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[1]J. C. Allred, D. K. Froman, A. M. Hudson, and L. Rosen, Phys. Rev. 82, 786 (1951).

[2]R. L. Henkel, J. E. Perry, Jr., and R. K. Smith, Phys. Rev. 99, 1050 (1955).

[3]G. F. Bogdanov, N. A. Vlasov, S. P. Kalinin, B. V. Rybakov, and V. A. Sidorov, Zh. Eksperim. i Teor. Fiz. 30, 981 (1956) [transl.: Soviet Phys.–JETP 3, 793 (1956)].

[4]E. W. Warburton and J. N. McGruer, Phys. Rev. 105, 639 (1957).

[5]K. P. Artemov and N. A. Vlasov, Zh. Eksperim. i Teor. Fiz. 39, 1612 (1960) [transl.: Soviet Phys.–JETP 12, 1124 (1961)].

[6]B. V. Rybakov, V. A. Sidorov, and N. A. Vlasov, Nucl. Phys. 23, 491 (1961).

[7]H. W. Lefevre, R. R. Borchers, and C. H. Poppe, Phys. Rev. 128, 1328 (1962).

[8]H. J. Erramuspe and R. J. Slobodrian, Nucl. Phys. 49, 65 (1963).

[9]G. G. Ohlsen and P. G. Young, Phys. Rev. 136, B1632 (1964).

[10]K. Nagatani, T. A. Tombrello, and D. A. Bromley, Phys. Rev. 140, B824 (1965).

[11]K. Fukunaga, H. Nakamura, T. Tanabe, K. Hosono, and S. Matsuki, J. Phys. Soc. Japan 22, 28 (1967).

[12]R. E. Warner and R. W. Bercaw, Nucl. Phys. A109, 205 (1967).

[13]T. Tanabe, J. Phys. Soc. Japan 25, 21 (1968).

[14]L. C. Northcliffe, C. W. Lewis, and D. P. Saylor, Nucl. Instr. Methods 69, 789 (1970).

[15]H. G. Blosser and R. St. Onge, Bull. Am. Phys. Soc. 14, 533 (1969).

[16]J. R. Stehn, M. D. Goldberg, B. A. Magurno, and R. Wiener-Chasman, Brookhaven National Laboratory Report No. BNL-325 (unpublished).

[17]C. W. Lewis and J. M. Blair, Nucl. Instr. Methods 35, 261 (1965).

[18]P. E. Shanley, Phys. Rev. 187, 1328 (1969).

[19]H. Nakamura, private communication.

[20]H. Nakamura, Nucl. Phys. A137, 51 (1969).

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Off-Shell t Matrix for the Boundary-Condition Model*

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An off-shell t matrix is developed for the boundary-condition model in the general case of coupled partial waves. This development is facilitated by the use of a method for solving the Lippmann-Schwinger equation directly for potentials of the square-well type. The t matrix obtained is shown to be unique under some rather mild assumptions as to analyticity and asymptotic behavior. An integral equation of the Lippmann-Schwinger type is obtained for the t matrix in the more realistic problem of boundary condition plus external potential.

INTRODUCTION

Perhaps the chief difficulty in constructing potential models to represent the effective nucleon-nucleon (N-N) interaction is the question of what to do about extremely short-range effects. In the region of small internucleon separation, it has long been recognized[1] that a local potential cannot adequately describe multimeson exchange and other inherently nonlocal higher-order effects suggested by the meson theory of nuclear forces. Consequently, attempts have been made to simulate the infinite complexity of the interaction in this region by introducing either highly repulsive short-range potentials (soft cores),[2] or simple nonlocal devices such as the hard core[3] and its generalization, the boundary-condition model (BCM).[4]

Together with appropriate longer-range components, all of these approaches can be employed to yield models[5] which give theoretical predictions in quite good agreement with the N-N scattering data up to the vicinity of 350 MeV. However, because of continuing improvements in computer facilities and computational techniques, it should be possible in the near future to discriminate between these models by employing them in the Faddeev equations[6] to calculate properties of the three-nucleon system. In doing so, one must learn how to properly incorporate singular two-body interactions such as the hard-core and BCM into the three-body framework. One of the virtues of the Faddeev formulation from this standpoint is that the dependence on the two-nucleon potentials can be entirely eliminated in favor of the off-shell two-nucleon t matrices.[7] On the other hand, it is not entirely clear what one should regard as the appropriate off-shell t matrix in such cases. It cannot, for example, be defined as the solution of the Lippmann-Schwinger (LS) equation with an energy-independent potential.[8]

In this paper we develop an off-shell t matrix for the BCM in the general case of coupled partial waves (tensor mixing). Furthermore, we show that the prescription is unique, providing one makes some mild assumptions regarding its analyticity and asymptotic behavior as a function of the energy. The BCM t matrix obtained is used to formulate a modified LS equation for the total t matrix in the general problem of BCM plus external potential; the convergence properties of the kernel of this equation are essentially identical to those of the LS kernel with the external potential alone.

We begin in Sec. II by developing a technique for directly solving the LS equation for potentials of the square-well type. In Sec. III we consider the special case of uncoupled partial waves; in this case it has been shown by Kim and Tubis[9] that the BCM can be formulated as the limit of a certain potential model. We give a derivation of the BCM t matrix for this case by employing the method of Sec. II to explicitly solve the LS equation for this potential, and subsequently going to the limit. Although necessarily equivalent, the expressions we obtain in this fashion are considerably simpler in form than those of Kim and Tubis. This simplicity of form, together with the considerations involved in this method of derivation, suggest a straightforward generalization to coupled partial waves. This is discussed in Sec. IV, where we also derive the modified LS equation described above.

The question of uniqueness is considered in Sec. V, as part of a discussion of our results. Here we note some special features of the BCM t matrix which should facilitate its use in the Faddeev equations.

II. SOLUTION OF LIPPMANN-SCHWINGER EQUATION

In this section we consider the direct solution of the LS equation for the off-shell t matrix in the case of a square-well potential. As is well known,[10] the square-well problem may be solved trivially in the differential-equation formulation (Schrödinger equation), and the off-shell t matrix constructed from the result. However, it is of some interest to learn to solve directly the type of integral equations which occur frequently in scattering theory, not all of which correspond to well-studied differential equations. The method to be described is based on a simple analyticity argument and is, in spirit, an extension of earlier work[11] by the author on quite different classes of potentials. The present problem, however, has rather special features of its own which require a modification of the earlier approach.

The method will first be developed in the context of the s-wave LS equation with a single square well, and will then be generalized to higher partial waves and superpositions of such potentials. The solutions thus derived are, of course, identical to those obtained by other means, although they are represented somewhat differently. However, it is not the solutions themselves but rather the considerations involved in their derivation which will be useful in the subsequent sections.

