Effect of Correlations on the Contribution of Three-Body Forces to the Binding Energy of Nuclear Matter

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We investigate the contribution of three-body forces to the binding energy of nuclear matter, including the effects of two- and three-body correlations induced by the two-nucleon forces, and find that it may by approximated using plane-wave three-nucleon states cut off when any interparticle distance is less than about 0.9 fm. Existing calculations can then be used to estimate that three-body forces contribute about 1 MeV to the binding energy.

1. INTRODUCTION

In the past few years there has been a revival of interest in theoretical estimates of three-nucleon forces and their contribution to the binding energy of nuclear matter E_3. As early as 1957, the relationship of the pion-exchange three-body potential W, which is illustrated in Fig. 1, to the virtual πN scattering amplitude was discussed by Miyazawa *et al.*[1] Brown, Green, and Gerace[2] suggested on the basis of Alder's partially conserved axial-vector current self-consistency condition on this amplitude, which demands that it vanish when the pion four-momentum is zero, that the contribution of W to E_3 would be small. This conclusion was immediately questioned by us[3] on the grounds that the soft-pion limit was only applicable to the contribution to E_3 which was of first order in W and zeroth order in V, the two-body potential. Higher-order terms in general introduce hard spacelike pions, requiring extrapolation of the πN amplitude to this region. Brown and Green[4] then extended their calculation to include first- and second-order terms in W, employing the device of regarding the first-order term in W as a renormalization of the pion mass. Nogami and his co-workers at McMaster University have also been working on this problem,[5] and in a recent paper[6] have clarified the nature of the effective-mass method and have suggested some improvements to the Brown-Green calculation. In these calculations the effects of correlations introduced by the two-body force V were often ignored. In some calculations a cutoff was arbitrarily introduced in the wave function at small values of one or more of the interparticle distances in the hope of simulating their effect.

In this paper we show how to include these correlations in a consistent way. Our formal result, derived in Sec. 2, is that the binding energy E_W contributed by a three-body potential W may be written as

$$E_W = \sum_{123}{}' (\langle \Psi_{123} | W | \Omega_{123} \rangle + \text{exchange terms}), \qquad (1)$$

where the prime on the summation indicates that it is over all different triples of occupied states, Ψ is a three-body wave function generated by the two-body force, and Ω is a three-body wave function generated by the two- and three-body forces. It should be emphasized that Eq. (1) represents the sum of *all* diagrams involving two- and three-body forces, some of which are shown in Fig. 2. The properties of the wave function Ψ may be extracted from the literature and are reviewed in Sec. 3. Its most important property for our purposes is that the strong short-range part of the two-nucleon interaction requires that it be "small" in an average sense when any interparticle separation is less than about 0.9 fm, and very close to a

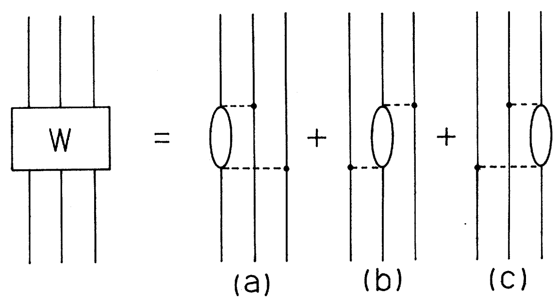

FIG. 1. Feynman diagrams for the pion-exchange three-body potential W. Solid lines represent nucleons; dotted lines, pions; and the blob, the virtual πN scattering amplitude with the iterated one-pion-exchange potential removed.

plane wave when all interparticle distances exceed 0.9 fm. This suggests using a cut-off plane wave as an approximation to Ψ.

After reviewing the properties of W in Sec. 4, in Secs. 5 and 6 we discuss the first- and second-order terms in the expansion of E_W in powers of W. We show that they can be estimated using this cut-off-plane-wave approximation to Ψ in the initial and final states, and using plane waves to approximate Ψ in the intermediate states.

This immediately lets us conclude that higher-order terms in W will contribute negligibly, since only the weak long-range part survives the cutoff. (The reason we have to go to second order is that W has a tensor component.) Similarly, three-body potentials from exchange of other particles will give negligible contributions having shorter range. For example, the $\pi\eta$ force of Fig. 3 will be significant only for $r_{23} \lesssim 1/m_\eta \approx 0.4$ fm, which is well within the cutoff. This result was, of course, anticipated on physical grounds—the hard cores keep nuclei far enough apart so that a short-range three-body force will not have a significant effect.

