Mixing of Collective States in an Exactly Soluble Three-Level Model*

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A general class of exactly soluble three-level models is presented for an N-fermion system. Assuming a monopole-monopole interaction among the fermions allows one to express the model Hamiltonian in terms of the generators of the SU_3 group. A specialized version of this class of models is adapted for the study of mixing two collective states of opposite parity, analogous to the mixing of giant-dipole-resonance and surface quadrupole vibrations in nuclei. Exact results of the model are studied as a function of interaction parameters and compared with approximations including up to three-particle–three-hole excitations. It is shown that in the model many-particle–many-hole states play an important role in the mixing of collective states and cannot be properly neglected. The implications of this result for the study of actual nuclei is discussed.

1. INTRODUCTION

When many-body-approximation methods are applied to the study of excited states of nuclei, one is often inhibited by the unmanageable number of states one should properly take into account in a rigorous calculation. Consequently one is often forced to make further approximations, often without any justification, in order to reduce the complexity of computational problems. It then becomes difficult to see which particular approximation made along the way is responsible for the success or failure of the theoretical result.

Various exactly soluble models have been shown to be useful in improving our understanding of the validity of approximation methods used in nuclear-structure studies.[1-9] Study of these models has also lead to invention of new approximation methods.[1]

The object of this paper is, first of all, to present a class of three-level soluble models which possess many of the features of actual nuclei, and secondly, to apply a specialized version of these models to examine the effects of mixing of two different collective states, such as the observed mixing of a giant dipole resonance with quadrupole surface vibrations in C^{12}.[10-12] The class of models we shall describe consists of N fermions distributed in three levels with a simple monopole interaction among the fermions. This particular simple form of the interaction enables us to classify the solutions according to irreducible representations of the group SU_3. The model has the following general features: It possesses collective states as well as single-particle states. The collective states are of two distinct types, with the possibility of mixing between them. Because of the symmetries of the Hamiltonian, the collective particle-hole states belong to a different representation of the SU_3 group than the single-particle states. The model can treat closed-shell systems as well as systems with several particles (or holes) outside of closed shell.

The main advantage of this class of models is that because of the symmetries of the model Hamiltonian, exact solutions are easily obtainable, and one can evaluate the validity of approximate methods simply by comparing approximate and exact results. The main limitations of this class of models are: (1) There is only one kind of fermion (either neutrons or protons for example). (2) The interaction is purely monopole; thus there are no angular momentum effects. (3) There is no mixing between collective and single-particle states. (4) The model possesses only discrete states. The class of models could easily be extended to include some of the effects left out, e.g., by introducing some simple form of SU_3 symmetry breaking.

The rest of the paper is organized as follows: Section 2 contains a detailed description of the general class of three-level models. Section 3 contains a description of the specialized form of the model used in the remainder of the paper, suitable for investigating the mixing of two types of collective states, where one collective state is of opposite parity and much higher in energy than the other. Section 4 discusses transition operators and sum rules. Conditions for the stability of the Hartree-Fock ground state are discussed in Sec. 5. Section 6 discusses the changes in the structure and the transition-strength distribution as the parameters of the model are varied. This section also contains a comparison of exact results with approximations including up to 3p-3h excitations. Summary of the results and concluding remarks are contained in Sec. 7, and computational details pertaining to Sec. 6 are given in the Appendix.

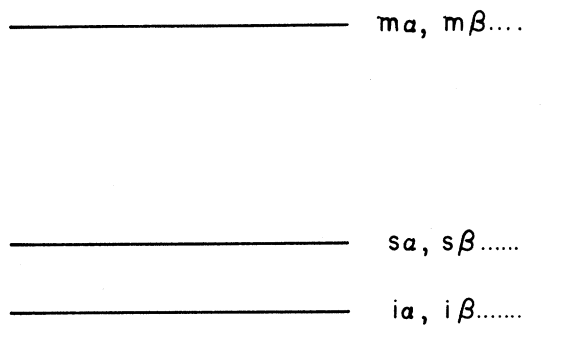

FIG. 1. Three energy levels of the model with corresponding particle labels.

2. DESCRIPTION OF THE GENERAL THREE-LEVEL MODEL

Let us consider a many-fermion system of N particles distributed in three levels, two of which have the same parity (see Fig. 1). These three levels could, for example, correspond to three major oscillator shells, or three different j-j coupling configurations. We shall let Greek letters α, β, γ, and δ be the quantum numbers necessary to distinguish between particles within the same shell, and Latin letters a, b, c, and d will label different shells. Furthermore we shall use the label i for the lowest level, the label s for an excited level of the same parity as the lowest level, and the label m for the excited level of opposite parity to that of the lowest level.

The general Hamiltonian that one can write for this N-particle system is

$$H=\sum_{a\alpha}T_{a\alpha}a^{\dagger}_{a\alpha}a_{a\alpha}+\tfrac{1}{4}\sum_{\substack{abcd\\ \alpha\beta\gamma\delta}}V_{a\alpha b\beta c\gamma d\delta}a^{\dagger}_{a\alpha}a^{\dagger}_{b\beta}a_{d\delta}a_{c\gamma}, \tag{2.1}$$

where the indices a, b, c, and d can run over all allowed values of i, s, m, and $V_{a\alpha b\beta c\gamma d\delta}$ is the antisymmetrized matrix element of the two-body interaction:

$$V_{a\alpha b\beta c\gamma d\delta}=(a\alpha b\beta|\,V|\,c\gamma d\delta)-(a\alpha b\beta|\,V|\,d\delta c\gamma).$$

We now construct a simple soluble model as follows. We require that our interaction be such that for each substate annihilated in level a there be a corresponding substate created in level b, i.e.,

$$V_{a\alpha b\beta c\gamma d\delta}=V_{a\alpha b\beta c\gamma d\delta}(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{2.2}$$

We further require that $V_{a\alpha b\beta c\gamma d\delta}$ be real and independent of α and β and depend only on level labels. Thus,

$$V_{a\alpha b\beta c\gamma d\delta}=v(abcd)(\delta_{\alpha\beta}\delta_{\gamma\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}), \tag{2.3}$$

and similarly

$$T_{a\alpha}=t(a). \tag{2.4}$$

The Hamiltonian H breaks up naturally into three parts: $H=H_0+H_1+H_2$. H_0 is the part of the Hamiltonian where the level labels before the interaction are the same as after the interaction; in H_1 the level of one particle is changed by the interaction; and in H_2 the levels of two particles are changed by the interaction.

With our assumption for the Hamiltonian we have

$$\begin{aligned}H_0=&\,t(i)\sum_{\alpha}a^{\dagger}_{i\alpha}a_{i\alpha}+t(s)\sum_{\alpha}a^{\dagger}_{s\alpha}a_{s\alpha}+t(m)\sum_{\alpha}a^{\dagger}_{m\alpha}a_{m\alpha}+v(iiii)\sum_{\alpha}a^{\dagger}_{i\alpha}a_{i\alpha}(\sum_{\beta}a^{\dagger}_{i\beta}a_{i\beta}-1)\\&+v(ssss)\sum_{\alpha}a^{\dagger}_{s\alpha}a_{s\alpha}(\sum_{\beta}a^{\dagger}_{s\beta}a_{s\beta}-1)+v(mmmm)\sum_{\alpha}a^{\dagger}_{m\alpha}a_{m\alpha}(\sum_{\beta}a^{\dagger}_{m\beta}a_{m\beta}-1)\\&+v(isis)[\sum_{\alpha}a^{\dagger}_{i\alpha}a_{i\alpha}(\sum_{\beta}a^{\dagger}_{s\beta}a_{s\beta}-1)+\sum_{\alpha}a^{\dagger}_{i\alpha}a_{s\alpha}\sum_{\beta}a^{\dagger}_{s\beta}a_{i\beta}]\\&+v(imim)[\sum_{\alpha}a^{\dagger}_{i\alpha}a_{i\alpha}(\sum_{\beta}a^{\dagger}_{m\beta}a_{m\beta}-1)+\sum_{\alpha}a^{\dagger}_{i\alpha}a_{m\alpha}\sum_{\beta}a^{\dagger}_{m\beta}a_{i\beta}]\\&+v(smsm)[\sum_{\alpha}a^{\dagger}_{s\alpha}a_{s\alpha}(\sum_{\beta}a^{\dagger}_{m\beta}a_{m\beta}-1)+\sum_{\alpha}a^{\dagger}_{s\alpha}a_{m\alpha}\sum_{\beta}a^{\dagger}_{m\beta}a_{s\beta}],\end{aligned} \tag{2.5a}$$

$$\begin{aligned}H_1=&\,v(iiis)(\sum_{\alpha}a^{\dagger}_{i\alpha}a_{s\alpha}\sum_{\beta}a^{\dagger}_{i\beta}a_{i\beta}+\sum_{\alpha}a^{\dagger}_{i\alpha}a_{i\alpha}\sum_{\beta}a^{\dagger}_{s\beta}a_{i\beta})+v(ssis)(\sum_{\alpha}a^{\dagger}_{s\alpha}a_{i\alpha}\sum_{\beta}a^{\dagger}_{s\beta}a_{s\beta}+\sum_{\alpha}a^{\dagger}_{s\alpha}a_{s\alpha}\sum_{\beta}a^{\dagger}_{i\beta}a_{s\beta})\\&+v(imsm)(\sum_{\alpha}a^{\dagger}_{i\alpha}a_{s\alpha}\sum_{\beta}a^{\dagger}_{m\beta}a_{m\beta}+\sum_{\alpha}a^{\dagger}_{s\alpha}a_{i\alpha}\sum_{\beta}a^{\dagger}_{m\beta}a_{m\beta}+\sum_{\alpha}a^{\dagger}_{m\alpha}a_{s\alpha}\sum_{\beta}a^{\dagger}_{i\beta}a_{m\beta}+\sum_{\alpha}a^{\dagger}_{m\alpha}a_{i\alpha}\sum_{\beta}a^{\dagger}_{s\beta}a_{m\beta}),\end{aligned} \tag{2.5b}$$