We consider the potential $V(r)=g\theta(a-r)$, for which the $l=0$ LS equation has the form

$$a_0(p',p;s)=v_0(p',p)-\int_0^\infty \frac{dq\,q^2}{q^2-s-i\epsilon}\times v_0(p',q)a_0(q,p;s)\,, \tag{1}$$

where

$$v_0(p',p)=\frac{ga}{\pi p p'}[j_0(a(p'-p))-j_0(a(p'+p))]. \tag{2}$$

Here for convenience we have taken the reduced mass to be $\frac{1}{2}$; $a_0(\kappa,\kappa;s)$ is the s-wave scattering amplitude ($\kappa=s^{1/2}$) with normalization such that

$$a_0(\kappa,\kappa;s)=-\frac{2}{\pi\kappa}e^{i\delta_0}\sin\delta_0\,.$$

Regarding p and s as fixed parameters, $a_0(p',p;s)$ can be considered as a function of the single variable p'. Assuming that a solution exists,[12] Eq. (1) serves to determine this function for real positive values of p'. However, Eq. (1) can also be used to extend the domain of $a_0(p',p;s)$ to negative and complex values of p'. To see this we first note that Eq. (2) implies that $v_0(p',p)$ is an even function of p', and hence, from Eq. (1),

$$a_0(-p',p;s)=a_0(p',p;s)\,, \tag{3}$$

which defines a_0 for negative values of p'. With this definition we can rewrite Eq. (1) in the form

$$a_0(p',p;s)=v_0(p',p)-(1/p')I(p')\,, \tag{4}$$

with

$$I(p')=\frac{ga}{\pi}\int_{-\infty}^{\infty}\frac{dq\,q}{q^2-s-i\epsilon}j_0(a(p'-q))a_0(q,p;s)\,.$$

Thus $a_0(p',p;s)$ can be expressed in terms of the known function $v_0(p',p)$ and a function $I(p')$ defined through a certain integral representation.

From the known properties of the spherical Bessel functions it is trivial to verify that $v_0(p',p)$ has the following properties:

$$\begin{aligned}&\text{(a) } v_0(p',p) \text{ is an entire function of } p'\,,\\ &\text{(b) } \lim_{\operatorname{Im}p'\to+\infty}[e^{iap'}v_0(p',p)]=0\,,\\ &\text{(c) } \lim_{\operatorname{Im}p'\to-\infty}[e^{-iap'}v_0(p',p)]=0\,.\end{aligned} \tag{5}$$

Furthermore, we assert that these same properties hold for $I(p')$ as well. To prove this, we first note that the integrand of the integral defining $I(p')$ is an entire function of p' for fixed q. Hence, by a standard property of integral representations, we infer that $I(p')$ will also be entire providing that the integral converges for all finite complex values of p'. It is clear that the existence of a solution to Eq. (1) insures this convergence for real p'. Thus, for real p', $|I(p')| < m$, where m is some positive bound. Moreover, since $v_0(p',p)$ is also bounded, Eq. (4) implies that for real p'

$$|p' a_0(p',p;s)| < M \,, \tag{6}$$

where M is finite. Applying Eq. (6) to the quantity $q a_0(q,p;s)$ appearing in the integrand of Eq. (4), and using the explicit formula for $j_0(x)$, it is straightforward to show that $I(p')$, for $|\mathrm{Im}p'| > 0$, satisfies a bound of the form

$$|I(p')| < \frac{1}{|\mathrm{Im}p'|}\left[c_1 e^{a\,\mathrm{Im}p'} + c_2 e^{-a\,\mathrm{Im}p'}\right]. \tag{7}$$

From Eq. (7) and the above comments we see immediately that $I(p')$ satisfies the properties stated in Eq. (5).

Returning to Eq. (4), we conclude that $a_0(p',p;s)$ can easily be extended to complex values of p'; indeed, it is an entire function of p' satisfying the properties of Eq. (5). With this information it is trivial to construct the solution to Eq. (1). We first define the functions[13]

$$h_l^{\pm}(x) = j_l(x) \pm i n_l(x) \,, \tag{8}$$

where $n_l(x)$ is the spherical Bessel function of the second kind. With this definition

$$h_l^{\pm}(x) \underset{x\to\infty}{\longrightarrow} \mp \frac{i e^{\pm ix}}{x} \,, \qquad h_0^{\pm}(x) \underset{x\to 0}{\longrightarrow} \mp \frac{i}{x} \,. \tag{9}$$

Thus $h_0^{\pm}(x)$ is analytic except for a simple pole at $x = 0$. We can now rewrite the definition of $I(p')$ in the form

$$I(p') = \frac{ga}{2\pi}\left[J^+(p') + J^-(p')\right], \tag{10}$$

with

$$J^{\pm}(p') = \int_{-\infty}^{\infty} \frac{dq\, q}{q^2 - s - i\epsilon} h_0^{\pm}(a(p'-q))\, a_0(q,p;s) \,.$$

We note that due to the pole of $h_0^{\pm}$ the functions $J^{\pm}(p')$ are not defined when p' is real; to avoid this ambiguity we will assume $\mathrm{Im}p' > 0$ when evaluating them. Our result will obviously not depend on this choice.

We next observe that, as a result of the analytic and asymptotic properties determined above, the integrals defining $J^{\pm}(p')$ can be evaluated by the method of residues. Thus the integral for $J^+(p')$ can be closed with a semicircular contour of infinite radius in the lower half plane; the contribution from this contour vanishes because of Eqs. (5) and (9). Within this contour the only singularity is a simple pole at $q = -s^{1/2}$ (the $i\epsilon$ condition is equivalent to $\mathrm{Im}s^{1/2} \geq 0$). Thus, recalling that we have used $\kappa = s^{1/2}$,

$$J^+(p') = -\pi i h_0^+(a(p'+\kappa))\, a_0(\kappa,p;s) \,. \tag{11}$$

Similarly, the integral for $J^-(p')$ is closed in the upper half plane and there are poles at $q = s^{1/2}$ and $q = p'$; hence

$$J^-(p') = \pi i\left[h_0^-(a(p'-\kappa))\, a_0(\kappa,p;s) - \frac{2i}{a}\frac{p'}{p'^2 - s} a_0(p',p;s)\right]. \tag{12}$$

Combining terms, we arrive at

$$I(p') = \frac{-p'g}{p'^2 - s}\left[-a_0(p',p;s) + B(p',a,\kappa)\, a_0(\kappa,p;s)\right], \tag{13}$$

where we have defined

$$B(x,a,y) = e^{iay}\left(\cos ax - iy\frac{\sin ax}{x}\right). \tag{14}$$

By inserting this result for $I(p')$ in Eq. (4) we obtain

$$a_0(p',p;s) = \frac{(p'^2 - s)\, v_0(p',p) + gB(p',a,\kappa)\, a_0(\kappa,p;s)}{p'^2 - s + g} \,, \tag{15}$$

which determines the off-shell amplitude in terms of the half-on-shell function $a_0(\kappa,p;s)$. To determine the latter, we note that if we insert the particular value $p' = \kappa$ into Eq. (15) it simply reduces to an identity. However, we also note that the right-hand side of Eq. (15) appears to have poles at $p' = \pm p_0$, where $p_0{}^2 = s - g$. Since $a_0(p',p;s)$ cannot have poles by the facts established above, it is clear that the numerator of Eq. (15) must also vanish at these points. This gives us a relation from which to determine $a_0(\kappa,p;s)$, and we obtain

$$a_0(\kappa,p;s) = v_0(p_0,p)/B(p_0,a,\kappa) \,. \tag{16}$$

Equations (15) and (16) provide the explicit solution to Eq. (1).