As only the short-range correlations in E_3 have been taken into account in our cutoff approximation, it is necessary to take the long-range two-body interaction v_l explicitly into account. This leads us to consider diagrams involving one three-body interaction W, and one long-range two-body interaction v_l with any number of short-range two-body interactions. Such diagrams are discussed in Sec. 6 in a way which precisely parallels the account of the term of second order in W.

If we adopt the cutoff approximation, we can use an existing calculation of Loiseau, Nogami, and Ross (LNR)[7] to estimate that $E_3 \approx -1$ MeV (i.e., 1-MeV attraction). Our result complements the vast literature on the contribution of two-body forces to the binding energy of nuclear matter,[8] and with it gives a coherent account of the two- and three-body effects. In Sec. 6 we discuss the present status of the binding-energy calculations.

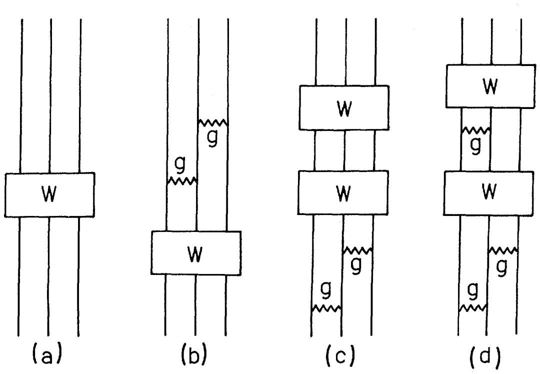

FIG. 2. Some typical Bethe-Goldstone-Rajaraman diagrams contributing to the three-body energy.

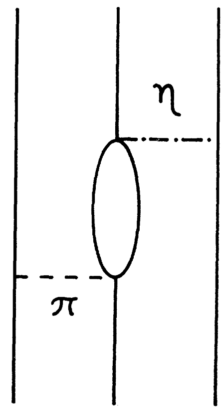

FIG. 3. The $\pi\eta$ three-body force.

2. BETHE-FADDEEV EQUATIONS WITH THREE-BODY FORCES

Bethe,[9] following the method used by Faddeev in his classic work on the three-body problem,[10] analyzed the sum T of all three-body diagrams in nuclear matter of the type shown in Fig. 4.[11] He wrote

$$T = T^{(1)} + T^{(2)} + T^{(3)}, \tag{2}$$

where $T^{(1)}$ is the sum of the subset of diagrams in which the last interaction is between particles 2 and 3, and obtained the coupled equations

$$T^{(1)} = g_{23} - g_{23}\frac{Q}{e}(T^{(2)} + T^{(3)}), \text{ etc.} \tag{3}$$

in which g is the two-body reaction matrix obtained from the Bethe-Goldstone equation, e is an energy denominator, and Q is the Pauli projection operator.

In the presence of a three-body force W we have to include diagrams of the type shown in Fig. 2. Let us call the sum of *all* three-body diagrams τ, and call τ_W the sum of all diagrams with at least one three-body interaction, i.e.,

$$\tau = \tau_W + T. \tag{4}$$

We can analyse τ and τ_W in much the same way as in Eq. (2), but we have an additional class of diagrams in which the last interaction is W. Figure 2(c) is an example. Calling $\tau^{(0)}$ the sum of such diagrams, we have

$$\tau = \tau^{(0)} + \tau^{(1)} + \tau^{(2)} + \tau^{(3)}. \tag{5}$$

Following Bethe's analysis we have the equations

$$\tau^{(1)} = g_{23} - g_{23}\frac{Q}{e}(\tau^{(0)} + \tau^{(2)} + \tau^{(3)}), \tag{6}$$

$$\tau^{(0)} = W - W\frac{Q}{e}\tau.$$

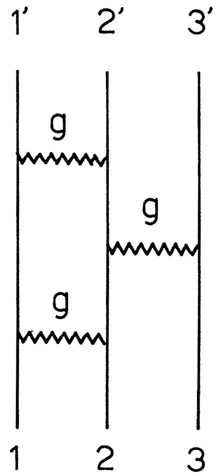

FIG. 4. A diagram contributing to the three-body T matrix in nuclear matter. This diagram contributes to $T^{(3)}$.

