is the complete two-body Hamiltonian, and s is a complex parameter. For a potential without a hard core the expansion for the lth partial wave is given by

$$T(s) = |F(s)\rangle\langle klm|T(s)|klm\rangle\langle F(s)| + \sum_{\nu=1}^{\infty} V|\Omega_\nu(s)\rangle \frac{\langle\Omega_\nu(s)|V|\Omega_\nu(s)\rangle^{-1}}{1-\eta_\nu(s)}\langle\Omega_\nu(s)|V. \tag{2.2}$$

In configuration space, the free wave $|klm\rangle$ is represented by

$$\langle\vec{r}|klm\rangle = (2\pi^2)^{-1/2} j_l(kr) Y_{lm}(\hat{r}). \tag{2.3}$$

$|F(s)\rangle$ is the ratio of the half-off-shell T matrix to the on-shell T matrix; i.e.,

$$|F(s)\rangle = T(s)|klm\rangle\langle klm|T(s)|klm\rangle^{-1}. \tag{2.4}$$

The eigenfunctions $|\Omega_\nu(s)\rangle$ and the eigenvalues $\eta_\nu(s)$ are obtained by solving the Schrödinger equation

$$[s - H_0 - \eta_\nu{}^{-1}V]|\Omega_\nu(s)\rangle = 0 \tag{2.5}$$

with the requirement that $|\Omega_\nu(s)\rangle$ remain finite everywhere and become a free wave for large r; i.e.,

$$\langle\vec{r}|\Omega_\nu(s)\rangle \underset{r\to\infty}{\sim} [(2\pi^2)^{1/2}kr]^{-1}\sin(kr - \tfrac{1}{2}l\pi)Y_{lm}(\hat{r}). \tag{2.6}$$

In the sum appearing in (2.2) the eigenfunction $|klm\rangle$ corresponding to $\eta_\nu \to \infty$ [see (2.5)] is not included.

This expansion has the advantage that no matter what order it is truncated at (assuming one keeps at least the first term), it has the following properties: It is exact half off the energy shell, it exactly satisfies the off-shell unitarity relation, and it reproduces the behavior of the exact T matrix when s is near a bound state or resonance energy. Furthermore, it is shown in Ref. 19 that the eigenfunctions and eigenvalues can be obtained as the solutions of nonsingular integral equations. Thus the terms in the expansion can be obtained using conventional numerical methods.

In the next section it is shown that inserting this expansion into the Faddeev equations leads to a coupled set of one-dimensional integral equations.

III. THREE-BODY EQUATIONS

The formulation of the three-body equations that is most convenient for our purposes is that of Alt, Grassberger, and Sandhas.[25] Their transition operators are defined by

$$Y_{\alpha\beta}(s) = (1 - \delta_{\alpha\beta})(s - H_0 - V_\beta) + V^\alpha + V^\alpha(s - H)^{-1}V^\beta. \tag{3.1}$$

Here H_0 is the kinetic energy operator for the three-particle system; H is the complete three-particle Hamiltonian; V_3, for example, is the potential acting between particles 1 and 2; V_0 is defined to be zero; and

$$V^\alpha = \sum_{\gamma\neq\alpha} V_\gamma. \tag{3.2}$$

These transition operators satisfy the equations

$$Y_{\alpha\beta}(s) = (1-\delta_{\alpha\beta})(s-H_0) + \sum_{\gamma\neq\alpha} T_\gamma(s)G_0(s)Y_{\gamma\beta}(s)\,,$$
$$= (1-\delta_{\alpha\beta})(s-H_0) + \sum_{\gamma\neq\beta} Y_{\alpha\gamma}(s)G_0(s)T_\gamma(s)\,, \quad \alpha,\beta = 0,1,2,3\,, \tag{3.3}$$

where

$$T_\gamma(s) = V_\gamma + V_\gamma G_0(s)T_\gamma(s)\,,$$
$$= V_\gamma + T_\gamma(s)G_0(s)V_\gamma\,, \tag{3.4}$$

and

$$G_0(s) = (s-H_0)^{-1}.$$

The operator $T_\gamma(s)$ is not identical to the two-body T matrix, since the particle not acted on by V_γ makes its presence felt through the three-particle kinetic energy operator H_0. If $\vec{p}_\gamma$ is the relative momentum of the two particles labeled by γ, and $\vec{q}_\gamma$ is the momentum of particle γ (we work in the c.m. system), then

$$\langle \vec{p}'_\gamma, \vec{q}'_\gamma | T_\gamma(s) | \vec{p}_\gamma, \vec{q}_\gamma\rangle = \delta(\vec{q}'_\gamma - \vec{q}_\gamma)\langle \vec{p}'_\gamma | \hat{T}_\gamma\left(s - \frac{q_\gamma^2}{2\mu^\gamma}\right)|\vec{p}_\gamma\rangle\,. \tag{3.5}$$