Before considering some simple generalizations of this result, we note the following useful integral formula. If $F(p')$ is any even function of p' which satisfies the properties stated in Eq. (5), then

$$\int_0^\infty \frac{dq\,q^2}{q^2-s-i\epsilon} v_0(p',q)F(q) = \frac{g}{p'^2-s}[F(p')-B(p',a,\kappa)F(\kappa)]\,. \tag{17}$$

This is a simple consequence of the fact that our evaluation of $I(p')$ above depended only on the analytic properties of $a_0(q,p;s)$.

We next consider the solution of the LS equation in partial wave L; for definiteness we will assume L to be an even integer (the method will work equally well for odd-integer L). The problem simplifies through use of the formula

$$v_L(p',p) = v_0(p',p) - \frac{2ga}{\pi p p'} \times \sum_{l=0,2,\ldots}^{L-2} (2l+3) j_{l+1}(ap') j_{l+1}(ap)\,, \tag{18}$$

which is valid for even L. In order to determine $a_L(p',p;s)$ it is necessary to evaluate the integral

$$\int_0^\infty \frac{dq\,q^2}{q^2-s-i\epsilon} v_L(p',q) a_L(q,p;s)\,. \tag{19}$$

This is accomplished by using the same reasoning as above to verify that a_L satisfies the conditions of Eq. (5); hence, by writing v_L in the form of Eq. (18), the part of the integral involving v_0 can be evaluated via Eq. (17). It only remains to determine the integrals

$$K_l = \int_0^\infty \frac{dq\,q}{q^2-s-i\epsilon} j_{l+1}(aq) a_L(q,p;s)\,. \tag{20}$$

Writing $2j_l(x) = h_l^+(x) + h_l^-(x)$, which follows from Eq. (8), it is easy to verify that

$$h_l^\pm(x) \underset{x\to 0}{\longrightarrow} c/x^{l+1}\,, \qquad a_L(q,p;s) \underset{q\to 0}{\longrightarrow} c' q^L\,, \tag{21}$$

because of the general properties of the partial-wave amplitudes. We only require K_l for $l \leq L-2$, and for such l Eq. (21) implies that $h_{l+1}^\pm(aq)a_L(q,p;s)$ is finite at $q=0$. The integrand of K_l is an even function of q, thus

$$K_l = \frac{1}{4}\int_{-\infty}^\infty \frac{dq\,q}{q^2-s-i\epsilon}[h_{l+1}^+(aq) + h_{l+1}^-(aq)] a_L(q,p;s)\,. \tag{22}$$

The two parts of the integral can again be evaluated by the method of residues through consideration of the properties given in Eqs. (5) and (9). In this case the only singularities are the simple poles at $q = \pm s^{1/2}$, and we obtain the result

$$K_l = \tfrac{1}{2}\pi i h_{l+1}^+(a\kappa) a_L(\kappa,p;s)\,. \tag{23}$$

Defining

$$f_L(p,a,\kappa) = B(p,a,\kappa) + (p^2-\kappa^2) i a^2 \times \sum_{l=0,2,\ldots}^{L-2} (2l+3) h_{l+1}^+(a\kappa) \frac{j_{l+1}(ap)}{ap}\,, \tag{24}$$

which implies that $f_L(\kappa,a,\kappa)=1$; we have determined that the integral of Eq. (19) has the value

$$\frac{g}{p'^2-s}[a_L(p',p;s) - f_L(p',a,\kappa) a_L(\kappa,p;s)]\,. \tag{25}$$

Thus,

$$a_L(p',p;s) = \frac{(p'^2-s)v_L(p',p) + g f_L(p',a,\kappa) a_L(\kappa,p;s)}{p'^2-p_0^2}\,. \tag{26}$$

As in the s-wave case, the half-on-shell amplitude is determined by requiring the numerator of Eq. (26) to vanish when $p'=p_0$. We then obtain

$$a_L(\kappa,p;s) = v_L(p_0,p)/f_L(p_0,a,\kappa)\,. \tag{27}$$

We conclude this section by outlining how the procedure may be extended to handle potentials of the form

$$V(r) = \sum_{m=1}^M g_m \theta(a_m - r)\,, \tag{28}$$

where we assume the ranges are ordered such that $a_1 > a_2 > \cdots > a_M$. For simplicity, we consider only the s-wave case. Defining $v_0(p',p;g,a)$ to be what we called $v_0(p',p)$ in Eq. (2), our present v_0 becomes

$$v_0(p',p) = \sum_{m=1}^M v_0(p',p;g_m,a_m)\,. \tag{29}$$

It is easy to verify that $a_0(p',p;s)$ now satisfies the properties of Eq. (5), but with a replaced by a_1. Henceforth we shall say that such a function satisfies the "a_1 condition." Thus, applying Eq. (17),

$$a_0(p',p;s) = v_0(p',p) - \frac{g_1}{p'^2-s}[a_0(p',p;s) - B(p',a_1,\kappa)a_0(\kappa,p;s)] - \int_0^\infty \frac{dq\,q}{q^2-s-i\epsilon} v_0^{(1)}(p',q) a_0(q,p;s)\,, \tag{30}$$

where we have defined

$$v_0^{(1)}(p',p) = v_0(p',p) - v_0(p',p;g_1,a_1)\,. \tag{31}$$

We now define a new function $a_0^{(1)}(p',p;s)$ by the relation

$$a_0(p',p;s)=\frac{p'^2-s}{p'^2-p_1^2}\left[a_0^{(1)}(p',p;s)+v_0(p',p;g_1,a_1)+g_1\frac{B(p',a_1,\kappa)}{p'^2-s}a_0(\kappa,p;s)\right], \tag{32}$$

where

$$p_1^2=s-g_1.$$

It follows from Eq. (30) that $a_0^{(1)}(p',p;s)$ satisfies the new integral equation

$$a_0^{(1)}(p',p;s)=Z_0^{(1)}(p',p;s)-\int_0^\infty\frac{dq\,q^2}{q^2-p_1^2-i\epsilon}v_0^{(1)}(p',q)a_0^{(1)}(q,p;s), \tag{33}$$

with

$$\begin{aligned}Z_0^{(1)}(p',p;s)&=v_0^{(1)}(p',p)-\int_0^\infty\frac{dq\,q^2}{q^2-p_1^2-i\epsilon}v_0^{(1)}(p',q)\left[v_0(q,p;g_1,a_1)+g_1\frac{B(q,a_1,\kappa)}{q^2-s-i\epsilon}a_0(\kappa,p;s)\right]\\&=\frac{p^2-s}{p^2-p_1^2}v_0^{(1)}(p',p)+g_1\left[\frac{B(p,a_1,p_1)}{p^2-p_1^2}-\frac{\pi i}{2}\frac{e^{ia_1(p_1+\kappa)}}{p_1+\kappa}a_0(\kappa,p;s)\right]v_0^{(1)}(p',p_1).\end{aligned} \tag{34}$$

In evaluating $Z_0^{(1)}$ we have used Eq. (17) and the fact that $v_0(q,p;g_1,a_1)=v_0(p,q;g_1,a_1)$, as well as the new formula

$$\int_0^\infty\frac{dq\,q^2}{q^2-x^2-i\epsilon}G(q)\frac{B(q,b,y)}{q^2-y^2-i\epsilon}=\frac{\pi i}{2}\frac{e^{ib(x+y)}}{x+y}G(x), \tag{35}$$

which holds providing that $G(q)$ is an even function of q satisfying the b condition.