and

$$\begin{aligned}H_2=&\,\tfrac{1}{2}v(iiss)[(\sum_{\alpha}a^{\dagger}_{i\alpha}a_{s\alpha})^2+(\sum a^{\dagger}_{s\alpha}a_{i\alpha})^2]+\tfrac{1}{2}v(iimm)[(\sum_{\alpha}a^{\dagger}_{i\alpha}a_{m\alpha})^2+(\sum_{\alpha}a^{\dagger}_{m\alpha}a_{i\alpha})^2]\\&+\tfrac{1}{2}v(ssmm)[(\sum_{\alpha}a^{\dagger}_{s\alpha}a_{m\alpha})^2+(\sum_{\alpha}a^{\dagger}_{m\alpha}a_{s\alpha})^2]+v(ismm)(\sum_{\alpha}a^{\dagger}_{i\alpha}a_{m\alpha}\sum_{\beta}a^{\dagger}_{s\beta}a_{m\beta}+\sum_{\alpha}a^{\dagger}_{m\alpha}a_{i\alpha}\sum a^{\dagger}_{m\beta}a_{s\beta}).\end{aligned} \tag{2.5c}$$

Let us now define the following nine operators:

$$N=\sum_\alpha (a^\dagger_{m\alpha}a_{m\alpha}+a^\dagger_{s\alpha}a_{s\alpha}+a^\dagger_{i\alpha}a_{i\alpha}),\quad Y=\tfrac{1}{3}\sum_\alpha (a^\dagger_{s\alpha}a_{s\alpha}+a^\dagger_{i\alpha}a_{i\alpha}-2a^\dagger_{m\alpha}a_{m\alpha}),$$
$$T_+=\sum_\alpha a^\dagger_{s\alpha}a_{i\alpha},\quad T_-=\sum_\alpha a^\dagger_{i\alpha}a_{i\alpha},\quad U_+=\sum_\alpha a^\dagger_{m\alpha}a_{m\alpha},\quad U_-=\sum_\alpha a^\dagger_{i\alpha}a_{m\alpha}, \tag{2.6}$$
$$V_+=\sum_\alpha a^\dagger_{m\alpha}a_{s\alpha},\quad V_-=\sum_\alpha a^\dagger_{s\alpha}a_{m\alpha},\quad T_0=\tfrac{1}{2}\sum_\alpha (a^\dagger_{s\alpha}a_{s\alpha}-a^\dagger_{i\alpha}a_{i\alpha}).$$

From the commutation properties of these operators listed in Table I, we see that the nine operators defined in Eq. (2.6) form the nine generators of the group SU_3.[13] Here N is the total number of particles in the system and Y is a measure of the parity: If $\frac{3}{2}Y$ is even for a given state, then the state has the same parity as ground state; if $\frac{3}{2}Y$ is odd, then the parity of the state is opposite to that of the ground state.

Furthermore, we can define additional operators which are linear combinations of N, Y, and T_0:

$$N_i=\sum_\alpha a^\dagger_{i\alpha}a_{i\alpha}\quad \text{No. of particles in } i \text{ level},$$
$$N_s=\sum_\alpha a^\dagger_{s\alpha}a_{s\alpha}\quad \text{No. of particles in } s \text{ level}, \tag{2.7}$$
$$N_m=\sum_\alpha a^\dagger_{m\alpha}a_{m\alpha}\quad \text{No. of particles in } m \text{ level},$$
$$U_0=\tfrac{1}{4}(-3Y+2T_0)=\tfrac{1}{2}(N_m-N_i),\quad V_0=-\tfrac{1}{4}(3Y+2T_0)=\tfrac{1}{2}(N_m-N_s).$$

It can be easily seen that each of the sets of operators (T_+, T_-, T_0), (U_+, U_-, U_0), and (V_+, V_-, V_0) satisfies angular momentum commutation rules.

We can now rewrite the model Hamiltonian [Eqs. (2.5)] in terms of the operators in Eqs. (2.6) and (2.7).

$$H_0=t(i)N_i+t(s)N_s+t(m)N_m+\tfrac{1}{2}v(i)N_i(N_i-1)+\tfrac{1}{2}v(s)N_s(N_s-1)+\tfrac{1}{2}v(m)N_m(N_m-1)$$
$$+v(isis)[N_i(N_s-1)+T_-T_+]+v(imim)[N_i(N_m-1)+U_-U_+]+v(smsm)[N_s(N_m-1)+V_-V_+], \tag{2.8a}$$
$$H_1=v(iiis)(T_-N_i+N_iT_+)+v(ssis)(T_+N_s+N_sT_-)+v(imsm)(T_-N_m+N_mT_++V_+U_-+U_+V_-), \tag{2.8b}$$
$$H_2=\tfrac{1}{2}v(iiss)(T_+^2+T_-^2)+v(iimm)(U_+^2+U_-^2)+\tfrac{1}{2}v(ssmm)(V_+^2+V_-^2)+v(ismm)(U_+V_++V_-U_-). \tag{2.8c}$$

Since our Hamiltonian H is a function of the generators of the group SU_3, it will commute with the Casimir operators of this group. Thus the eigenstates of the Hamiltonian can be simultaneously eigenstates of the Casimir operators. We know from group-theoretical considerations that to each set of eigenvalues of Casimir operators there corresponds a set of quantum numbers (λ, μ) which describes the symmetry of a given irreducible representation. For each set (λ, μ) we can construct a multiplet on a two-dimensional plot as follows. We plot Y along the vertical axis, and T_0 along the horizontal axis. Each pair of values Y, T_0 is represented by a lattice point in the diagram. Such multiplets are shown in Fig. 2. Since for a given total number of particles N, T_0, and Y, occupation numbers of each level are completely specified, each lattice point on a diagram can be specified by the occupation numbers N_i, N_s, and N_m for the levels i, s, and m, respectively, provided the total number of particles N is given.

Thus all points lying on the line parallel to the T_0 axis have the same value of N_m, all points lying on the axis inclined at a 60° angle to the T_0 axis have the same value of N_i, and all points on an axis of 120° to the T_0 axis have the same N_s value. (See Fig. 2.) The operators T_+, T_-, U_+, U_-, V_+, and V_- move one particle from one level to the other one as shown in Fig. 2. The first part of the Hamiltonian H_0 is diagonal in occupation

TABLE I. Commutation properties of operators in Eqs. (2.6) and (2.7).

$[T_0, T_\pm]=\pm T_\pm$	$[Y, T_\pm]=0$	
$[T_0, V_\pm]=\mp\frac{1}{2}V_\pm$	$[Y, V_\pm]=\mp V_\pm$	
$[T_0, U_\pm]=\pm\frac{1}{2}U_\pm$	$[Y, U_\pm]=\mp U_\pm$	
$[T_+, V_-]=[T_-, U_-]=[T_+, U_+]=[T_-, V_+]=[U_+, V_+]=[U_-, V_-]=0$		
$[T_+, U_-]=V_-$	$[T_-, V_-]=U_-$	$[V_-, U_+]=T_+$
$[T_+, V_+]=-U_+$	$[T_{-1}U_+]=-V_+$	$[U_-, V_+]=T_-$
$[T_+, T_-]=2T_0$	$[V_+, V_-]=2V_0$	$[U_+, U_-]=2U_0$