The caret denotes a two-body transition operator and μ^1, for example, is given by

$$\mu^1 = m_1(m_2+m_3)/(m_1+m_2+m_3)\,.$$

In the three-particle Hilbert space the expansion (2.2) becomes

$$T_\gamma(s) = \sum_{\nu=0}^{\infty}\int \left|g_{\gamma\nu}\left(s-\frac{q_\gamma^2}{2\mu^\gamma}\right)\right\rangle |\vec{q}_\gamma\rangle\, d\vec{q}_\gamma\, \tau_{\gamma\nu}\left(s-\frac{q_\gamma^2}{2\mu^\gamma}\right)\langle \vec{q}_\gamma|\left\langle g_{\gamma\nu}\left(s-\frac{q_\gamma^2}{2\mu^\gamma}\right)\right|, \tag{3.6}$$

where

$$|g_{\gamma 0}(s)\rangle = |F_\gamma(s)\rangle\langle k_{0\gamma} l_\gamma m_\gamma | V_\gamma | B_\gamma\rangle, \quad k_{0\gamma}{}^2 = -B_\gamma\,, \tag{3.7}$$
$$\tau_{\gamma 0}(s) = \langle k_\gamma l_\gamma m_\gamma | \hat{T}_\gamma(s) | k_\gamma l_\gamma m_\gamma\rangle / |\langle k_{0\gamma} l_\gamma m_\gamma | V_\gamma | B_\gamma\rangle|^2\,, \tag{3.8}$$
$$|g_{\gamma\nu}(s)\rangle = V_\gamma |\Omega_{\gamma\nu}(s)\rangle\,, \quad \nu = 1,2,3,\ldots\,, \tag{3.9}$$
$$\tau_{\gamma\nu}(s) = \frac{\langle \Omega_{\gamma\nu}(s)|V_\gamma|\Omega_{\gamma\nu}(s)\rangle^{-1}}{1-\eta_{\gamma\nu}}, \quad \nu = 1,2,3,\ldots\,. \tag{3.10}$$

We have assumed for the sake of definiteness in writing that each two-body subsystem γ has a bound state $|B_\gamma\rangle$ with binding energy B_γ. Furthermore, we have assumed that only the lth partial wave is being considered. It is easy to show from (2.1) and (2.4) that

$$|g_{\gamma 0}(s)\rangle \underset{s\to -B_\gamma}{\longrightarrow} V_\gamma |B_\gamma\rangle\,, \tag{3.11}$$

$$\tau_{\gamma 0}(s) \underset{s\to -B_\gamma}{\longrightarrow} \frac{1}{s+B_\gamma}\,. \tag{3.12}$$

We now introduce a new set of transition operators by the relation

$$X_{\alpha\beta}(s) = G_0(s)Y_{\alpha\beta}(s)G_0(s)\,, \quad \alpha,\beta = 0,1,2,3\,. \tag{3.13}$$

From (3.3) these operators satisfy the equations

$$X_{\alpha\beta}(s) = (1-\delta_{\alpha\beta})G_0(s) + \sum_{\gamma\neq\alpha} G_0(s)T_\gamma(s)X_{\gamma\beta}(s)\,, \quad \alpha,\beta = 0,1,2,3\,. \tag{3.14}$$

Following Lovelace[26] we define scattering amplitudes and potentials by the relations

$$X_{\alpha\nu,\,\beta\mu}(\vec{q}'_\alpha, \vec{q}_\beta; s) \equiv \langle \vec{q}'_\alpha |\left\langle g_{\alpha\nu}\left(s-\frac{q'^2_\alpha}{2\mu^\alpha}\right)\right| X_{\alpha\beta}(s) \left| g_{\beta\mu}\left(s-\frac{q_\beta^2}{2\mu^\beta}\right)\right\rangle |\vec{q}_\beta\rangle, \tag{3.15}$$

$$Z_{\alpha\nu,\,\beta\mu}(\vec{q}'_\alpha, \vec{q}_\beta; s) = \langle \vec{q}'_\alpha |\left\langle g_{\alpha\nu}\left(s-\frac{q'^2_\alpha}{2\mu^\alpha}\right)\right| G_0(s) \left| g_{\beta\mu}\left(s-\frac{q_\beta^2}{2\mu^\beta}\right)\right\rangle |\vec{q}_\beta\rangle(1-\delta_{\alpha\beta})\,. \tag{3.16}$$