The half-on-shell function $a_0(\kappa,p;s)$ is determined from Eq. (32) by requiring the right-hand side to be finite at $p'=p_1$; this gives us

$$a_0(\kappa,p;s)=\frac{a_0^{(1)}(p_1,p;s)+v_0(p_1,p;g_1,a_1)}{B(p_1,a_1,\kappa)}. \tag{36}$$

Considering Eq. (33), we note that p_1 is the "on-shell" value for the function $a_0^{(1)}$. Thus, the problem has been reduced to the determination of $a_0^{(1)}$ through the solution of Eq. (33). However, this equation is equivalent in structure to Eq. (1), and our by now familiar argument shows us that $a_0^{(1)}$ satisfies the a_2 condition. That is, we may repeat the procedure which led us from the equation for a_0 to the equation for $a_0^{(1)}$, obtaining a similar equation for a new function $a_0^{(2)}$, etc. With the solution of the equation for $a_0^{(M-1)}$ the solution of the total problem is complete.

III. BCM t MATRIX: A SPECIAL CASE

In this section we consider a derivation for the BCM t matrix in the particular case of uncoupled partial waves, i.e., in the absence of tensor forces. For this case it has been shown by Kim and Tubis[9] that the boundary condition can be formulated in terms of the potential

$$V(r)=g\theta(a-r)+g'a\delta(r-a), \tag{37}$$

with $g>0$, $g'<0$, in the limit as g, $g'\to\infty$ such that

$$\lambda_l=\sqrt{g}+ag'-1/a \tag{38}$$

remains finite (clearly the limit must be taken separately in each partial wave). This fact enables us to obtain the BCM t matrix by solving the LS equation with potential $V(r)$ and going explicitly to the above limit in our result. This approach is facilitated by using the method of the previous section to solve the appropriate LS equation.

In obtaining the solution we shall make use of the generalization of Eq. (17) to arbitrary l,

$$\int_0^\infty\frac{dq\,q^2}{q^2-s-i\epsilon}v_l(p,q)G(q)=\frac{g}{p^2-s}[G(p)-f_l(p,a,\kappa)G(\kappa)], \tag{39}$$

which holds providing that $G(p)=(-1)^lG(-p)$, behaves like p^l as $p\to0$ and satisfies the a condition. This expression has been derived for even l in Sec. II [compare Eqs. (19) and (25)], in which case f_l is given by Eq. (24). The derivation for odd l is similar, and one obtains

$$f_l(p,a,\kappa)=C(p,a,\kappa)+(p^2-\kappa^2)ia^2\times\sum_{m=1,3,\ldots}^{l-2}(2m+3)h_{m+1}^{+}(a\kappa)\frac{j_{m+1}(ap)}{ap},$$

where

$$C(p,a,\kappa)=f_1(p,a,\kappa)=\frac{e^{ia\kappa}}{a\kappa}[(1-ia\kappa)\sin ap-a^2\kappa^2j_1(ap)]. \tag{40}$$

As is the case for even l, $f_l(\kappa, a, \kappa) = 1$.

The LS equation now has the form

$$a_l(p',p;s) = w_l(p',p) - \int_0^\infty \frac{dq\,q^2}{q^2 - s - i\epsilon} \times w_l(p',q) a_l(q,p;s), \qquad (41)$$

where

$$w_l(p',p) = v_l(p',p) + \frac{2}{\pi} g' a^3 j_l(ap) j_l(ap'). \qquad (42)$$

As before, $v_l(p',p)$ corresponds to the square-well potential. It is easy to see that the analyticity argument introduced in the preceding section can be carried over without change. Thus a_l satisfies the a condition and

$$\int_0^\infty \frac{dq\,q^2}{q^2 - s - i\epsilon} j_l(aq) a_l(q,p;s) = \frac{\pi i \kappa}{2} h_l^+(a\kappa) a_l(\kappa,p;s). \qquad (43)$$

Together with Eq. (39), this implies that the solution to Eq. (41) is given by

$$a_l(p',p;s) = \frac{(p'^2 - s) w_l(p',p) + [g f_l(p', a, \kappa) - i\kappa g' a^3 (p'^2 - s) h_l^+(a\kappa) j_l(ap')] a_l(\kappa,p;s)}{p'^2 - p_0^2} \qquad (44)$$

with $p_0^2 = s - g$.

The value of $a_l(\kappa,p;s)$ is obtained by requiring the numerator to vanish when $p' = p_0$; hence

$$a_l(\kappa,p;s) = \frac{w_l(p_0,p)}{f_l(p_0, a, \kappa) + i\kappa g' a^3 h_l^+(a\kappa) j_l(ap_0)}. \qquad (45)$$

It is straightforward, but somewhat tedious, to show that

$$t_l^{BC}(s) = \lim_{g, g' \to \infty} a_l(\kappa, \kappa; s), \qquad (46)$$

the limits being taken as prescribed in Eq. (38), where $t_l^{BC}(s)$ is the usual BCM scattering amplitude. Likewise, the limits can be performed in Eq. (44) to obtain the off-shell BCM t matrix

$$t_l^{BC}(p',p;s) = \lim_{g, g' \to \infty} a_l(p',p;s) \qquad (47)$$

in terms of the half-on-shell quantity $t_l^{BC}(\kappa,p;s)$. Thus

$$t_l^{BC}(p',p;s) = (p'^2 - s)\tilde{v}_l(p',p) + f_l(p', a, \kappa) t_l^{BC}(\kappa,p;s), \qquad (48)$$

where we have defined

$$\tilde{v}_l(p',p) = v_l(p',p;1,a), \qquad (49)$$

corresponding to a square-well potential of range a and unit strength.

The half-on-shell t matrix $t_l^{BC}(\kappa,p;s)$ may be obtained by taking the limit of Eq. (45); however, it is simpler to set $p = \kappa$ in Eq. (48) and use the symmetry of $t_l^{BC}(p',p;s)$ under exchange of p' and p to infer that

$$t_l^{BC}(\kappa,p;s) = (p^2 - s)\tilde{v}_l(p, \kappa) + f_l(p, a, \kappa) t_l^{BC}(s). \qquad (50)$$

Finally, substitution of Eq. (50) into Eq. (48) gives the result

$$t_l^{BC}(p',p;s) = R_l(p',p;s) + f_l(p', a, \kappa) f_l(p, a, \kappa) t_l^{BC}(s), \qquad (51)$$

where

$$R_l(p',p;s) = (p'^2 - s)\tilde{v}_l(p',p) + f_l(p', a, \kappa)(p^2 - s)\tilde{v}_l(p, \kappa).$$

Clearly $R_l(p',p;s)$ must be symmetric in p' and p, as may be verified directly. Using the relation

$$f_l(p, a, \kappa_+) - (-1)^l f_l(p, a, \kappa_-) = -i\pi\kappa(p^2 - s)\tilde{v}_l(p, \kappa), \quad \kappa_\pm = (s \pm i\epsilon)^{1/2} = \pm\kappa, \qquad (52)$$

which is easily established via Eq. (39); it is straightforward to show that our expression for t_l^{BC} satisfies the off-shell unitarity relation

$$t_l^{BC}(p',p;s + i\epsilon) - t_l^{BC}(p',p;s - i\epsilon) = -i\pi\kappa t_l^{BC}(p', \kappa; s + i\epsilon) t_l^{BC}(\kappa,p;s - i\epsilon). \qquad (53)$$

We have thus obtained a remarkably simple formula for the off-shell BCM t matrix which has a number of interesting properties. For example, the dependence of the off-shell t matrix on the logarithmic-derivative parameter λ_l is contained entirely in its explicit dependence on $t_l^{BC}(s)$. Secondly, contrary to what one might expect from a pseudopotential formulation of the type proposed by Hoenig and Lomon,[14] the BCM t matrix obtained is not separable. Also, although our t matrix is analytic in s except for the right-hand cut, and (possibly) bound-state poles contained in $t_l^{BC}(s)$, it approaches $-sv_l(p',p)$ as $s \to \infty$. In contrast, t matrices resulting from conventional potentials approach the corresponding potential in this limit.