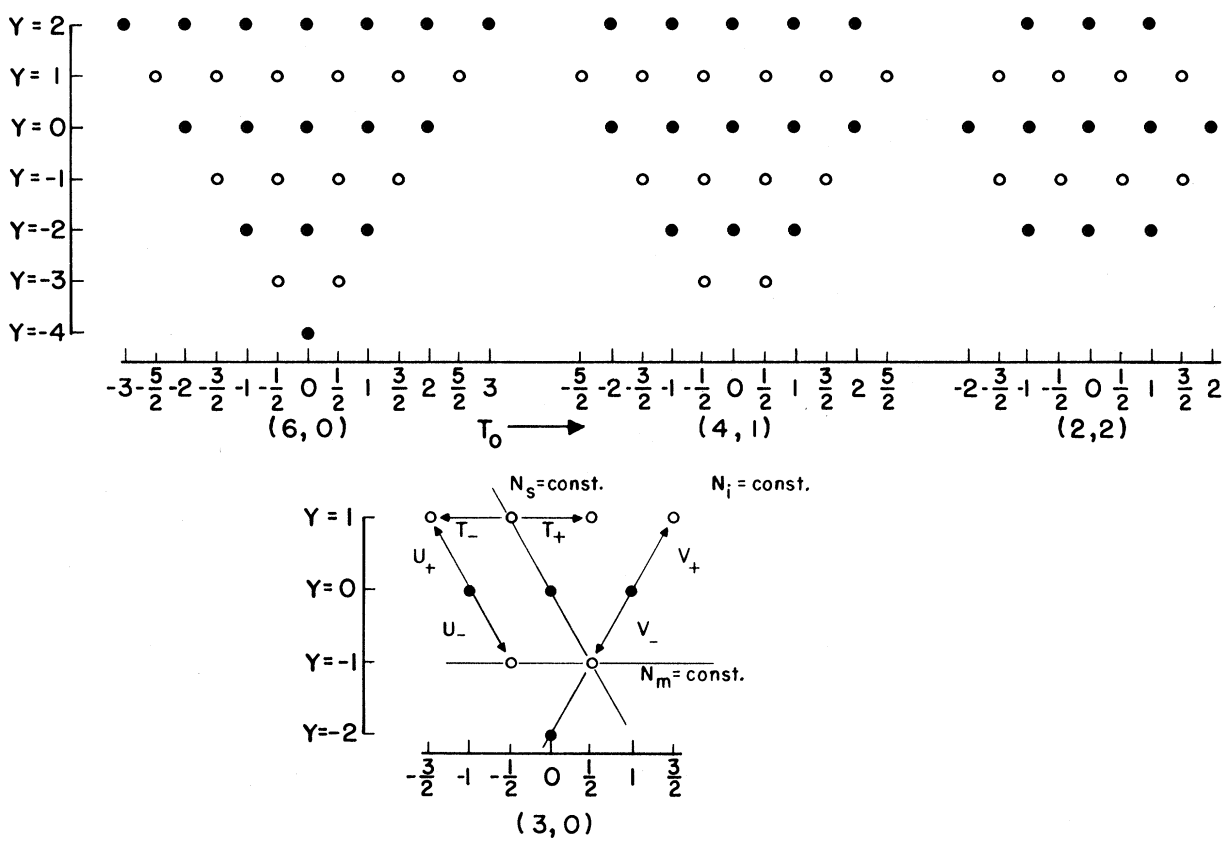

FIG. 2. Some of the possible multiplets for the six-particle system. The (3,0) multiplet is blown up to indicate the axes of constant N_i, N_s, and N_m and the action of operators T_+, T_-, U_+, U_-, V_+, and V_-. For simplicity the diagrams (4,1) and (2,2) do not display repeated states.

numbers N_i, N_s, and N_m, thus only H_1 and H_2 will mix states of the same multiplet represented by lattice points.

One more symmetry is provided by parity conservation. States with even values of N_m have opposite parity to those with odd N_m. Since our Hamiltonian conserves parity, it will not mix odd-N_m states with even-N_m states. In the multiplet diagrams we shall represent even-N_m states by circles and odd-N_m states by dots.

The symmetry (λ, μ) is a measure of the "collectiveness" of a given state. Thus the most symmetric representation $(\lambda, \mu) = (N, 0)$ corresponds to a band of most collective states in the following sense.

First of all we note that for an N-particle system in the absence of interaction terms H_1 and H_2 the $(\lambda, \mu) = (N, 0)$ representation contains a state $Y = \frac{1}{3}N$, $T_0 = -\frac{1}{2}N$, which corresponds to all particles being in the ith level. This state is completely symmetric in the substate labels, $\alpha, \beta, \ldots$. All other states in this multiplet are generated by the operators of Eq. (2.6), which themselves are symmetric in substate labels; thus all of the states of $(N, 0)$ representation are completely symmetric in substate labels. In other words, all states of the $(N, 0)$ multiplet are linear combinations of single-particle states with equal weights. Since the interaction terms H_1 and H_2 [Eqs. (2.8)] consist of symmetric operators alone, turning on H_1 and H_2 will not change the symmetry of the given representation. The states of lower symmetry are correspondingly less collective.

If the number of different substates α in all levels is at least as large as the number of particles in the system, all representations are allowed, including the symmetric one, $(\lambda, \mu) = (N, 0)$. However, if the number of particles N is larger than the number of substates α, some values of (λ, μ) cannot occur. For example, let us consider a system which forms a closed shell for four particles in the ith level. If we add more particles to this system the additional particles will have to go into the sth and mth level. Thus, for five particles the representation $(\lambda, \mu) = (5, 0)$ is forbidden in this case; the lowest allowed symmetry is $(\lambda, \mu) = (3, 1)$.

More generally, for a system in which n particles form a closed shell, Young tableaux of al-

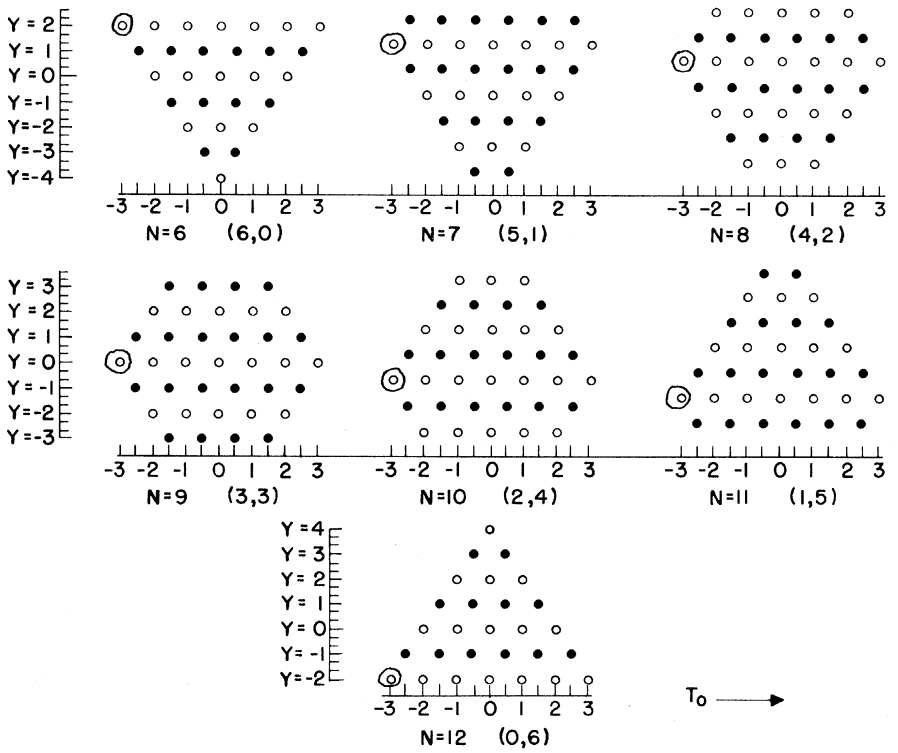

FIG. 3. Lowest symmetry multiplets for an N-particle system with six particles forming a closed shell. The shell-model ground state is indicated by a large circle. For simplicity the diagrams (5,1), (4,2), (3,3), (2,4), and (1,5) do not display repeated states.

lowed representations can have at most n columns. Figure 3 shows highest symmetry multiplets for a system in which six particles form a closed shell. Different multiplets correspond to different values of the total number of particles. These multiplets contain the shell-model ground state. We envisage a nucleus where the first unfilled shell above the closed shell has parity opposite to that of the closed shell. This is the case for nuclei such as O^{16} or Ca^{40} which correspond to major harmonic-oscillator shell closure. Additional nucleons will fill the next major shell of opposite parity. In Fig. 3 the "shell-model" ground state is thus the state with $N_i=6$, $N_s=0$, $N_m=N-N_i$, or $T_0=-3$, $Y=6-\frac{3}{2}N$. This is the state where the closed shell is filled and remaining particles are in the mth level.

3. SPECIALIZATION OF THE MODEL

In the following we shall restrict our discussion to the case where we have as many particles in the system as there are substates. Furthermore, we assume a Hartree-Fock (HF) representation, the HF ground state being the state in which all substates α are occupied in the ith level.

The choice of the HF representation leads to several simplifications of the Hamiltonian. First of all, from the condition that the HF Hamiltonian has no matrix elements between occupied and unoccupied states we have

$$v(iiss)=0\,. \tag{3.1}$$

In addition, the HF single-particle energies are given by

$$\epsilon_a=t(a)+\sum_i v(iaia)\,,$$

or

$$\begin{aligned}\epsilon_i&=t(i)+(N-1)v(i)\,,\\ \epsilon_s&=t(s)+(N-1)v(isis)\,,\\ \epsilon_m&=t(m)+(N-1)v(imim)\,.\end{aligned} \tag{3.2}$$

The expectation value of H in the HF ground state is

$$E_0=(\psi_{\rm HF}|H|\psi_{\rm HF})=N[\epsilon_i-\tfrac{1}{2}(N-1)v(i)]\,. \tag{3.3}$$

Making use of the Casimir operator C, where

$$C=T_-T_++V_-V_++U_-U_++T_0+U_0+V_0+\tfrac{4}{3}(T_0^2+U_0^2+V_0^2)\,, \tag{3.4}$$

we can rewrite the H_0 part of the Hamiltonian as

$$H_0 - E_0 - (C - \tfrac{1}{3}N^2 - N)v(imim)$$
$$= N_s\lambda_s + N_m\lambda_m + \tfrac{1}{2}N_s(N_s-1)[v(s)+v(i)-4v(isis)] + \tfrac{1}{2}N_m(N_m-1)[v(m)+v(i)-4v(imim)]$$
$$+ 2N_sN_m[\tfrac{1}{2}v(i) - v(isis) + v(smsm) - v(imim)] + [T_-T_+ - N_i(N_s+1)][v(isis) - v(imim)]$$
$$+ [V_-V_+ - N_s(N_m+1)][v(smsm) - v(imim)], \tag{3.5}$$

where

$$\lambda_s = \epsilon_s - \epsilon_i + (N-1)v(isis), \quad \lambda_m = \epsilon_m - \epsilon_i + (N-1)v(imim). \tag{3.6}$$

The last term on the left and the last two terms on the right of Eq. (3.5) vanish for the symmetric representation $(\lambda, \mu) = (N, 0)$. The last term on the right is the only one that does not conserve T.

Let us now consider the physical situation represented by this model. We shall neglect H_1 and H_2 for the time being. The ground state is then the state with $N_s = N_m = 0$, $N = N_i$ and its energy is the HF ground-state energy E_0. We have seen that it belongs to the symmetric $(N, 0)$ representation. Assume it has even parity.