Inserting the expansion (3.6) in (3.14) and using the definitions (3.15) and (3.16) we find

$$X_{\alpha\nu,\,\beta\mu}(\vec{q}'_\alpha,\vec{q}_\beta;s)=Z_{\alpha\nu,\,\beta\mu}(\vec{q}'_\alpha,\vec{q}_\beta;s)+\sum_{\gamma=1}^{3}\sum_{\sigma=0}^{\infty}\int Z_{\alpha\nu,\,\gamma\sigma}(\vec{q}'_\alpha,\vec{q}''_\gamma;s)d\vec{q}''_\gamma\tau_{\gamma\sigma}\left(s-\frac{q''^2_\gamma}{2\mu^\gamma}\right)X_{\gamma\sigma,\,\beta\mu}(\vec{q}''_\gamma,\vec{q}_\beta;s)\,. \tag{3.17}$$

Using (3.1), (3.11), (3.13), and (3.15), and the fact that $|B_\gamma\rangle$ is a solution of the two-body Schrödinger equation, it is straightforward to show that

$$X_{\alpha 0,\,\beta 0}(\vec{q}'_\alpha,\vec{q}_\beta;s)\underset{\text{on shell}}{\longrightarrow}\langle\vec{q}'_\alpha|\langle B_\alpha|[V^\alpha+V^\alpha(s-H)^{-1}V^\beta]|B_\beta\rangle|\vec{q}_\beta\rangle\,, \tag{3.18}$$

where on shell means

$$s=-B_\alpha+q_\alpha{}^2/2\mu^\alpha\,,\quad \alpha=1,2,3\,. \tag{3.19}$$

Thus, the physical elastic scattering amplitude is obtained directly from the solutions of (3.17). For the case of three identical particles it is easy to show[26] that the properly symmetrized amplitudes are defined by

$$X_{\nu\mu}(\vec{q}',\vec{q};s)=\sum_{\alpha=1}^{3}X_{\alpha\nu,\,\beta\mu}(\vec{q}',\vec{q};s)\,. \tag{3.20}$$

From (3.16) and (3.17), it follows that the symmetrized amplitudes satisfy the equation

$$X_{\nu\mu}(\vec{q}',\vec{q};s)=2Z_{\nu\mu}(\vec{q}',\vec{q};s)+2\sum_{\sigma=0}^{\infty}\int Z_{\nu\sigma}(\vec{q}',\vec{q}'';s)d\vec{q}''\,\tau_\sigma(s-\tfrac{3}{4}q''^2)X_{\sigma\mu}(\vec{q}'',\vec{q};s)\,. \tag{3.21}$$

These equations can also be used for spin-dependent forces if one thinks of the indices ν, μ, and σ as cover indices which include spin and isospin indices. We do not bother to write out these equations, since they can easily be obtained following Lovelace.[26]

After a partial-wave analysis of (3.17) or (3.21), we are left with a set of coupled integral equations in one continuous variable. This infinite set of coupled equations must, of course, in practice be truncated. We will show in the next section that the scattering amplitudes obtained from the truncated equations satisfy the three-body off-shell unitarity relations.

IV. OFF-SHELL UNITARITY

It is easy to convince oneself that the T matrix obtained by truncating the expansion presented in Sec. II does not correspond to an energy-independent Hermitian potential. Thus it is not obvious that the three-particle scattering amplitudes obtained by using the truncated expansions satisfy the three-particle off-shell unitarity relations. These unitarity relations have their simplest form when expressed in terms of the transition operators given by (3.1). The relations are[25]

$$Y_{\alpha\beta}(s)-Y_{\alpha\beta}(s^*)=-2\pi i\sum_{\gamma=0}^{3}Y_{\alpha\gamma}(s)\Delta_\gamma(E)Y_{\gamma\beta}(s^*)\,,\quad \alpha,\beta=0,1,2,3\,, \tag{4.1}$$

with

$$s=E+i\epsilon\,,\quad \epsilon>0\,,$$

and

$$\Delta_\gamma(E)=\int|B_\gamma\rangle|\vec{q}_\gamma\rangle d\vec{q}_\gamma\,\delta\left(E+B_\gamma-\frac{q_\gamma{}^2}{2\mu^\gamma}\right)\langle\vec{B}_\gamma|\langle\vec{q}_\gamma|\,,\quad \gamma=1,2,3\,, \tag{4.2}$$