It is instructive to examine the manner in which the BCM t matrix produces the desired boundary condition on the wave function $\psi_{\kappa l}(r)$. The two are related by the expression

$$\psi_{\kappa l}(r) = \left(\frac{2}{\pi}\right)^{1/2} i^l \left[j_l(\kappa r) - \int_0^\infty \frac{dp\,p^2}{p^2 - s - i\epsilon} j_l(pr) t_l^{BC}(p, \kappa; s) \right]. \qquad (54)$$

The integral in Eq. (54) can be performed analytically with the aid of the following simple formulas:

$$\int_0^\infty \frac{dq\, q^2}{q^2-s-i\epsilon} f_l(q,a,\kappa)G(q)=0\,;$$
$$\int_0^\infty \frac{dq\, q^2}{q^2-s-i\epsilon} j_l(bq)G(q)=\frac{\pi i\kappa}{2}h_l^+(b\kappa)G(\kappa)\,,\quad (b>a)\,;$$
$$\int_0^\infty dq\, q^2\tilde{v}_l(q,p)G(q)=G(p)\,. \tag{55}$$

These formulas hold provided that $G(q)$ satisfies the conditions discussed in relation to Eq. (39) and are easily obtained in a similar fashion.

Using the expression of Eq. (50) for $t_l^{\rm BC}(p,\kappa;s)$, we therefore obtain

$$\psi_{\kappa l}(r)=0\,,\quad r<a\,;$$
$$=\left(\frac{2}{\pi}\right)^{1/2} i^l\left[j_l(\kappa r)-\frac{\pi i\kappa}{2}h_l^+(\kappa r)t_l^{\rm BC}(s)\right],\quad r>a\,. \tag{56}$$

The wave function then satisfies the boundary condition

$$\lim_{\epsilon\to 0}\frac{\psi_{\kappa l}{}'(a+\epsilon)}{\psi_{\kappa l}(a+\epsilon)}=\lambda_l\,, \tag{57}$$

provided that

$$t_l^{\rm BC}(s)=\frac{2}{\pi i\kappa}\,\frac{\lambda_l j_l(\kappa a)-\kappa j_l{}'(\kappa a)}{\lambda_l h_l^+(\kappa a)-\kappa h_l^+{}'(\kappa a)}\,, \tag{58}$$

the prime denoting differentiation with respect to the argument. Equation (58), of course, is the usual expression for the on-shell BCM t matrix.

In a sense, therefore, the expression given in Eq. (50) "works" because, in addition to satisfying unitarity and reducing to $t_l^{\rm BC}(s)$ when $p=\kappa$, the integration properties summarized in Eq. (55) lead to Eq. (56). This observation provides the key to the generalization considered in the next section.

IV. BCM t MATRIX: GENERAL CASE

In the case of coupled partial waves, there is at present no counterpart to the development given in the previous section. That is, one does not know of a potential analogous to that of Eq. (37) which gives the BCM t matrix in some appropriate limit. Instead, we shall adopt an alternative approach based on the considerations discussed in the latter part of Sec. III. The t matrix obtained in this fashion is a simple generalization of Eq. (51), and reduces to that expression when the parameter coupling the partial waves tends to zero. In the latter part of this section we shall employ this result to obtain a modified LS equation for the problem of a boundary condition plus an external potential.

We first note that in a state of total angular momentum J, the wave function and t matrix are related by

$$\psi_{\kappa l}^J(r)=\left(\frac{2}{\pi}\right)^{1/2} i^l\left[j_l(\kappa r)-\sum_{l'=|J-S|}^{J+S}\int_0^\infty \frac{dp\,p^2}{p^2-s-i\epsilon}j_{l'}(pr)t_{l'l}^{J;\rm BC}(p,\kappa;s)\right], \tag{59}$$

where S is the total spin. In analogy with Eq. (56), we want a form for $t_{l'l}^{J;\rm BC}(p,\kappa;s)$ such that

$$\psi_{\kappa l}^J(r)=0\,,\quad r<a\,;$$
$$=\left(\frac{2}{\pi}\right)^{1/2} i^l\left[j_l(\kappa r)-\frac{\pi i\kappa}{2}\sum_{l'}h_{l'}^+(\kappa r)t_{l'l}^{J;\rm BC}(s)\right],\quad r>a\,. \tag{60}$$

[The relation between $t_{l'l}^{J;\rm BC}(s)$ and the BCM amplitude of Feshbach and Lomon (FL) is given in the Appendix.] Such a form is immediately suggested by Eq. (50) and the integral formulas of Eq. (55). We consider

$$t_{l'l}^{J;\rm BC}(p,\kappa;s)=\delta_{l'l}(p^2-s)\tilde{v}_l(p,\kappa)+f_{l'}(p,a,\kappa)t_{l'l}^{J;\rm BC}(s)\,. \tag{61}$$

It is easy to verify that this expression, when substituted into Eq. (59), results in Eq. (60). Furthermore, it reduces to $t_{l'l}^{J;\rm BC}(s)$ when $p=\kappa$ and satisfies the unitarity relation

$$t_{l'l}^{J;\rm BC}(p,\kappa;s+i\epsilon)-t_{l'l}^{J;\rm BC}(p,\kappa;s-i\epsilon)=-i\pi\kappa\sum_m t_{l'm}^{J;\rm BC}(p,\kappa;s+i\epsilon)t_{ml}^{J;\rm BC}(s-i\epsilon)\,, \tag{62}$$

provided that $t_{l'l}^{J;\rm BC}(s)$ satisfies on-shell unitarity, i.e., Eq. (62) with $p=\kappa$. Clearly, Eq. (61) reduces to Eq. (50) as the coupling between partial waves vanishes; that is, as $t_{l'l}^{J;\rm BC}\to 0$ for $l'\neq l$.