Equations for $\tau_W^{(i)}$ may be obtained by subtracting Eq. (3) from Eq. (6). They are,

$$\tau_W^{(1)} = -g_{23}\frac{Q}{e}\tau^{(0)} - g_{23}\frac{Q}{e}(\tau_W^{(2)} + \tau_W^{(3)}) . \tag{7}$$

The similarity between these equations and the Bethe-Faddeev Eq. (3) suggests the ansatz

$$\tau_W^{(1)} = -T^{(1)}\frac{Q}{e}\tau^{(0)} \tag{8}$$

which is readily verified to be a solution to Eq. (7). In this way we obtain

$$\tau = T + \left(1 - T\frac{Q}{e}\right)W\left(1 - \frac{Q}{e}\tau\right), \tag{9}$$

which could be used as the basis of an iterative solution. Equation (9) has a simple interpretation in terms of diagrams, as shown in Fig. 5. The sum of all three-body diagrams with interactions is the sum of diagrams without three-body interactions, Fig. 5(a), and those with such interactions. Select the last such interaction in the diagram. It can be followed by no interactions [Figs. 5(b) and 5(d)] or by two-body interactions only [Figs. 5(c) and 5(e)], and be preceded by no interactions [Figs. 5(b) and 5(c)] or by interactions [Figs. 5(d) and 5(e)]. Putting all these together we obtain the various terms in Eq. (9).

To obtain the contribution E_W of W to the binding energy, we must take the matrix element of τ_W between plane-wave states $|1,2,3\rangle = |\Phi\rangle$ and $|1',2',3'\rangle = |\Phi'\rangle$, where to allow for exchange the states $1'$, $2'$, $3'$ are a permutation of the states 1, 2, 3, and then sum over all occupied triples and all permutations, with appropriate phases to take account of Fermi statistics. Thus, denoting this summation by $\sum''$,

$$E_W = \sum_{123}'' \left\langle \Phi'_{123} \left| \left(1 - T\frac{Q}{e}\right) W \left(1 - \frac{Q}{e}\tau\right) \right| \Phi_{123} \right\rangle . \tag{10}$$

To simplify Eq. (10) we define three-body wave functions Ψ and Ω by

$$\Psi = \left(1 - \frac{Q}{e}T\right)\Phi \tag{11}$$

and

$$\Omega = \left(1 - \frac{Q}{e}\tau\right)\Phi . \tag{12}$$

Ψ is the correlated three-body wave function introduced by Moskowski.[12] In terms of the partial-defect wave functions $Z^{(1)}$, $Z^{(2)}$, $Z^{(3)}$ introduced in the solution of the Bethe-Faddeev equation,[8] and defined by

$$Z^{(1)} = \frac{Q}{e}(T^{(2)} + T^{(3)})\Phi , \tag{13}$$

$$\Psi = \Phi - \tfrac{1}{2}\sum_i Z^{(i)} , \tag{14}$$

Ω is the obvious generalization of Ψ to include the three-body interaction. We have now derived Eq. (1),

$$E_W = \sum_{123}'' \langle \Psi'_{123} | W | \Omega_{123} \rangle , \tag{1}$$

expressing the energy shift due to W in the familiar form

$$\text{energy shift} = \langle \text{unperturbed wave function} | \text{perturbation} | \text{perturbed wave function} \rangle . \tag{15}$$

Here Ψ is called the unperturbed wave function because it does not contain the effects of the three-body potential W. However, it does contain all the effects of the short-range two-body potential and could equally well be referred to as the correlated wave function. In the rest of this paper we will select one or the other name according to which property of Ψ we wish to emphasize.

Calculation of E_3 thus involves three ingredients, namely
(i) the unperturbed correlated wave function Ψ,
(ii) the three-body potential W, and
(iii) the perturbed wave function Ω,
which we now discuss in turn.

3. CORRELATED THREE-BODY WAVE FUNCTION

There is by now an extensive literature on three-body correlations in nuclear matter. This has largely been concerned with calculating the three-body correlation energy, and it has therefore been found convenient to develop the properties of the defect wave functions $Z^{(i)}$ rather than the correlated wave function Ψ. However, it is a trivial matter to obtain Ψ from the discussion in the literature.