$$\begin{aligned}\Delta_0(E)&=\int|\vec{p}_\gamma,\vec{q}_\gamma\rangle d\vec{p}_\gamma d\vec{q}_\gamma\,\delta\left(E-\frac{p_\gamma{}^2}{2\mu_\gamma}-\frac{q_\gamma{}^2}{2\mu^\gamma}\right)\langle\vec{p}_\gamma,\vec{q}_\gamma|\\&=\delta(E-H_0)\,.\end{aligned} \tag{4.3}$$

Here μ_γ is the reduced mass of subsystem γ, e.g.,

$$\mu_1=m_2m_3/(m_2+m_3)\,.$$

We will now prove the following theorem: If the two-particle T matrix has the properties

$$\hat{T}(s)\underset{s\to -B}{\longrightarrow}\frac{\hat{V}|B\rangle\langle B|\hat{V}}{s+B}\,, \tag{4.4}$$

$$\begin{aligned}\hat{T}(s)-\hat{T}(s^*)&=-2\pi i\hat{T}(s)\delta(E-\hat{H}_0)\hat{T}(s^*)\,,\\&=-2\pi i\hat{T}(s^*)\delta(E-\hat{H}_0)\hat{T}(s)\,,\end{aligned} \tag{4.5}$$

where $\hat{V}$ is the two-body potential and $|B\rangle$ is the two-body bound state at energy $-B$ then the scattering amplitudes obtained by solving (3.3) satisfy the off-shell unitarity relations (4.1). In other words, any two-body T matrix or approximation to a two-body T matrix which has the correct behavior near the bound-state poles and which satisfies the two-body off-shell unitarity relations will lead to a three-body transition operator which satisfies the three-body off-shell unitarity relations. The proof follows.

Using (4.4) and (4.5), it is easy to show that the three-body transition operator $T_\gamma(s)$ that arises from the two-body operator $\hat{T}_\gamma(s)$ [see (3.5)] satisfies the relations

$$\begin{aligned} T_\gamma(s) - T_\gamma(s^*) &= -2\pi i(E - H_0)\Delta_\gamma(E)(E - H_0) - 2\pi i T_\gamma(s)\Delta_0(E)T_\gamma(s^*) \\ &= -2\pi i(E - H_0)\Delta_\gamma(E)(E - H_0) - 2\pi i T_\gamma(s^*)\Delta_0(E)T_\gamma(s)\,. \end{aligned} \tag{4.6}$$

We will need the discontinuities across the right-hand cuts of the kernels appearing in (3.3). Using the relation

$$G_0(s) - G_0(s^*) = -2\pi i\delta(E - H_0)\,, \tag{4.7}$$

it follows from (4.6) that

$$T_\gamma(s)G_0(s) - T_\gamma(s^*)G_0(s^*) = -2\pi i\{(E - H_0)\Delta_\gamma(E) + T_\gamma(s)\delta(E - H_0)[1 + T_\gamma(s^*)G_0(s^*)]\}\,. \tag{4.8}$$

From (3.3) we obtain

$$Y_{\alpha\beta}(s) - Y_{\alpha\beta}(s^*) = \sum_{\gamma\neq\alpha}[T_\gamma(s)G_0(s) - T_\gamma(s^*)G_0(s^*)]\,Y_{\gamma\beta}(s^*) + \sum_{\gamma\neq\alpha} T_\gamma(s)G_0(s)[Y_{\gamma\beta}(s) - Y_{\gamma\beta}(s^*)]\,. \tag{4.9}$$

By comparing (3.3) and (4.9) one sees that the discontinuity in $Y_{\alpha\beta}(s)$ satisfies equations with the same kernels as $Y_{\alpha\beta}(s)$ itself. With this observation, it is easy to show that

$$Y_{\alpha\beta}(s) - Y_{\alpha\beta}(s^*) = \sum_{\gamma=1}^{3} Y_{\alpha\gamma}(s)G_0(s)[T_\gamma(s)G_0(s) - T_\gamma(s^*)G_0(s^*)]Y_{\gamma\beta}(s^*)\,. \tag{4.10}$$