Let us now consider one-particle–one-hole states. There are N such states of even parity where one particle is excited into the sth level, and N more odd-parity states for one particle excited into the mth level. In the symmetric representation there is one even-parity particle-hole state and one of odd parity. Their excitation energies are given, respectively, by λ_s and λ_m. All the remaining one-particle–one-hole states belong to the $N-1$ multiplets $(\lambda, \mu) = (N-2, 1)$. Thus there are $(N-1)$ degenerate even-parity states $(N_s = 1, N_m = 0)$ with excitation energy $\Delta E_s = \epsilon_s - \epsilon_i - v(isis)$ and $(N-1)$ degenerate odd-parity states with excitation energy $\Delta E_m = \epsilon_m - \epsilon_i - v(imim)$.

We see that the physical situation described by this model is like that of the schematic model of Brown[14] where we have one "collective" particle-hole state of each parity shifted in energy from the HF value by the amount $(N-1)v(isis)$ or $(N-1)v(imim)$, while the rest of the particle-hole states remain close to their unperturbed values given by Eqs. (3.2). In addition, these collective particle-hole states exhaust all the transition strength from the ground state. This can be seen as follows. Since in this model all the transition operators have to be symmetric in the substate labels $\alpha, \beta, \gamma, \ldots$, we can have transition matrix elements only between states of the same multiplet. Thus transition matrix elements vanish between the ground state, which is in the symmetric representation, and all the excited p-h states in the $(N-2, 1)$ multiplet.

In this paper we shall be primarily concerned with the collective levels; thus we shall deal only with the symmetric representation. We would like to simulate a nucleus such as C^{12}, for example, in which there is a low-lying collective quadrupole state and a relatively high-lying giant-dipole state. We then identify our even-parity p-h collective state as "the quadrupole vibration" and the odd-parity p-h state as the "giant-dipole state." We expect a situation where $\lambda_m \gg \lambda_s$; thus $v(imim)$ is positive and $v(isis)$ is negative.

Neglecting H_1 and H_2, we obtain an energy spectrum as shown in Fig. 4. We have an even-parity vibrational band on top of the ground state, another one on top of the "giant-dipole state," a third one on top of the two-phonon "giant-dipole" state, etc. The terms in H_0 proportional to $N_s(N_s-1)$, $N_m(N_m-1)$, and N_sN_m are responsible for "anharmonicity" in the spectrum. For simplicity we

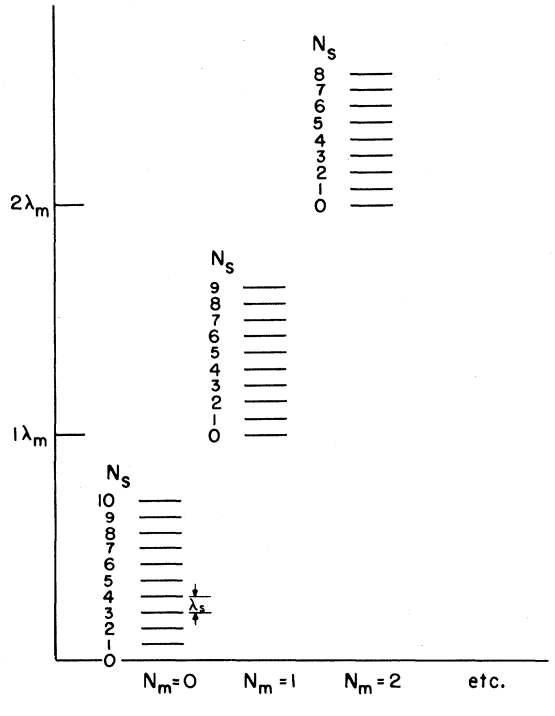

FIG. 4. The energy level structure for the model Hamiltonian (3.7).

TABLE II. Exact eigenvalues E of the Hamiltonian of Eqs. (3.9) for the two lowest bands for four particles and various values of the interaction parameters $v_1 = N^{1/2}v(ssis)/\lambda_s$, $v_2 = 2N^{1/2}v(imsm)/\lambda_s$, and $v_3 = Nv(iiss)/\lambda_s$. The energies E are in the units of λ_s.

v_1	v_2 [a]	v_3	Ground-state band, $N_m=0$				
0	···	0	0.000	1.000	2.000	3.000	4.000
0.3	···	0	0.000	0.845	1.661	2.906	4.588
0.3	···	0.4	−0.034	0.879	1.646	2.828	4.681
0.3	···	0.8	−0.128	0.789	1.718	2.804	4.816
			Excited-state band, [b] $N_m=1$				
0	0.2	0	−0.030	0.990	2.010	3.030	
0	0.6	0	−0.249	0.917	2.083	3.249	
0	1.0	0	−0.621	0.793	2.207	3.621	
0.3	0.2	0	−0.033	0.787	1.869	3.377	
0.3	0.6	0	−0.302	0.599	1.956	3.748	
0.3	0.2	0.4	−0.035	0.796	1.818	3.420	
0.3	0.6	0.4	−0.244	0.553	1.875	3.816	
0.3	0.2	0.8	−0.073	0.792	1.795	3.485	
0.3	0.6	0.8	−0.227	0.520	1.806	3.901	

[a] The ground-state band is independent of v_2.
[b] The energy λ_m has been subtracted from E.

shall neglect anharmonic terms. Thus the spectrum plotted in Fig. 4 is that of

$$H_0 - E_0 = N_s\lambda_s + N_m\lambda_m . \tag{3.7}$$

We shall further simplify the model as follows. Since $\lambda_m \gg \lambda_s$, we neglect all interaction terms in H_2 which do not conserve N_m. Furthermore, since we shall be working in the symmetric representation

$$V_+U_- + U_+V_- = T_-N_m + N_mT_+ , \tag{3.8}$$

our model Hamiltonian finally becomes $H = H_0 + H_1 + H_2$, with

$$H_0 = N_s\lambda_s + N_m\lambda_m , \tag{3.9a}$$

$$H_1 = v(ssis)(T_+N_s + N_sT_-) + 2v(imsm)(T_-N_m + N_mT_+) , \tag{3.9b}$$

and

$$H_2 = \tfrac{1}{2}v(iiss)(T_+^2 + T_-^2) . \tag{3.9c}$$

The second term in the Hamiltonian H_1 gives rise to "dipole-quadrupole" coupling, since it will mix the pure "giant-dipole" state $N_m=1$, $N_s=0$ with the two-particle–two-hole state $N_m=1$, $N_s=1$, corresponding to quadrupole oscillation on top of the "giant-dipole" state. In the ground-state band $N_m=0$, H_1 will have no matrix elements between the shell-model ground state $N_m=0$, $N_s=0$ and any of the other $N_m=0$ states. If $H_2=0$, the shell-model ground state remains completely uncoupled from the other states. Thus H_2 is the term that gives rise to ground-state correlations. For $H_1=0$ the model reduces to a monopole model considered previously.[1,3] If the $v(imsm)$ term is zero in H_1, then the structure of the ground-state band will be the same as that of the excited-state band except that the excited-state band will be shifted by an energy λ_m relative to the ground state.

4. TRANSITION OPERATORS

There are two types of one-body transition operators in this model: the "dipole" operator, i.e., the operator which changes the parity of the state, given by

$$D = \mathfrak{D}(U_+ + V_+ + U_- + V_-) , \tag{4.1}$$

TABLE III. Exact eigenvalues E of the Hamiltonian of Eqs. (3.9) for the two lowest bands for seven particles and various values of the interaction parameters $v_1 = N^{1/2}v(ssis)/\lambda_s$, $v_2 = 2N^{1/2}v(imsm)/\lambda_s$, and $v_3 = Nv(iiss)/\lambda_s$. The energies E are in the units of λ_s.

v_1	v_2 [a]	v_3	Ground-state band, $N_m=0$							
0	···	0	0.000	1.000	2.000	3.000	4.000	5.000	6.000	7.000
0.3	···	0	0.000	0.775	1.270	2.136	3.336	4.864	6.719	8.900
0.3	···	0.4	−0.040	0.845	1.464	2.054	3.132	4.691	6.701	9.152
0.3	···	0.8	−0.157	0.647	1.489	2.222	3.042	4.569	6.727	9.462
			Excited-state band [b] $N_m=1$							
0	0.2	0	−0.034	0.977	1.989	3.000	4.011	5.023	6.034	
0	0.6	0	−0.294	0.804	1.902	3.000	4.098	5.196	6.294	
0	1.0	0	−0.761	0.493	1.746	3.000	4.254	5.507	6.761	
0.3	0.2	0	−0.042	0.583	1.303	2.371	3.765	5.486	7.533	
0.3	0.6	0	−0.469	0.099	1.053	2.336	3.947	5.885	8.148	
0.3	0.2	0.4	−0.046	0.707	1.321	2.240	3.620	5.446	7.712	
0.3	0.6	0.4	−0.287	0.188	0.946	2.146	3.783	5.857	8.366	
0.3	0.2	0.8	−0.120	0.648	1.402	2.194	3.510	5.435	7.931	
0.3	0.6	0.8	−0.282	0.273	0.913	1.987	3.638	5.851	8.619	

[a] The ground-state band is independent of v_2.
[b] The energy λ_m has been subtracted from E.

and the "quadrupole" operator, which is a transition operator between states of the same parity,

$$Q = \mathcal{Q}(T_+ + T_-) . \tag{4.2}$$

The sum rules for these operators

$$\sum_n (\varphi_n | O | \varphi_g)^2 = (\varphi_g | O^2 | \varphi_g) \tag{4.3}$$

are, respectively, given by

$$\sum_n (\varphi_n | D | \varphi_g)^2 = \mathcal{D}^2 (N + 2\langle T_+ \rangle) , \tag{4.4}$$

$$\sum_n (\varphi_n | Q | \varphi_g)^2 = \mathcal{Q}^2 (N + 2N\langle N_s \rangle - 2\langle N_s^2 \rangle + 2\langle T_+^2 \rangle) , \tag{4.5}$$

where the expectation values are taken with respect to the ground state which is assumed to be a member of the $N_m = 0$ band of the symmetric representation. For the uncorrelated ground state φ_0 ($N_m = 0$, $N_s = 0$), the sum rules reduce to

$$\sum_n (\varphi_n | D | \varphi_0) = N\mathcal{D}^2 , \tag{4.6}$$

$$\sum_n (\varphi_n | Q | \varphi_0) = N\mathcal{Q}^2 . \tag{4.7}$$

TABLE IV. Exact solutions of the Hamiltonian (3.9) for 10 particles and various values of the interaction parameters v_1, v_2, and v_3, defined in Table II. The energies E are in units of λ_s. The eigenvectors are in the columns below the corresponding eigenvalues. Eigenvectors are given by the eigenstates of $N_s = E$.