Day[13] has constructed an approximate analytic solution to the Bethe-Faddeev Eq. (3), for the case of a central spin-independent potential. Dahlbolm[14] has given a solution which also includes the effects of two-body tensor forces. For simplicity we use Day's solution, constructed for the spin-independent standard hard-core potential of Moskowski and Scott,[15] with a core radius c. Expressed in terms of Ψ, this solution is

$$\begin{aligned}\Psi = [&1-(\eta_{12}+\eta_{23}+\eta_{31})+\eta_{12}(\zeta_{13}+\zeta_{23})+\eta_{23}(\zeta_{21}+\zeta_{31})+\eta_{31}(\zeta_{32}+\zeta_{12})-(\zeta_{12}\zeta_{13}+\zeta_{12}\zeta_{23}+\zeta_{13}\zeta_{32})\\ &+(\eta_{23}\zeta_{12}\zeta_{13}+\eta_{12}\zeta_{13}\zeta_{23}+\eta_{13}\zeta_{12}\zeta_{32})-2\zeta_{12}\zeta_{23}\zeta_{31}]\chi_{123}\frac{1}{\mathcal{V}^{3/2}}\\ =&\frac{1}{\mathcal{V}^{3/2}}\psi_{123}\chi_{123},\end{aligned} \tag{16}$$

where $\eta_{ij}(r_{ij})$ is the on-shell defect wave function for the two-nucleon system, $\zeta_{ij}(r_{ij})$ is the corresponding off-shell wave function, χ_{123} is the spin-isospin wave function, and $\mathcal{V}$ is the normalization volume. In deriving Eq. (16) the approximation of neglecting the momentum of the hole states is made, so that the space part of the uncorrelated wave function, $\exp i(\vec{k}_1\cdot\vec{x}_1+\vec{k}_2\cdot\vec{x}_2+\vec{k}_3\cdot\vec{x}_3)$, is replaced by unity.

Figure 6 shows the form of the space wave function ψ_{123} and the corresponding density $|\psi_{123}|^2$ for various configurations. It was calculated using quadratic approximations to the defect wave functions with healing at $2.2c$ for η and $1.9c$ for ζ. Since these are not very different, it aids visualization of the results to consider the further approximation $\eta=\zeta$, when Day's ansatz for ψ reduces to that proposed by Moszkowski,[12]

$$\psi_D \approx \psi_M = (1-\eta_{12})(1-\eta_{23})(1-\eta_{31}). \tag{17}$$

Both ψ_D and ψ_M vanish exactly when *any* r_{ij} is less than the core radius c, and both become unity when *all* r_{ij} exceed the healing distance $2.2c$. From the graphs of Fig. 6, and from the approximation of Eq. (17), we also see that when one or more of the r_{ij} takes on an intermediate value, ψ is in general small.

Thus we can approximate ψ by assigning the plane-wave value to it when *all* r_{ij} exceed a cutoff which we somewhat arbitrarily set at 0.9 fm, and setting $\psi=0$ otherwise. Since it will be seen that $|\psi|^2$ always appears in the expressions we write down for E_3, the cutoff approximation should be quite good. We shall see that E_3 estimated in the cutoff approximation will turn out to be small. Only those interested in a precise value of this small energy need evaluate the integrals in Eqs. (27) and (32) with the full expression (16) for Ψ. It should be noted that the fact that Ψ vanishes identically when $r_{ij}<c$ removes the difficulties associated with contact interactions discussed by Bhaduri, Loiseau, and Nogami.[5]

Our discussion of Ψ has concentrated on the correlations induced by the short-range part of the potential. As shown by one of us,[16] the long-range part may be taken into account perturbatively.

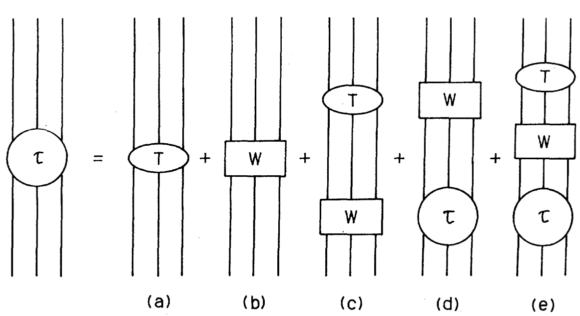

FIG. 5. Diagrammatic interpretation of Eq. (9).