We note that the fully off-shell t matrix we are looking for must satisfy the unitarity relation

$$t_{l'l}^{J;\rm BC}(p',p;s+i\epsilon)-t_{l'l}^{J;\rm BC}(p',p;s-i\epsilon)=-i\pi\kappa\sum_m t_{l'm}^{J;\rm BC}(p',\kappa;s+i\epsilon)t_{ml}^{J;\rm BC}(\kappa,p;s-i\epsilon)\,, \tag{63}$$

the right-hand side of which is known because of Eq. (61), and the relation

$$t_{l'l}^{J;\mathrm{BC}}(p',p;s)=t_{ll'}^{J;\mathrm{BC}}(p,p';s)\,, \tag{64}$$

which is a consequence of time-reversal invariance. Analogy with Eq. (51) leads us to consider the form

$$t_{l'l}^{J;\mathrm{BC}}(p',p;s)=\delta_{l'l}R_l(p',p;s)+f_{l'}(p',a,\kappa)f_l(p,a,\kappa)t_{l'l}^{J;\mathrm{BC}}(s)\,, \tag{65}$$

which satisfies Eqs. (63) and (64) and reduces to Eq. (61) when p is on shell. Except for the right-hand cut, this expression is analytic in s except for (possibly) bound-state poles contained in $t_{l'l}^{J;\mathrm{BC}}(s)$. Furthermore, the residue at such a pole is consistent with that of the half-on-shell form and the factorization property.[15] We can thus assert that Eq. (65) is unique, given Eq. (61), provided that we make some mild additional assumptions concerning asymptotic behavior. We shall return to this point in Sec. V.

We now turn to the physically more interesting problem of a boundary condition plus an external potential $V_e(r)$, where we assume

$$\begin{aligned} &V_e(r)=0\,, \quad r<a\,;\\ &\lim_{r\to\infty}[rV_e(r)]=0\,. \end{aligned} \tag{66}$$

In general, $V_e(r)$ will contain tensor and spin-orbit as well as central terms. For our two-particle state characterized by J and S, it is convenient to introduce the states $|lp\rangle$, such that

$$\begin{aligned} &\langle l'p'|lp\rangle=\delta_{l'l}\delta(p'-p)/p^2\,,\\ &\langle r|lp\rangle=(2/\pi)^{1/2}i^l j_l(pr)\,. \end{aligned} \tag{67}$$

We will regard t^J as an operator on these states, in terms of which V_e has the representation $\langle l'p'|V_e|lp\rangle$ (the dependence of this matrix element on J and S is implicit).

In order to derive an integral equation for t^J we shall take the point of view that there is a well-behaved potential $V_c(r)$, analogous to that of Eq. (37), which in some limit gives rise to the BCM t matrix of Eq. (65). This allows us to perform the standard manipulations relevant to scattering from two potentials to the equation

$$\tau^J=V-VG_0\tau^J\,, \tag{68}$$

where

$$V=V_e+V_c\,.$$

Thus, introducing

$$a^J=V_c-V_cG_0a^J\,, \tag{69}$$

we obtain

$$\tau^J=a^J+K-KG_0\tau^J\,, \tag{70}$$

where

$$K=(1-a^JG_0)V_e\,.$$

By assumption, $a^J\to t^{J;\mathrm{BC}}$ when the appropriate limit is taken. Since only a^J (and not V_c) appears in Eq. (70), we pass to this limit to obtain

$$\begin{aligned} t^J&=\lim\tau^J\\ &=t^{J;\mathrm{BC}}+\tilde K-\tilde KG_0t^J\,, \end{aligned} \tag{71}$$

with $\tilde K=(1-t^{J;\mathrm{BC}}G_0)V_e$. Taking matrix elements between the states $|lp\rangle$, Eq. (71) becomes an integral equation for $t^J_{l'l}(p',p;s)$. By using the explicit form for $t^{J;\mathrm{BC}}$ given in Eq. (65), it is easy to verify that $\tilde K$ has the same convergence properties as V_e. Together with the second part of Eq. (66) and standard arguments,[16] this implies that t^J as given from Eq. (71) is well defined.

Since our derivation of Eq. (71) was not rigorous, it is worth checking that t^J does give rise to the proper boundary condition with arbitrary V_e. The wave function $\psi^J_{\kappa l}(r)$ and t^J are related by

$$\psi^J_{\kappa l}(r)=\langle r|1-G_0t^J|l\kappa\rangle\,. \tag{72}$$

Hence, defining

$$\psi_l^{J;\mathrm{BC}}(r,p)=\langle r|1-G_0t^{J;\mathrm{BC}}|lp\rangle\,, \tag{73}$$

we have

$$\psi^J_{\kappa l}(r)=\psi_l^{J;\mathrm{BC}}(r,\kappa)-\sum_{l'}\int_0^\infty dp\,p^2\psi_{l'}^{J;\mathrm{BC}}(r,p)\langle l'p|G_0V_e(1-G_0t^J)|l\kappa\rangle\,. \tag{74}$$

Explicitly,

$$\begin{aligned} \psi_l^{J;\mathrm{BC}}(r,p)&=0\,, \quad r<a\,;\\ &=\left(\frac{2}{\pi}\right)^{1/2}i^l\left[j_l(pr)-\frac{\pi i\kappa}{2}\sum_{l'}h^+_{l'}(\kappa r)t_{l'l}^{J;\mathrm{BC}}(\kappa,p;s)\right], \quad r>a\,. \end{aligned} \tag{75}$$

Clearly, $\psi_l^{J;\text{BC}}(r,\kappa)$ is the wave function for the pure BCM given in Eq. (60).

In order to show that $\psi_{\kappa l}^J(r)$ does indeed satisfy the boundary condition, we write

$$\psi_l^{J;\text{BC}}(r,p)=\chi_{\kappa l}^J(r,p)+f_l(p,a,\kappa)\psi_l^{J;\text{BC}}(r,\kappa)\,, \tag{76}$$

where

$$\chi_{\kappa l}^J(r,p)=\left(\frac{2}{\pi}\right)^{1/2} i^l\left[j_l(pr)-f_l(p,a,\kappa)j_l(\kappa r)-\frac{\pi i\kappa}{2}h_l^+(\kappa r)(p^2-s)\tilde{v}_l(p,\kappa)\right],\quad (r>a)\,, \tag{77}$$

as a consequence of Eq. (61). We now assert that $\chi_{\kappa l}^J(a,p)=\chi_{\kappa l}^{J}{}'(a,p)=0$. This can be verified directly with some labor, but one can make a much simpler argument based on analyticity. We note that for fixed r and p, $\chi_{\kappa l}^J(r,p)$ is an entire function of s; that is, there is an apparent cut due to the dependence on κ, but the discontinuity vanishes by Eq. (52). In other words, $\chi_{\kappa l}^J(r,p)$ is an even function of κ. Checking next the asymptotic behavior, it is easy to see that $\chi_{\kappa l}^J(a,p)$ is bounded by a constant as $s\to\infty$. However, the only entire function with this property is a constant, and the constant must be zero since $\chi_{\kappa l}^J(a,p)$ vanishes when $\kappa=p$. Similarly, we observe that $\chi_{\kappa l}^{J}{}'(a,p)$ is bounded by a first-degree polynomial in κ; this implies that $\chi_{\kappa l}^{J}{}'(a,p)$ is a first-degree polynomial. However, the above argument implies that $\chi_{\kappa l}^{J}{}'(a,p)$ is an even function of κ, and hence it is a constant. Again, $\chi_{\kappa l}^{J}{}'(a,p)$ vanishes when $\kappa=p$, completing the proof.