Ground-state band, $N_m = 0$
$v_1 = 0$, $v_2 = 0.2, 0.6, 1.0$, $v_3 = 0$

$N_s \backslash E$	0.000	1.000	2.000	3.000	4.000	5.000	6.000	7.000	8.000	9.000	10.000

Excited-state band, $N_m = 1$, with the energy λ_m subtracted

$v_1 = 0$, $v_2 = 0.2$, $v_3 = 0$

$N_s \backslash E$	−0.036	0.972	1.980	2.988	3.996	5.004	6.012	7.012	8.028	9.036
0	0.982	0.185	0.023	0.002	0.000	0.000	0.000	0.000	0.000	0.000
1	−0.185	0.951	0.244	0.035	0.003	0.000	0.000	0.000	0.000	0.000
2	0.023	−0.244	0.928	0.277	0.043	0.004	0.000	0.000	0.000	0.000
3	−0.023	0.036	−0.277	0.913	0.294	0.005	0.005	0.000	0.000	0.000
4	0.000	−0.004	0.044	−0.294	0.905	0.300	0.047	0.005	0.000	0.000
5	0.000	0.000	0.005	0.047	−0.300	0.905	0.294	0.044	0.004	0.000
6	0.000	0.000	0.000	−0.005	0.047	−0.294	0.913	0.277	0.036	0.002
7	0.000	0.000	0.000	0.000	−0.004	0.043	0.277	0.928	0.244	0.023
8	0.000	0.000	0.000	0.000	0.000	−0.003	−0.035	−0.244	0.951	0.185
9	0.000	0.000	0.000	0.000	0.000	0.000	0.002	0.023	−0.185	0.982

$v_1 = 0$, $v_2 = 0.6$, $v_3 = 0$

$N_s \backslash E$	−0.313	0.756	1.826	2.896	3.965	5.035	6.104	7.174	8.244	9.313
0	0.862	0.474	0.174	−0.049	0.010	−0.002	0.000	0.000	0.000	0.000
1	−0.474	0.630	0.558	−0.247	0.076	−0.018	0.003	0.000	0.000	0.000
2	0.174	−0.557	0.477	−0.582	0.289	−0.092	0.021	0.004	0.000	0.000
3	−0.049	0.248	−0.582	−0.384	0.587	−0.308	0.099	0.021	0.003	0.000
4	0.011	−0.076	0.289	0.587	0.339	−0.588	0.308	0.093	0.018	0.002
5	−0.002	0.017	−0.093	−0.308	−0.588	−0.339	0.587	0.289	0.076	0.011
6	0.000	−0.003	0.021	0.098	0.308	−0.587	0.383	0.582	0.248	0.049
7	0.000	0.000	0.004	−0.021	−0.092	−0.290	−0.582	0.477	0.557	0.174
8	0.000	0.000	0.000	0.003	0.018	0.076	0.247	−0.558	0.630	0.474
9	0.000	0.000	0.000	0.000	−0.002	−0.010	−0.486	0.174	−0.474	0.862

$v_1 = 0$, $v_2 = 1.0$, $v_3 = 0$

$N_s \backslash E$	−0.824	0.359	1.542	2.725	3.908	5.092	6.275	7.458	8.641	9.824
0	0.696	0.605	0.350	−0.155	0.055	0.016	0.004	−0.001	0.000	0.000
1	−0.605	0.229	0.570	−0.445	0.227	0.086	0.025	−0.006	−0.001	0.000
2	0.350	−0.570	−0.018	−0.491	0.480	0.265	0.101	−0.028	−0.006	0.001
3	−0.155	0.445	−0.491	0.143	0.434	0.490	0.276	−0.101	−0.025	0.004
4	0.054	−0.227	0.479	0.435	−0.194	0.414	0.490	−0.264	−0.086	0.016
5	−0.016	0.086	−0.264	−0.490	−0.414	−0.194	0.435	−0.479	−0.227	0.055
6	0.004	−0.025	0.101	0.276	−0.489	−0.434	−0.143	−0.491	−0.445	0.155
7	−0.001	0.006	−0.028	−0.101	−0.265	0.480	−0.491	0.018	−0.569	0.350
8	0.000	−0.001	0.006	0.025	0.086	−0.227	0.445	0.510	−0.229	0.605
9	0.000	0.000	−0.001	−0.004	−0.016	0.055	−0.155	−0.350	0.605	0.696

5. STABILITY OF THE HF GROUND STATE

We shall now consider the ground-state band, i.e., the $N_m=0$ band of the $(\lambda, \mu)=(N, 0)$ representation. In the absence of the interaction terms H_1 and H_2 the ground state is the $N_s=0$ state. As we turn on the interaction H_1 the states with $N_s \geq 1$ will mix with each other while the $N_s=0$ state remains uncoupled. For strong enough values of the interaction parameter $v(isis)/\lambda_s$, another state can become lower in energy than the $N_s=0$ state. At this point the original HF state becomes unstable, and a new HF representation must be found. If $H_2 \neq 0$, then the $N_s=0$ state will mix with other states. It can be shown that in order for the $N_s=0$ state to be a stable HF solution, the interaction strength for H_1 must satisfy the condition

$$(N-1)v(ssis)<\lambda_s, \tag{5.1}$$

and the interaction strength for H_2 must satisfy the condition

$$(N-1)v(iiss)<\lambda_s. \tag{5.2}$$

The stability of HF solution in the absence of the H_1 term has been discussed previously.[3] The influence of the interaction strength H_1 on the stability of the HF ground state will be discussed in a future publication.

TABLE V. Exact solutions of the Hamiltonian (3.9) for 10 particles and various values of the interaction parameters v_1, v_2, and v_3, defined in Table II. The energies E are in units of λ_s. The eigenvectors are in the columns below the corresponding eigenvalues.

Ground-state band, $N_m=0$

$v_1=0.3$, $v_2=0.2$, 0.6, $v_3=0$

$N_s \backslash E$	0.	0.628	0.910	1.562	2.475	3.666	5.131	6.868	8.879	11.162	13.718
0	1.000	0	0	0	0	0	0	0	0	0	0
1	0	0.462	0.776	0.393	−0.163	0.051	0.012	0.024	0	0	0
2	0	−0.426	−0.174	0.550	−0.596	0.336	0.129	0.036	0.007	0.001	0
3	0	0.430	−0.132	−0.430	−0.233	0.581	0.428	0.187	0.055	0.011	0.001
4	0	−0.414	0.291	0.071	0.449	0.049	0.526	0.459	0.211	0.059	0.009
5	0	0.360	−0.337	0.228	−0.160	−0.428	−0.024	0.497	0.455	0.196	0.044
6	0	−0.275	0.297	−0.359	−0.204	0.181	−0.422	−0.009	0.510	0.418	0.140
7	0	0.180	−0.212	0.332	0.376	0.230	0.142	−0.432	0.096	0.547	0.321
8	0	−0.097	0.122	−0.222	−0.334	−0.403	0.309	0.261	−0.417	0.313	0.533
9	0	0.041	−0.054	0.109	0.194	0.310	−0.419	0.427	−0.211	−0.245	0.622
10	0	−0.012	0.016	−0.035	−0.069	−0.132	0.232	−0.368	0.507	−0.570	0.452

Excited-state band, $N_m=1$, with energy λ_m subtracted

$v_1=0.3$, $v_2=0.2$, $v_3=0$

$N_s \backslash E$	−0.047	0.348	0.853	1.661	2.740	4.093	5.719	7.616	9.787	12.230
0	0.960	0.223	−0.156	0.055	0.015	0.003	0	0	0	0
1	−0.241	0.413	−0.703	0.475	0.213	0.069	0.017	0.003	0	0
2	0.111	−0.493	0.210	0.480	0.582	0.338	0.127	0.033	0.006	0.001
3	−0.066	0.477	0.176	−0.400	0.255	0.572	0.397	0.159	0.040	0.006
4	0.042	−0.407	−0.364	−0.012	−0.435	0.137	0.547	0.408	0.156	0.032
5	−0.026	0.304	0.383	0.321	0.051	−0.434	0.119	0.544	0.377	0.114
6	0.015	−0.195	−0.296	−0.396	0.322	0.034	−0.428	0.198	0.555	0.287
7	0.007	0.104	0.178	0.302	−0.408	0.372	−0.067	−0.384	0.382	0.513
8	0.003	−0.043	−0.081	−0.160	0.277	−0.406	0.455	−0.278	−0.187	0.636
9	0.001	0.012	0.024	0.054	−0.109	0.204	−0.342	0.496	−0.586	0.486