Using (4.8) this becomes

$$Y_{\alpha\beta}(s) - Y_{\alpha\beta}(s^*) = -2\pi i\sum_{\gamma=1}^{3} Y_{\alpha\gamma}(s)\Delta_\gamma(E)Y_{\gamma\beta}(s^*) - 2\pi i\sum_{\gamma=1}^{3} Y_{\alpha\gamma}(s)G_0(s)T_\gamma(s)\delta(E - H_0)[1 + T_\gamma(s^*)G_0(s^*)]Y_{\gamma\beta}(s^*)\,. \tag{4.11}$$

From (3.3) it follows that

$$[1 + T_\gamma(s)G_0(s)]Y_{\gamma\beta}(s) = (\delta_{0\beta} - \delta_{\gamma\beta})(s - H_0) + Y_{0\beta}(s)\,, \tag{4.12}$$

and

$$Y_{\gamma 0}(s) = (1 - \delta_{\alpha 0})(s - H_0) + \sum_{\gamma=1}^{3} Y_{\alpha\gamma}(s)G_0(s)T_\gamma(s)\,. \tag{4.13}$$

Combining (4.12) and (4.13) with (4.11), we finally arrive at (4.1). Since the T matrices obtained by truncating the expansions of Sec. II satisfy (4.4) and (4.5), the three-body amplitudes obtained by solving the truncated versions of (3.17) or (3.21) satisfy the three-particle off-shell unitarity relations (4.1). This property is desirable for at least two reasons: It makes the definition of phase shifts unambiguous, and the relations (4.1) give a useful check on numerical work.

V. NUMERICAL CALCULATIONS

We now apply the expansion of Sec. II and the equations of Sec. III to the problem of calculating the properties of the three-nucleon system. We assume the nucleons interact via pairwise square-well potentials in the singlet and triplet spin states. We use the same parameters as Brayshaw and Buck.[6] For the singlet state the depth is $V_s = 14.017$ MeV and the width is $b_s = 2.5895$ fm; for the triplet state we have $V_t = 34.406$ MeV and $b_t = 2.0719$ fm. We only include the force in relative two-body s states. The expansion for the square-well T matrix is given by Eqs. (3.2) of Ref. 19.

Table I gives the results for the triton binding energy and the n-d scattering lengths as a function of the number of terms retained in the separable expansion (2.2). We always retain the same number of terms in each two-body spin state. It is seen that the results for the triton binding energy have converged using three terms for each spin state. Two terms are enough to give accuracies of the order of 1%. Our final result for the triton binding energy (9.12 MeV) is about 6% larger than that of Brayshaw and Buck,[6] who find a value of 8.72 MeV. In order to check our program we have repeated one of Kharchenko and Storozhenko's[17] calculations for the triton binding energy. They

TABLE I. Triton binding energy and n-d scattering lengths for the square-well potential. N is the number of terms in the separable expansion of the T matrix for each two-body spin state (triplet or singlet).

N	E_T (MeV)	$a_{1/2}$ (fm)	$a_{3/2}$ (fm)
1	8.05	1.34	6.40
2	9.05	0.52	6.39
3	9.12	0.48	6.39

use the Weinberg series for the square-well T matrix and consider four different sets of parameters for the well depths and widths. For their parameter set c they obtain a binding energy of 9.08 MeV. Using the same parameters we find a value of 9.13 MeV. The two results agree to within about $\frac{1}{2}$%. Kharchenko and Storozhenko[17] have compared a calculation of the triton binding energy using the K-harmonic method[4] with a calculation using the Weinberg series. They[17] find that the Weinberg series gives a binding energy of 8.84 MeV, whereas the K-harmonic method[4] gives a value of 8.43 MeV. The discrepancy of about 5% is comparable to what we have found. These results indicate better convergence for the separable-expansion methods than for the K-harmonic method. For all of the methods further corrections to the binding energy can only increase its value. Our final results of 9.12 MeV for the triton binding energy is about 7% greater than the experimental value of 8.49 MeV.