This result, when combined with Eqs. (74) and (76), shows that

$$\frac{\psi_{\kappa l}^{J}{}'(a)}{\psi_{\kappa l}^J(a)}=\frac{\psi_l^{J;\text{BC}}{}'(a,\kappa)}{\psi_l^{J;\text{BC}}(a,\kappa)}\,. \tag{78}$$

Thus $\psi_{\kappa l}^J(r)$ satisfies the same boundary condition as the pure BCM function $\psi_l^{J;\text{BC}}(r,\kappa)$.

V. DISCUSSION

In the preceding sections we have developed an expression for the off-shell BCM t matrix. Our result, Eq. (65), was chosen such as to produce the correct wave function and to satisfy off-shell unitarity; the particular form being suggested by analogy to the special case discussed in Sec. III. We have used this result to formulate an LS-type integral equation for the off-shell t matrix in the more realistic problem of BCM plus external potential. However, up until this point we have put aside questions as to the uniqueness of the expressions we have obtained. We now consider to what extent our results are ambiguous in the sense that one could make the replacement $t^{J;\text{BC}}\to t^{J;\text{BC}}+\Delta^J$ without affecting the desired characteristics.

We first note that if such a replacement is to leave the wave function $\psi_l^{J;\text{BC}}(r,\kappa)$ unchanged, Eq. (73) requires that

$$0=\langle r|-G_0\Delta^J|l\kappa\rangle\,, \tag{79}$$

for all r. This implies that $\Delta^J|l\kappa\rangle=0$, or that $\Delta_{l'l}^J(p',p;s)$ vanishes half on shell. Let us now consider $\Delta_{l'l}^J(p',p;s)$, for fixed p' and p, as a function of the complex variable s. It seems reasonable to suppose that the proper $t_{l'l}^{J;\text{BC}}(p',p;s)$ should be analytic in s except for the unitarity cut and (possibly) poles for negative s corresponding to bound states. This is certainly the case for t matrices arising from energy-independent potentials, or limits of such potentials such as considered in Sec. III. Moreover, if one allowed the possibility of additional singularities, it would be necessary to relate them to some dynamical mechanism in the core region. It is just our ignorance of such effects which leads us to consider the BCM in the first place. Therefore, since we have previously shown that the form given in Eq. (65) for $t^{J;\text{BC}}$ has precisely the correct singularities, we must necessarily conclude that $\Delta_{l'l}^J(p',p,s)$ is an entire function of s.

It follows from the above that Δ^J may be written in the form

$$\Delta_{l'l}^J(p',p;s)=(p'^2-s)(p^2-s)\tilde{\Delta}_{l'l}^J(p',p;s)\,, \tag{80}$$

where $\tilde{\Delta}_{l'l}^J(p',p;s)$ is an entire function of s; here we have used the fact that $\Delta_{l'l}^J(p',p;s)$ vanishes half on shell and is either even or odd in the variables p' and p. As a consequence of Eq. (80) and the standard properties of entire functions,[17] we observe that as $s\to\infty$ in some directions, $\Delta_{l'l}^J(p',p;s)$ increases at least like s^2. Thus, if we additionally require that

$$\lim_{|s|\to\infty}\left|\frac{t_{l'l}^{J;\text{BC}}(p',p;s)}{s}\right|<\infty\,, \tag{81}$$

which is satisfied by the form of Eq. (65), we can conclude that $\Delta^J\equiv 0$. In support of the assumption stated in Eq. (81) we recall that our intention is to use $t^{J;\text{BC}}$ in the Faddeev equations, the kernel of which contains terms of the type

$$t_{l'l}^J(p',p;s)/(p'^2-s)\,, \tag{82}$$

where $s = W - q^2/M_r$. Therefore, if one does not require Eq. (81), there will be directions in the variable q, or in the three-body energy W, in which the Faddeev kernel is unbounded. Although this would not necessarily be pathological, it is difficult to think of exceptions which would not produce unusual and undesirable effects in the three-body amplitudes.

We have thus shown that our result for $t^{J;\mathrm{BC}}$ is unique if one makes rather reasonable assumptions as to analyticity and asymptotic behavior. It is worth noting that a corresponding question of ambiguity arises in attempts to represent the BCM by a pseudopotential.[18] In such cases the only argument one has with which to discriminate between pseudopotentials is simplicity of form. In comparison, our restrictions on the t matrix seem much cleaner and more closely related to the physical situation.

In conclusion, we note two aspects of our BCM t matrix which make it well suited for practical three-nucleon calculations. Most important is the fact that one can define a very natural and accurate separable approximation[19] to $t_{l'l}^{J;\mathrm{BC}}(p',p;s)$. We observe from Eq. (65) that it *is* separable for $l' \neq l$, while for the diagonal elements $t_{ll}^{J;\mathrm{BC}} - t_{l'l'}^{J+2;\mathrm{BC}}$ is separable because of Eqs. (51) and (18). Since the two-nucleon data require that t^{J+2} be much smaller than t^J, it is natural to approximate $t_{ll}^{J;\mathrm{BC}}$ by just the separable difference between it and $t_{l'l'}^{J+2;\mathrm{BC}}$. This is consistent with the fact that one commonly ignores t^{J+2} in comparison with t^J in the Faddeev equations anyway. This rather unusual property is shared by the t matrix resulting from the square-well potential, and was successfully employed by this author in a comparable calculation.[20] Secondly, the BCM t matrix for the triplet s- and d-wave system is extremely efficient in incorporating both repulsive core and tensor effects into a very simple form. For these reasons the pure BCM would appear to combine ease of calculation with a relatively sophisticated two-nucleon interaction.

APPENDIX

In this Appendix we give the connection between our amplitude $t_{l'l}^{J;\mathrm{BC}}(s)$ and the BCM t matrix of FL[4] in the case of $S=1$. The values of l' and l are thus restricted to $J \pm 1$ for the coupled partial waves, and the S matrix S^J is a 2×2 matrix with the elements

$$S_{\alpha\beta}^J = \delta_{\alpha\beta} - i\pi\kappa t_{J+2\alpha-3,\, J+2\beta-3}^{J;\mathrm{BC}}(s)\,, \quad \alpha, \beta = 1, 2\,. \tag{A1}$$

S^J may be expressed in terms of the eigenphase shifts $\eta_J^{(\alpha)}$ and the mixing parameter ϵ_J in the convention of Blatt and Biedenharn.[21] Thus

$$\begin{aligned} S_{11}^J &= \cos^2\epsilon_J\, e^{2i\eta_J^{(1)}} + \sin^2\epsilon_J\, e^{2i\eta_J^{(2)}}\,,\\ S_{12}^J &= S_{21}^J = \cos\epsilon_J\, \sin\epsilon_J\, (e^{2i\eta_J^{(1)}} - e^{2i\eta_J^{(2)}})\,,\\ S_{22}^J &= \sin^2\epsilon_J\, e^{2i\eta_J^{(1)}} + \cos^2\epsilon_J\, e^{2i\eta_J^{(2)}}\,. \end{aligned} \tag{A2}$$