$v_1=0.3$, $v_2=0.6$, $v_3=0$

$N_s \backslash E$	−0.781	−0.294	0.395	1.380	2.640	4.174	5.980	8.059	10.411	13.035
0	0.270	0.753	0.536	−0.253	0.088	−0.023	0.005	0.001	0	0
1	−0.371	−0.389	0.372	−0.611	0.407	−0.174	0.053	0.012	0.002	0
2	0.445	0.065	−0.465	−0.078	0.543	0.472	0.228	0.072	0.015	0.002
3	−0.469	0.169	0.186	0.428	−0.067	−0.476	0.487	0.244	0.073	0.012
4	0.430	−0.288	0.139	−0.240	−0.396	0.112	0.454	0.477	0.222	0.052
5	−0.340	0.296	−0.328	−0.129	0.244	0.396	−0.082	0.481	0.440	0.157
6	0.227	−0.231	0.343	0.353	0.169	−0.199	−0.422	0.035	0.537	0.341
7	−0.125	0.141	−0.246	−0.349	−0.392	−0.266	0.079	−0.429	0.270	0.543
8	0.054	−0.064	0.126	0.214	0.327	0.422	0.406	−0.169	−0.277	0.612
9	−0.016	0.020	−0.042	−0.080	−0.146	−0.249	0.382	0.512	−0.559	0.432

6. RESULTS AND DISCUSSION

Here we shall restrict the values of the interaction parameters of the H_1 and H_2 terms of the Hamiltonian to the values given by Eqs. (5.1) and (5.2). The eigenvalues and eigenvectors of the Hamiltonian [Eqs. (3.9)] have been obtained numerically up to 10 particles for $N_m=0$ (ground-state band) and $N_m=1$ (excited-state band). The results are presented in Tables II to VII.

Let us first set $H_2=0$ and look at the effect of the interaction term H_1 on both the ground- and the excited-state bands. There are two terms contributing to H_1; one is characterized by interaction strength $v(ssis)$, and the other one by $v(imsm)$. The $v(imsm)$ term vanishes in the ground-state band ($N_m=0$). Thus if $v(ssis)=0$, the ground-state band is purely vibrational, i.e., we have N_s equally spaced excited states on top of the ground state with a separation λ_s as in Fig. 4. Let us now consider the excited "dipole" band $N_m=1$ with $v(ssis)=0$. If $v(imsm)$ is also zero, then the excited-state band will be equally spaced with energy separation equal to λ_s. The dipole strength from the ground state, i.e., the matrix element $(\varphi_n|D|\varphi_g)^2$ [see Eq. (4.6)] will be exhausted by the first ex-

TABLE VI. Exact solutions of the Hamiltonian (3.9) for 10 particles and various values of the interaction parameters v_1, v_2, and v_3, defined in Table II. The energies E are in units of λ_s. The eigenvectors are in the columns below the corresponding eigenvalues.

Ground-state band, $N_m=0$

$v_1=0.3$, $v_2=0.2$, 0.6, $v_3=0.4$

$N_s\backslash E$	−0.043	0.822	1.395	1.618	2.264	3.345	4.797	6.606	8.768	11.282	14.146
0	0.985	−0.077	0.133	0.041	0.050	0.029	−0.013	0.004	0.001	−0.003	0
1	0.045	0.938	0.229	0.215	0.215	0.140	−0.071	0.029	0.009	−0.002	0
2	−0.159	−0.238	0.694	0.247	0.424	0.359	−0.227	0.110	0.043	−0.011	0.002
3	0.039	−0.169	−0.651	0.082	0.244	0.444	−0.430	0.280	0.131	−0.043	0.009
4	0.011	0.158	0.269	−0.512	−0.415	0.011	−0.390	0.454	0.297	−0.125	0.032
5	−0.011	−0.064	−0.028	0.597	−0.067	−0.440	0.096	0.368	0.460	−0.275	0.089
6	0.004	0.014	−0.039	−0.436	0.454	0.088	0.424	−0.102	0.402	−0.448	0.203
7	−0.001	0	0.032	0.240	−0.473	0.380	−0.107	−0.432	−0.011	0.486	0.373
8	0.114	−0.001	−0.015	−0.104	0.297	−0.466	−0.396	0.021	−0.435	−0.213	0.540
9	0.0	0	0.005	0.035	−0.126	0.283	0.451	0.475	−0.202	0.297	0.587
10	0.0	0	−0.002	−0.008	0.034	−0.096	−0.213	−0.380	0.538	0.580	0.417

Excited-state band, $N_m=1$, with energy λ_m subtracted

$v_1=0.3$, $v_2=0.2$, $v_3=0.4$

$N_s\backslash E$	−0.052	0.670	1.066	1.577	2.513	3.818	5.480	7.494	9.859	12.574
0	0.983	0.557	−0.122	0.105	0.065	0.031	0.011	0.003	0.001	0
1	−0.136	0.847	−0.231	0.342	0.259	0.146	0.065	0.022	0.006	0.001
2	−0.107	−0.514	−0.361	0.420	0.475	0.371	0.211	0.089	0.027	0.053
3	0.066	0.097	0.645	−0.240	0.217	0.470	0.422	0.240	0.092	0.021
4	−0.016	0.050	−0.522	−0.314	−0.416	0.672	0.448	0.435	0.229	0.068
5	0	−0.051	0.300	0.539	−0.087	−0.437	0.028	0.462	0.418	0.170
6	0.002	0.025	−0.138	−0.431	0.467	−0.026	−0.435	0.097	0.512	0.339
7	0	−0.009	0.052	0.233	−0.442	0.459	−0.087	−0.408	0.292	0.526
8	0	0.002	−0.016	−0.089	0.236	−0.423	0.503	−0.281	−0.237	0.607
9	0	0	0.003	0.022	−0.073	0.180	−0.347	0.525	−0.600	0.454

$v_1=0.3$, $v_2=0.6$, $v_3=0.4$

$N_s\backslash E$	−0.307	−0.072	0.370	1.184	2.345	3.865	5.730	7.945	10.510	13.423
0	0.809	−0.318	0.385	0.266	0.145	−0.065	0.023	0.007	−0.001	0
1	−0.534	−0.169	0.424	0.513	0.411	−0.245	0.111	0.039	−0.010	0.002
2	0.233	0.496	−0.413	0.096	0.445	−0.462	0.297	0.134	−0.043	0.009
3	−0.077	−0.552	−0.044	−0.454	−0.925	−0.369	0.472	0.310	−0.129	0.032
4	0.018	0.447	0.386	0.139	−0.414	0.163	0.346	0.472	−0.285	0.091
5	−0.003	−0.296	−0.449	0.294	0.198	0.397	−0.149	0.388	−0.458	0.210
6	0	0.163	0.332	−0.445	0.277	−0.179	−0.418	−0.048	−0.481	0.382
7	0	−0.074	−0.180	0.340	−0.451	−0.332	0.077	−0.439	−0.188	0.546
8	0	0.026	0.072	0.167	0.314	0.452	0.445	−0.162	0.317	0.582
9	0	−0.006	−0.019	0.050	−0.118	−0.235	−0.392	0.537	0.567	0.404

cited state of the $N_m = 1$ band. No other "dipole" transitions from the ground state can occur in this case. As we turn on the interaction term $v(imsm)$ the states of the $N_m = 1$ band start mixing with each other, while the ground-state band remains unperturbed.

As a result of this mixing the dipole strength becomes distributed among the states of $N_m = 1$ band, the sum rule remaining the same as in the noninteracting case. As long as $v(ssis) = 0$ the vibrational character of the $N_m = 1$ band is preserved. (See Tables II–IV.) It is shown in the Appendix that in this case the energy splitting of the $N_m = 1$ band is given by

$$\Delta E = [\lambda_s^2 + 16v^2(imsm)]^{1/2}, \tag{6.1}$$

with the lowest $N_m = 1$ state given by

$$E = \tfrac{1}{2}(N-1)(\lambda_s - \Delta E), \tag{6.2}$$

and the dipole strength S_n to the state φ_n

$$S_n = (\varphi_n | D | \varphi_g)^2 = \left(\frac{N-1}{n}\right)\left(\frac{\Delta E + \lambda_s}{2\Delta E}\right)^{N-1}\left(\frac{\Delta E - \lambda_s}{\Delta E + \lambda_s}\right)^n, \tag{6.3}$$

where $n = 0, 1, \ldots, N-1$ specifies the member of the $N_m = 1$ band corresponding to the eigenvalue

TABLE VII. Exact solutions of the Hamiltonian (3.9) for 10 particles and various values of the interaction parameters v_1, v_2, and v_3, defined in Table II. The energies E are in units of λ_s. The eigenvectors are in the columns below the corresponding eigenvalues.