Column 2 of the table illustrates the rate of convergence of the expansion in a calculation of the doublet scattering length. It appears that three terms give a satisfactory result; however, it should be noted that a calculation with four terms was not attempted, since it would have taken a prohibitive amount of machine time. Also, some difficulty was encountered in achieving stability against mesh variation. We estimate our results to be accurate to about 5%. Our result of 0.48 fm for the doublet scattering length is somewhat higher than the value of 0.408 fm found by Brayshaw and Buck.[6] Their calculation includes both s- and d-wave two-body interactions. The additional attraction supplied by the d wave is consistent with their scattering length being lower than ours. Furthermore, our result is consistent with the results of Kharchenko and Storozhenko.[17] For their parameter set c they find a triton binding energy of 9.08 MeV and a scattering length of 0.49 fm; for their parameter set a they find 9.20 MeV and 0.46 fm. If we assume a linear relation between the doublet scattering length and the triton binding energy[27] and interpolate, using our binding energy of 9.12 MeV, we find $a_{1/2} = 0.48$ fm, which agrees with the final result in the table. Our calculated value of 0.48 fm agrees better with the older experimental value of 0.7 ± 0.3 fm,[28] than with the newer value of 0.11 ± 0.07 fm.[29] Column 3 of the table shows that one term in the separable expansion gives a satisfactory result for the quartet scattering length. The theoretical result is in reasonable agreement with both the older experimental result of $a_{3/2} = 6.38 \pm 0.06$ fm[28] and the newer one of $a_{3/2} = 6.14 \pm 0.06$ fm.[29]

Figures 1 and 2 present results of calculations for $k \cot[\mathrm{Re}(\delta)]$ for neutron lab energies ranging from 0 to 14.1 MeV. k is the relative wave number and is related to the neutron lab energy by $k^2 = (0.02141\ \mathrm{MeV}^{-1}) E_{\mathrm{lab}}\ \mathrm{fm}^{-2}$. It appears (see Fig. 1) that it is necessary to retain at least three terms

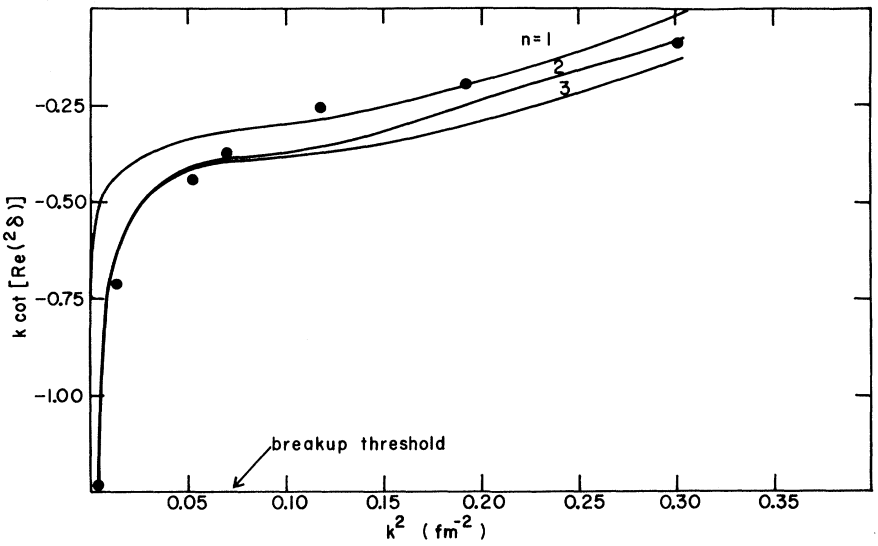

FIG. 1. Values of $k \cot[\mathrm{Re}(\delta)]$ for the doublet n-d scattering states versus k^2, where k is the relative wave number. The solid lines are calculated values for various separable approximations. Solid circles are experimental values taken from van Oers and Seagrave (Ref. 19).

in the separable expansion for each two-body spin state in order to get satisfactory results for $k\cot[\mathrm{Re}(\delta)]$ in the doublet state. The agreement between theory and experiment is good below the breakup threshold, but deteriorates somewhat above the threshold. The theoretical results below threshold are in satisfactory agreement with those obtained by Brayshaw and Buck.[6] Figure 2 gives the results for the quartet state. One term in the separable expansion gives satisfactory results. The agreement between theory and experiment is reasonable below threshold, but becomes progressively poorer above threshold.

VI. SUMMARY AND DISCUSSION

We have found that retaining two to three terms in the separable expansion of the T matrix for each two-particle spin state (triplet and singlet) gives satisfactory convergence in the three-nucleon doublet state for the triton binding energy, the doublet n-d scattering length, and the effective-range quantity $k\cot[\mathrm{Re}(\delta)]$ for neutron lab energies up to 14.1 MeV. The simple square-well force model we have employed overbinds the triton by 0.63 MeV. This is no surprise, since we have left out effects due to short-range repulsion and tensor coupling in the two-nucleon force. Both of these effects tend to decrease the binding energy of the triton. Our final result for the doublet scattering length (0.48 fm) would be increased by these effects, thereby pushing the theoretical result even closer to the old experimental result of 0.7 fm.[28] The theoretical results for $k\cot[\mathrm{Re}(\delta)]$ in the doublet state agree well with those obtained by van Oers and Seagrave[29] from experiment below the breakup threshold, as well as with the theoretical results obtained by Brayshaw and Buck[6] using a different calculational technique. Our results for $k\cot[\mathrm{Re}(\delta)]$ are lower than the experimental ones above the breakup threshold. Using a more repulsive force model would lower the phase shifts and bring the theoretical values of $k\cot[\mathrm{Re}(\delta)]$ closer to the experimental values.