In the boundary-condition model, ϵ_J and the $\eta_J^{(\alpha)}$ are determined by requiring that

$$a\Phi_J^{(\alpha)\prime}(a) = F\Phi_J^{(\alpha)}(a)\,, \tag{A3}$$

where

$$F = \begin{pmatrix} f_{J,J-1} & f_J^{(t)} \\ f_J^{(t)} & f_{J,J+1} \end{pmatrix}, \tag{A4}$$

and $\Phi_J^{(\alpha)}(r)$ is the (two-component) wave function

$$\Phi_J^{(\alpha)}(r) = \begin{pmatrix} \varphi_{J,J-1}^{(\alpha)}(r) \\ \varphi_{J,J+1}^{(\alpha)}(r) \end{pmatrix}. \tag{A5}$$

Here

$$\varphi_{J,J+2\beta-3}^{(\alpha)}(r) = U_{\alpha\beta}^J \cos\eta_J^{(\alpha)} e^{i\eta_J^{(\alpha)}} \times [j_{J+2\beta-3}(\kappa r) - \tan\eta_j^{(\alpha)} n_{J+2\beta-3}(\kappa r)]\,, \tag{A6a}$$

with

$$U^J = \begin{pmatrix} \cos\epsilon_J & \sin\epsilon_J \\ -\sin\epsilon_J & \cos\epsilon_J \end{pmatrix}. \tag{A6b}$$

It is straightforward to show from (A3) that

$$\tan\eta_J^{(\alpha)} = -\frac{1}{2A}[B + (-1)^\alpha (B^2 - 4AC)^{1/2}]\,, \tag{A7}$$

with

$$\begin{aligned} A &= 1 - \rho_t n_{J-1}(\kappa a) n_{J+1}(\kappa a)\,,\\ B &= -\tan\eta_J^+ - \tan\eta_J^- \\ &\quad + \rho_t[n_{J-1}(\kappa a) j_{J+1}(\kappa a) + n_{J+1}(\kappa a) j_{J-1}(\kappa a)]\,,\\ C &= \tan\eta_J^+ \tan\eta_J^- - \rho_t j_{J-1}(\kappa a) j_{J+1}(\kappa a)\,. \end{aligned} \tag{A8}$$

The quantities in (A8) are defined by

$$\begin{aligned} \rho_t &= (f_J^{(t)})^2 \{[f_{J,J-1} n_{J-1}(\kappa a) - \kappa a n_{J-1}{}'(\kappa a)] \\ &\quad \times [f_{J,J+1} n_{J+1}(\kappa a) - \kappa a n_{J+1}{}'(\kappa a)]\}^{-1}\,,\\ \tan\eta_J^\pm &= \frac{\kappa a j_{J\pm1}{}'(\kappa a) - f_{J,J\pm1} j_{J\pm1}(\kappa a)}{\kappa a n_{J\pm1}{}'(\kappa a) - f_{J,J\pm1} n_{J\pm1}(\kappa a)}\,. \end{aligned} \tag{A9}$$

Note that as $f_J^{(t)} \to 0$, $\eta_J^{(1)} \to \eta_J^-$, and $\eta_J^{(2)} \to \eta_J^+$. Finally,

$$\tan\epsilon_J = \frac{f_{J,J-1} n_{J-1}(\kappa a) - \kappa a n_{J-1}{}'(\kappa a)}{j_{J+1}(\kappa a) - \tan\eta_J^{(1)} n_{J+1}(\kappa a)} \frac{\tan\eta_J^{(1)} - \tan\eta_J^-}{f_J^{(t)}}\,. \tag{A10}$$

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[1]M. M. Lévy, Phys. Rev. 88, 725 (1952).

[2]A. B. Volkov, Nucl. Phys. 74, 33 (1965); I. R. Afnan and Y. C. Tang, Phys. Rev. 175, 1337 (1968); Phys. Rev. C 1, 750(E) (1970).

[3]R. Jastrow, Phys. Rev. 79, 389 (1950); 81, 165 (1951).

[4]G. Breit and W. G. Bouricius, Phys. Rev. 75, 1029 (1949); H. Feshback and E. Lomon, Phys. Rev. 102, 891 (1956).

[5]R. V. Reid, Jr., Ann. Phys. (N.Y.) 50, 411 (1968); T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1962); K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. 126, 881 (1962); D. P. Saylor, R. A. Bryan, and R. E. Marshak, Phys. Rev. Letters 5, 266 (1960); H. Feshbach and E. Lomon, *ibid.* 6, 635 (1961).

[6]L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. 39, 1459 (1960) [transl.: Soviet Phys.—JETP 12, 1014 (1961)].

[7]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967); D. D. Brayshaw, Phys. Rev. 176, 1855 (1968).

[8]If this were not so, one could prove in the usual fashion that the t matrix would approach the potential as the energy approached infinity. However, we show later in this paper that the BCM t matrix must increase at least linearly with the energy in this limit.

[9]Y. E. Kim and A. Tubis, Phys. Rev. C 1, 414 (1970).

[10]See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), or any equivalent textbook on elementary quantum mechanics.

[11]D. D. Brayshaw, Phys. Rev. 167, 1505 (1968).

[12]We shall regard the fact that Eq. (1) has a unique solution as well established by many other sources. In our derivation we assume for convenience that our fixed value of s is not an eigenvalue.

[13]The functions $h_l^+(x)$ are sometimes called spherical Hankel functions of the first kind, and are often denoted by $h_l(x)$ in the literature.

[14]M. M. Hoenig and E. L. Lomon, Ann. Phys. (N.Y.) 36, 363 (1966). This involves the combination of a δ function and its derivative, implying that the pseudopotential is separable in momentum space. Usually separable potentials imply separable t matrices; the fallacy here is due to the fact that the BCM t matrix is not the solution of the LS equation with this potential. Indeed, the solution does not even exist because of lack of convergence.

[15]Due to the factorization of the t matrix at a bound-state pole, the off-shell residue is completely determined if the half-on-shell residue is known.

[16]See, for example, M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

[17]R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1954).

[18]E. L. Lomon and M. McMillan, Ann. Phys. (N.Y.) 23, 439 (1963).

[19]The virtues of separable approximations in three-body calculations are by now well documented; for a review see D. D. Brayshaw, Phys. Rev. 182, 1658 (1969).

[20]D. D. Brayshaw and B. Buck, Phys. Rev. Letters 24, 733 (1970).

[21]J. M. Blatt and L. C. Biedenharn, Phys. Rev. 86, 399 (1952).

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Binding Energy of a Λ Particle in Nuclear Matter

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The results of a complete calculation of the binding energy D of a Λ particle in nuclear matter using the method of the independent-pair approximation, which systematically take into account the second- and third-order Born corrections, are presented. It is found that these corrections are small and that the Born series converges rapidly. A comparison of our results with those obtained using other methods based on the Brueckner theory shows that they are identical.

I. INTRODUCTION

Calculations of the binding energy D of a Λ particle in nuclear matter have lately increased in tempo,[1-7] for they provide information about the Λ-nucleon interaction in angular momentum states higher than zero and the possible presence of noncentral components.[8] The calculations which have been performed using central Λ-N potentials with hard cores have primarily used two approaches – the variational approach of Jastrow[9] and various versions[10] of the Brueckner-Bethe theory. Both of these approaches give results for D which are much larger than the experimental estimates of about 30 MeV.[11] Calculations using the Jastrow method give values for D about 20 MeV higher than those obtained by methods based on the Brueckner theory.

The disparity of about 20 MeV between the two approaches was first noticed by Ram and James[12]