Ground-state band, $N_m = 0$

$v_1 = 0.3$, $v_2 = 0.2$, 0.6, $v_3 = 0.8$

$N_s \backslash E$	−0.174	0.560	1.288	1.973	2.473	3.156	4.527	6.392	8.076	11.458	14.643
0	0.944	−0.087	0.230	−0.002	−0.088	0.053	0.033	0.015	0.005	−0.001	0
1	0.070	0.869	−0.023	0.427	−0.034	0.187	0.125	0.064	0.026	−0.008	0.001
2	−0.306	−0.091	0.719	−0.009	−0.406	0.312	0.282	0.177	0.083	−0.029	0.006
3	0.049	−0.416	−0.356	0.504	0.136	0.333	0.395	0.333	0.198	−0.082	0.022
4	0.081	0.207	0.320	−0.616	−0.249	−0.522	0.267	0.416	0.349	−0.186	0.059
5	−0.045	0.056	0.382	0.054	0.611	−0.273	−0.142	0.247	0.435	−0.331	0.134
6	−0.001	−0.091	−0.112	0.322	−0.561	−0.179	−0.377	−0.169	0.297	−0.452	0.256
7	0.009	0.033	−0.041	−0.263	0.236	0.595	0.028	−0.404	−0.098	−0.417	0.411
8	−0.004	0	0.044	0.093	−0.021	−0.494	0.500	0.010	−0.434	−0.122	0.540
9	0.000	−0.003	−0.015	−0.010	−0.024	0.207	−0.478	0.523	−0.186	0.339	0.552
10	0	0.001	0.002	−0.003	0.010	−0.043	0.187	−0.393	0.567	0.583	0.383

Excited-state band, $N_m = 1$, with energy λ_m subtracted

$v_1 = 0.3$, $v_2 = 0.2$, $v_3 = 0.8$

$N_s \backslash E$	−0.148	0.540	1.210	1.770	2.403	3.595	5.277	7.407	9.973	12.972
0	0.953	0.007	0.273	0.049	−0.096	0.060	0.286	0.010	−0.003	0
1	−0.086	0.866	0.070	0.353	−0.240	0.199	0.113	0.048	−0.015	0.003
2	−0.261	−0.330	0.661	0.041	−0.386	0.371	0.269	0.143	−0.054	0.013
3	0.113	−0.257	−0.620	0.322	−0.103	0.370	0.417	0.298	−0.142	0.041
4	0.029	0.260	0.024	−0.702	0.180	0.008	0.350	0.435	−0.286	0.105
5	−0.040	−0.060	0.256	0.503	0.386	−0.378	−0.044	0.378	−0.437	0.219
6	0.013	−0.030	−0.175	−0.137	−0.634	−0.134	0.415	0.010	−0.461	0.380
7	0	0.025	0.048	−0.028	0.411	0.561	−0.115	−0.416	−0.209	0.532
8	−0.001	−0.007	−0.001	0.033	−0.140	−0.430	0.553	−0.276	0.280	0.577
9	0	0.001	−0.003	−0.009	0.023	0.146	−0.352	0.554	0.607	0.423

$v_1 = 0.3$, $v_2 = 0.6$, $v_3 = 0.8$

$N_s \backslash E$	−0.320	0.140	0.560	1.073	2.096	3.582	5.505	7.860	10.645	13.860
0	0.876	−0.184	0.301	−0.244	0.187	0.104	0.466	0.017	−0.005	0.001
1	−0.409	−0.623	0.109	−0.400	0.394	0.290	0.163	0.071	−0.022	0.004
2	−0.100	0.685	0.223	−0.072	0.348	0.432	0.340	0.189	−0.074	0.018
3	0.212	−0.190	0.645	0.260	−0.095	0.271	0.438	0.353	−0.179	0.054
4	−0.101	−0.162	0.593	0.208	−0.377	−0.189	0.249	0.449	−0.332	0.129
5	0.005	0.190	−0.277	−0.579	0.070	−0.376	−0.196	0.300	−0.460	0.255
6	0.018	0.091	0.047	0.508	0.436	0.133	−0.399	−0.115	−0.423	0.414
7	−0.017	0.022	0.021	−0.254	−0.508	0.414	0.068	−0.437	−0.114	0.544
8	0.003	0	−0.016	0.076	0.278	−0.481	0.485	−0.151	0.349	0.552
9	0	0.001	0.004	−0.012	−0.079	0.216	−0.404	0.561	0.571	0.377

$$E_n = \lambda_m + \tfrac{1}{2}(N-1)(\lambda_s - \Delta E) + n\Delta E \,. \tag{6.4}$$

The first five excited band energies and corresponding dipole strengths are given in Table VIII for 10 particles and different values of the interaction parameter $v_2 = 2N^{1/2}v(imsm)/\lambda_s$. It is seen from Eq. (6.1) that the energy splitting $\Delta E/\lambda_s$ depends only on the interaction strength $2v(imsm)/\lambda_s$ and does not depend on the number of particles N. The dipole strength, on the other hand, depends on N. The ratio of the dipole strengths of the one-phonon ($n=1$) state to that of the zero-phonon ($n=0$) one is given by

$$R = \frac{(\varphi_1|D|\varphi_g)^2}{(\varphi_0|D|\varphi_g)^2} = (N-1)\frac{\Delta E/\lambda_s - 1}{\Delta E/\lambda_s + 1} \,. \tag{6.5}$$

It is seen that for a given $\Delta E/\lambda_s$ this ratio is directly proportional to $(N-1)$. For $2v(imsm)/\lambda_s \ll 1$,

$$R \approx (N-1)\left[\frac{2v(imsm)}{\lambda_s}\right]^2 \,. \tag{6.6}$$

One possible approximation method one might be tempted to consider[11] is to include two-particle–two-hole (2p-2h) or three-particle–three-hole (3p-3h) configurations as indicated in Fig. 5. This is done in our model by simply diagonalizing the Hamiltonian matrix in a two-dimensional (2p-2h) or a three-dimensional (3p-3h) subspace of the N-dimensional space of the $N_m = 1$ band.

Let us first consider the approximation where we only include up to 2p-2h states. The energy splitting between no-phonon and one-phonon states is given in this approximation by

$$\Delta E|_{2p2h} = [\lambda_s^2 + 16(N-1)v^2(imsm)]^{1/2} \,, \tag{6.7}$$

and the dipole strength ratio is

$$R|_{2p2h} = \frac{\Delta E|_{2p2h} - \lambda_s}{\Delta E|_{2p2h} + \lambda_s} \,. \tag{6.8}$$

For $(N-1)[2v(imsm)/\lambda_s]^2 \ll 1$,

$$R|_{2p2h} \approx (N-1)[2v(imsm)/\lambda_s]^2 \,. \tag{6.9}$$

Note, that the 2p-2h approximation depends only on the interaction parameter $v(imsm)$. Comparing these approximate results with the exact expressions for ΔE and R in Eqs. (6.1) and (6.3), we see that for a given interaction strength the approximate energy splitting will be larger than the exact one, while the corresponding approximate dipole-strength ratio will be too low. In a calculation on an actual nucleus one is likely to adjust the parameters of the calculation to obtain proper energy splitting. If this is done in this model, then the resulting dipole-strength ratio $R|_{2p2h}$ will be too low by a factor of $(N-1)$.

Extending the approximation up to 3p-3h states improves the agreement with exact results for the zero-phonon and one-phonon states (see Table VIII). However, the two-phonon state is not given accurately.

We thus conclude that an approximation which only includes a few p-h excitations could lead to a qualitative explanation of giant-resonance splitting, but it might be too optimistic to expect quantitative predictions from such approximations.

We should note here that with H_0 given by Eq. (3.9), the effect of the $v(imsm)$ term in H_1 is al-

TABLE VIII. Five lowest $N_m = 1$ band energies, ΔE, relative to the ground state and corresponding transition strengths, S, from the ground state. The energy λ_m has been subtracted from ΔE.

	$N=10$, $v_1=0$,[a] $v_3=0$[a]					
	2p-2h approximation		3p-3h approximation		Exact	
v_2[a]	ΔE	S	ΔE	S	ΔE	S
0.2	−0.035	0.967	−0.036	0.964	−0.036	0.964
	1.035	0.033	0.974	0.035	0.972	0.034
			2.061	0	1.980	0
					2.988	0
					3.996	0
0.6	−0.258	0.830	−0.306	0.740	−0.313	0.743
	1.258	0.170	0.866	0.246	0.756	0.225
			2.440	0.014	1.826	0.030
					2.896	0
					3.965	0
1.0	−0.572	0.739	−0.761	0.514	−0.824	0.484
	1.572	0.261	0.798	0.449	0.359	0.365
			2.964	0.037	1.542	0.122
					2.725	0.024
					3.908	0

[a] Parameters v_1, v_2, and v_3 are defined in Table II.

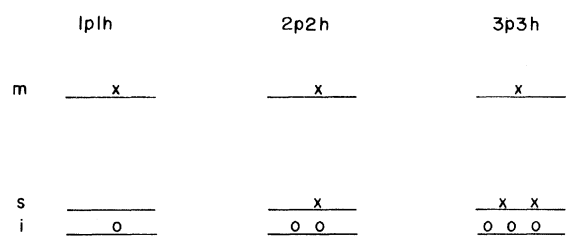

FIG. 5. Particle-hole configurations included in approximations of Sec. 6. Circles represent holes, crosses represent particles.

ways to increase the energy level spacing in the excited-state band relative to the ground state. However, in the original Hamiltonian [Eqs. (2.8) and (3.5)] there are other terms present in H_0, in particular a term proportional to $N_s N_m$:

$$2N_s N_m = [\tfrac{1}{2}v(i) - v(isis) + v(smsm) - v(imim)]. \tag{6.10}$$

The above term vanishes in the ground-state band ($N_s = 0$) but contributes a term proportional to N_s to the excited-state band, changing the unperturbed energy splitting λ_s to

$$\lambda_s' = \lambda_s + v(i) - 2v(isis) - 2v(imim) + 2v(smsm). \tag{6.11}$$

If $\lambda_s' < \lambda_s$ it is possible to obtain energy splittings in the $N_m = 1$ band smaller than those in $N_m = 0$ band.