We have found that one term in the separable expansion gives satisfactory convergence in the three-nucleon quartet state for the scattering length as well as for $k\cot[\mathrm{Re}(\delta)]$ up to neutron lab energies of 14.1 MeV. The agreement with experiment is good below the breakup threshold, but quite poor above. The theoretical values of $k\cot[\mathrm{Re}(\delta)]$ above threshold are higher than the experimental values; just the opposite of the situation in the doublet state. We find that $k\cot[\mathrm{Re}(\delta)]$ changes sign from negative to positive as we go from lower to higher energies. This feature also appears in the calculation of Aaron, Amado, and Yam,[30] who use a very different force model; namely, the well-known Yamaguchi separable potential.[31] The experimental values[29] of $k\cot[\mathrm{Re}(\delta)]$ stay negative for neutron lab energies up to 14.1 MeV. We have no satisfactory explanation for this discrepancy.

The calculations in this paper have only dealt with the three-nucleon states of total orbital angular momentum zero. We are presently extending the calculations to states of higher angular momentum. Obtaining the n-d scattering amplitude in the other partial waves will allow us to make direct comparisons with the experimental cross sections. This might help pin down the discrepancies between the theoretical and experimental s-wave phase shifts. It is expected that the separable expansion will converge faster in the higher

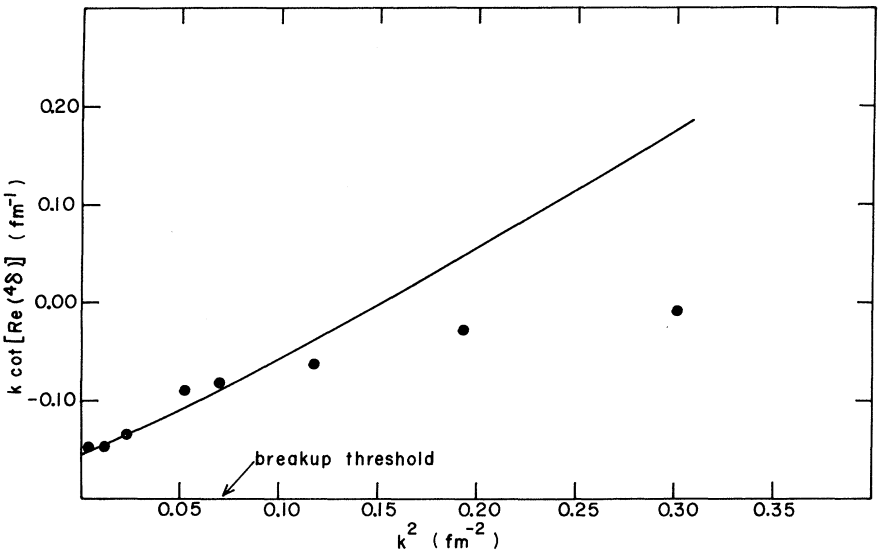

FIG. 2. Values of $k\cot[\mathrm{Re}(\delta)]$ for the quartet n-d scattering state versus k^2, where k is the relative wave number. The solid line is the calculated values. Solid circles are experimental values taken from van Oers and Seagrave (Ref. 29).

n-d partial waves, since the centrifugal barrier should prevent the particles from scattering too far off the energy shell. Recall that the first term in the expansion gives the exact half-off-shell two-particle T matrix.

It is doubtful whether it is worth extending the simple square-well force model to very high energies, since the phase parameters of this model are not realistic at high energies. However, tests of the separable-expansion method should be carried out at higher energies. We are planning to do this.

ACKNOWLEDGMENT

The author would like to thank O. Bahethi for help with the calculations. Thanks are also due to the Computing Center at the State University of New York at Buffalo, which is partially supported by National Institute of Health Grant No. FR-00126 and National Science Foundation Grant No. GP-7318, for providing the machine time.

[1]L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [transl.: Soviet Phys. – JETP 12, 1014 (1961)].

[2]A. J. Dragt, J. Math. Phys. 6, 533 (1965); W. Zickendraht, Ann. Phys. (N.Y.) 35, 18 (1965); Yu. A. Simonov, Yadern. Fiz. 3, 630 (1966) [transl.: Soviet J. Nucl. Phys. 3, 461 (1966)]; A. M. Badalyan and Yu. A. Simonov, Yadern. Fiz. 3, 1032 (1966) [transl.: Soviet J. Nucl. Phys. 3, 155 (1966)].

[3]Yu. A. Simonov and A. M. Badalyan, Yadern. Fiz. 5, 88 (1967) [transl.: Soviet J. Nucl. Phys. 5, 60 (1967)].

[4]A. M. Badalyan, E. S. Galpern, and V. N. Lyakhovitskiĭ, Yadern. Fiz. 8, 313 (1968) [transl.: Soviet J. Nucl. Phys. 8, 180 (1969)].

[5]A. M. Badalyan, Yadern. Fiz. 8, 1128 (1968) [transl.: Soviet J. Nucl. Phys. 8, 656 (1969)].

[6]D. D. Brayshaw and B. Buck, Phys. Rev. Letters 24, 733 (1970).

[7]G. Erens, J. L. Visschers, and R. van Wageningen, to be published.

[8]B. N. Zakharyev, V. V. Pustovalov, V. D. E'fros, Yadern, Fiz. 8, 406 (1968) [transl.: Soviet J. Nucl. Phys. 8, 234 (1969)].

[9]A. Ahmezadeh and J. A. Tjon, Phys. Rev. 139, B1085 (1965); T. A. Osborn and H. P. Noyes, Phys. Rev. Letters 17, 215 (1966); E. El-Baz, C. Fayard, G. H. Lamot, and J. LaFoucrière, Nuovo Cimento 64A, 13 (1969); E. P. Harper, Y. E. Kim, and A. Tubis, Phys. Rev. C 2, 877 (1970).

[10]T. Osborn, Stanford Linear Accelerator Report No. SLAC-79 (unpublished); J. W. Humberston, R. L. Hull, and T. A. Osborn, Phys. Letters 27B, 195 (1968).

[11]Y. E. Kim, J. Math. Phys. 10, 1491 (1969); Phys. Letters 29B, 411 (1969).

[12]Y. E. Kim and A. Tubis, Phys. Rev. C 1, 414, 1627 (1970).

[13]R. A. Malfliet and J. A. Tjon, Nucl. Phys. A127, 161 (1969), Phys. Letters 29B, 391 (1969); 30B, 293 (1969).

[14]J. S. Ball and D. Y. Wong, Phys. Rev. 169, 1362 (1968); J. R. Fulco and D. Y. Wong, *ibid.* 172, 1062 (1968).

[15]S. Weinberg, Phys. Rev. 131, 440 (1963).

[16]V. F. Kharchenko and N. M. Petrov, Nucl. Phys. A137, 417 (1969).

[17]V. F. Kharchenko and S. A. Storozhenko, Nucl. Phys. A137, 437 (1969).

[18]M. G. Fuda, Phys. Rev. 174, 1134 (1968); *ibid.* 178, 1682 (1969).

[19]M. G. Fuda, Phys. Rev. 186, 1078 (1969).

[20]E. Harms, Phys. Rev. C 1, 1667 (1970).

[21]L. P. Kok, Ph. D. thesis, State University of Groningen, 1969 (unpublished).

[22]D. D. Brayshaw, Phys. Rev. 182, 1658 (1969).

[23]T. A. Osborn, Nucl. Phys. A138, 305 (1969).

[24]K. L. Kowalski, Phys. Rev. Letters 15, 798 (1965); H. P. Noyes, *ibid.* 15, 538 (1965).

[25]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967).

[26]C. Lovelace, Phys. Rev. 135, B1225 (1964).

[27]A. C. Phillips, Nucl. Phys. A107, 209 (1968); L. M. Delves and A. C. Phillips, Rev. Mod. Phys. 41, 497 (1969); T. Brady, E. Harms, L. Laroze, and J. S. Levinger, Phys. Rev. C 2, 59 (1970).

[28]D. G. Hurst and J. Alcock, Can. J. Phys. 29, 36 (1951).

[29]W. T. H. van Oers and J. D. Seagrave, Phys. Letters 24B, 562 (1967).

[30]R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. 140, B1291 (1965).

[31]Y. Yamaguchi, Phys. Rev. 95, 1628 (1954).