Consider the $v(ssis)$ term in H_1 [Eqs. (3.9)]. As was pointed out previously, this term alone leaves the lowest state of each band unperturbed but mixes all the other states, i.e., $N_s = 0$ remains a good quantum number, but $N_s > 0$ is not. In the absence of the $v(imsm)$ term the $N_s = 0$, $N_m = 1$ state will exhaust all the "dipole strength" from the ground state. In order to obtain spreading of the dipole strength over excited states we must include $v(imsm)$. Tables II, III, V, and IX show the results for $v_1 = N^{1/2}v(ssis)/\lambda_s = 0.3$ and different values of $v(imsm)$. We see that both the ground- and excited-state bands lose their vibrational character, i.e., the levels within each band are no longer equally spaced. For small values of the $v(imsm)$ term most of the dipole strength is still concentrated in the lowest member of the excited-state band; however, for $2N^{1/2}v(imsm)/\lambda_s = 0.6$, we see that the dipole strength is distributed over four states with 57% going to the second-highest state of the excited-state band (see Table IX). The 2p-2h and 3p-3h approximations are no longer valid.

The effect of H_2 part of the Hamiltonian is displayed in Tables II, III, VI, VII, X, and XI. Since the H_2 term mixes the ground state (the lowest state of the ground-state band) with other members of the ground-state band, the dipole sum rule is no longer given by Eq. (4.6), since the ground-state expectation value of T_+ does not vanish in this case.

From Tables X and XI we see that as the strength of the H_2 term increases, the relative amount of dipole strength going to the lowest member of excited-state band is also increased. The H_2 term thus seems to counteract the effect of the $v(ssis)$ term as far as the distribution of dipole strength is concerned. Again the 2p-2h and 3p-3h approximations give poor results for $v_2 = 0.6$.

7. SUMMARY AND CONCLUSIONS

The general class of models presented in this paper is especially useful in investigating systems possessing different collective modes. A version of this class of models was adapted here for the study of the mixing of two collective states, analogous to the mixing of giant-dipole-resonance with

TABLE IX. Five lowest $N_m = 1$ band energies, ΔE, relative to the ground state and corresponding transition strengths, S, from the ground state. The energy λ_m has been subtracted from ΔE.

$N = 10$, $v_1 = 0.3$, $v_3 = 0$

	2p-2h approximation		3p-3h approximation		Exact	
v_2	ΔE	S	ΔE	S	ΔE	S
0.2	−0.035	0.967	−0.042	0.947	−0.048	0.922
	1.035	0.033	0.733	0.051	0.348	0.050
			2.309	0.001	0.854	0.024
					1.661	0
					2.740	0
0.6	−0.258	0.830	−0.385	0.612	−0.781	0.073
	1.258	0.170	0.606	0.375	−0.294	0.567
			2.779	0.013	0.395	0.287
					1.380	0.064
					2.640	0.001

TABLE X. Five lowest $N_m=1$ band energies, ΔE, relative to the ground state and corresponding transition strengths, S, from the ground state. The energy λ_m has been subtracted from ΔE.

			$N=10$, $v_1=0.3$, $v_4=0.4$			
	2p-2h approximation		3p-3h approximation		Exact	
v_2	ΔE	S	ΔE	S	ΔE	S
0.2	−0.035	0.967	−0.044	0.976	−0.009	1.003
	1.035	0.033	0.699	0.008	0.714	0.010
			2.344	0.016	1.109	0.002
					1.621	0.001
					2.556	0
0.6	−0.258	0.830	−0.243	0.778	−0.263	0.628
	1.258	0.170	0.646	0.190	−0.028	0.155
			2.597	0.031	0.413	0.180
					1.227	0.048
					2.393	0.006

surface quadrupole vibrations in nuclei. The physical picture described by this model is that of a set of vibrational bands, each band being separated from the next one by an energy λ_m, much larger than the energy splitting λ_s within each band (Fig. 4). The model Hamiltonian [Eqs. (3.9)] mixes states within each band but does not mix states belonging to different bands. In this work we are mainly concerned with the two lowest bands, the lowest member of the first-excited-state band being a collective state analogous to the giant dipole resonance. The interaction term primarily responsible for the mixing of the two types of collective excitations is shown to be the $v(imsm)$ term of the Hamiltonian (3.9), the term that promotes one particle form the lowest level i to the middle level s, leaving the other particle in the mth level. This term vanishes in the ground-state band. Furthermore, the value of $v(imsm)$ must be at least such that $2N^{1/2}v(imsm)/\lambda_s = O(0.5)$, in order that there be any appreciable spread of transition strength over several excited states. It was also shown that in the excited-state band the contribution of the many-particle–many-hole states is quite important for the above value of $v(imsm)$; the 2p-2h and 3p-3h approximations do not give satisfactory results. We are led to conclude that in treating this type of mixing of collective states the many-phonon vibrations must be properly taken into account. One possible approach might be a search for a different representation in the excited-state band from that of the ground-state one, as we did in the Appendix for the model. The result of this approach is a vibrational band on top of the dipole state with a different frequency than that of the ground-state vibrational band.

There are other possible applications of this class of models. One could look at double-closed-shell systems simulating nuclei, such as O^{16} or Ca^{40}, by reversing the order of the s and m levels in Fig. 1. In this case other terms in the Hamiltonian (2.8) will become more important than the ones included in Eqs. (3.9). One could also consider a system, in which n particles form a closed shell, and look at the effects on the structure as other particles are added outside the

TABLE XI. Five lowest $N_m=1$ band energies, ΔE, relative to the ground state and corresponding transition strengths, S, from the ground state. The energy λ_m has been subtracted from ΔE.

			$N=10$, $v_1=0.3$, $v_3=0.8$			
	2p-2h approximation		3p-3h approximation		Exact	
v_2	ΔE	S	ΔE	S	ΔE	S
0.2	−0.035	0.967	−0.111	0.955	0.026	1.008
	1.035	0.033	0.695	0.002	0.714	0
			2.416	0.043	1.384	0
					1.944	0
					2.576	0
0.6	−0.258	0.830	−0.258	0.808	−0.147	0.839
	1.258	0.170	0.340	0.129	0.314	0.110
			2.918	0.062	0.734	0.039
					1.247	0.019
					2.270	0.002

closed shell. This would involve studying representations other than the symmetric one, $(\lambda, \mu) = (N, 0)$, as shown in Fig. 3.

We are grateful to C. A. Levinson and M. K. Banerjee for numerous fruitful discussions.

APPENDIX

Consider Eqs. (3.9) for $v(ssis) = v(iiss) = 0$ and $N_m = 1$. Since if $N_m = 1$, then $N_s = T_0 + \frac{1}{2}(N-1)$, Eqs. (3.9) become

$$H = \lambda_m + \lambda_s T_0 + \tfrac{1}{2}(N-1)\lambda_s + 2v(imsm)(T_- + T_+)$$
$$= \lambda_m + \lambda_s T_0 + \tfrac{1}{2}\lambda_s(N-1) + 4v(imsm)T_x . \qquad (A.1)$$

We shall seek a representation in which H is diagonal in the following way. Let us perform a rotation through an angle θ about the y axis in T space:

$$T_0 = \sin\theta T'_x + \cos\theta T'_0 ,$$
$$T_x = \cos\theta T'_x - \sin\theta T'_0 . \qquad (A.2)$$

We now express Eq. (A.1) in terms of the operators T'_0 and T'_x [Eq. (A.2)]:

$$H = \lambda_m + \tfrac{1}{2}(N-1) + [\lambda_s \sin\theta + 4v(imsm)\cos\theta]T'_x$$
$$+ [\lambda_s \cos\theta - 4v(imsm)\sin\theta]T'_0 . \qquad (A.3)$$

Since the angle θ is at our disposal we let the coefficient of T'_x in Eq. (A.3) vanish:

$$\sin\theta = -\frac{4v(imsm)}{\lambda_s}\cos\theta ,$$
$$\pm\sin\theta = \frac{4v(imsm)}{[\lambda_s^2 + 16v^2(imsm)]^{1/2}} , \qquad (A.4)$$
$$\pm\cos\theta = -\frac{1}{[\lambda_s^2 + 16v^2(imsm)]^{1/2}} .$$

Choosing the minus sign in Eqs. (A.4), and letting

$$\Delta E = [\lambda_s^2 + 16v^2(imsm)]^{1/2} \qquad (A.5)$$

we get

$$H = \lambda_m + \tfrac{1}{2}\lambda_s(N-1) + \Delta E T'_0 . \qquad (A.6)$$

The eigenvalues of T'_0 are

$$-\tfrac{1}{2}(N-1) + n , \quad n = 0, 1, \ldots, (N-1) . \qquad (A.7)$$

Therefore, the eigenvalues of H [Eq. (A.5)] become

$$E_n = \lambda_m + \tfrac{1}{2}(N-1)(\lambda_s - \Delta E) + n\Delta E . \qquad (A.8)$$

Since in this case, the ground state φ_g is the lowest state of the $N_m = 0$ band, the dipole strength S_n into the eigenstate φ_n of T'_0 with eigenvalue $-\frac{1}{2}(N-1) + n$ is

$$S_n = (\varphi_n | D | \varphi_g)^2 = | d_{\frac{1}{2}(N-1), \frac{1}{2}(N-1)-n}(\theta) |^2 ,$$

where $d_{\frac{1}{2}(N-1), \frac{1}{2}(N-1)-n}(\theta)$, the matrix elements of a finite rotation, are defined, e.g., in the work of Edmonds.[15] Using Ref. 15

$$| d_{\frac{1}{2}(N-1), \frac{1}{2}(N-1)-n}(\theta) |^2$$
$$= \frac{(N-1)!}{n!\,(N-1-n)!}\left(\frac{1+\cos\theta}{2}\right)^{N-1-n}\left(\frac{1-\cos\theta}{2}\right)^{n}$$

and Eq. (A.4) with the minus sign, we finally get

$$S_n = \left(\frac{N-1}{n}\right)\left(\frac{\Delta E + \lambda_s}{2\Delta E}\right)^{N-1}\left(\frac{\Delta E - \lambda_s}{\Delta E + \lambda_s}\right)^{n} . \qquad (A.9)$$

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