4. PION-EXCHANGE THREE-BODY POTENTIAL

The three-body potential we will use consists of the three terms of Fig. 1. It obviously suffices to consider just one term, say W_2 of Figs. 1(b) and 7. This potential has the isospin structure

$$W_2 \sim \tau_i^{(1)}[\delta_{ij}f^{(+)}+\epsilon_{ijk}\tau_k^{(2)}f^{(-)}]\tau_j^{(3)}, \tag{18}$$

where $f^{(+)}$ and $f^{(-)}$ are related to the parts of the πN scattering amplitude which are symmetric and antisymmetric in isospin. Since the Day wave function Ψ has the same isospin structure as the uncorrelated wave function Φ, the arguments used previously[4-6] will apply. We find that only the isospin-symmetric part $f^{(+)}$ contributes in first order, and it also dominates in second order.

This part of the potential has been examined in detail by Brown and Green[4] and Nogami and his

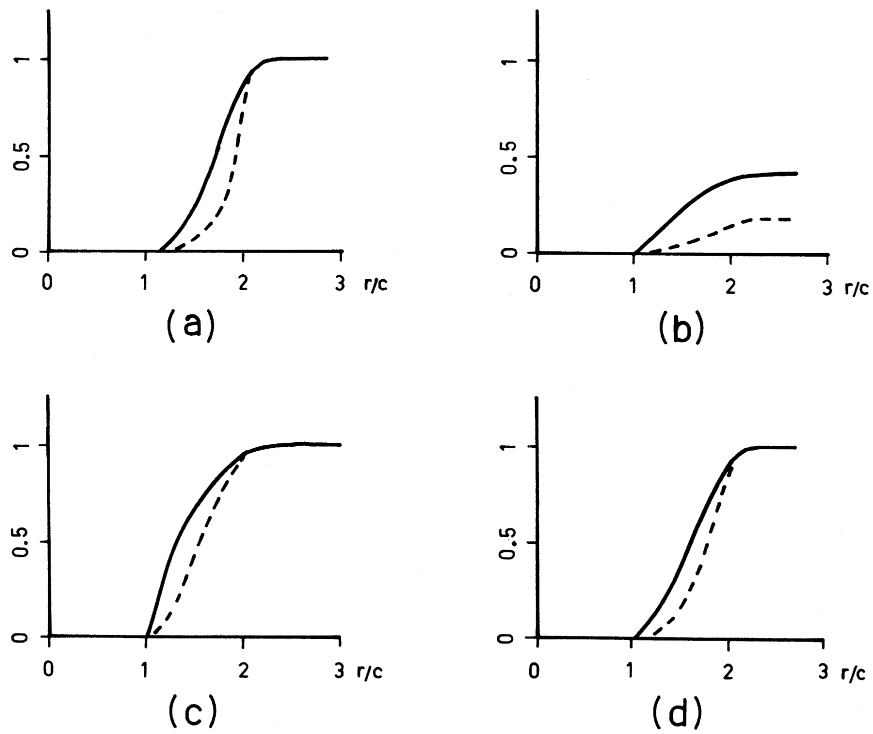

FIG. 6. The spatial part of three-body wave function in Day's approximation. The solid line represents ψ and the dashed line $|\psi|^2$. (a) $r_{12}=r_{23}=r_{31}=r$; (b) $r_{12}=r_{23}=1.5c$, $r_{31}=r$; (c) $r_{12}=r_{23}=2.2c$, $r_{31}=r$; (d) $r_{12}=r_{23}=r$, $r_{31}=2c$.

colleagues.[5,6] It depends on how one chooses to extrapolate the πN amplitude and the vertex function and propagator corrections, off the pion energy shell $k^2=q^2=\mu^2$. We do not add to this discussion of how to extrapolate, but simply use the method adopted by Brown and Green and Bhaduri, Nogami, and Ross, which seems to us the best available. Our method is readily adapted to any other choice.

Bhaduri *et al.* write the isosymmetric πN amplitude with $q_0=k_0\simeq 0$ as

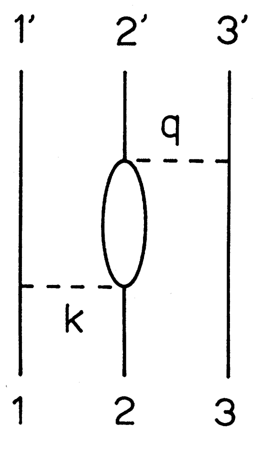

FIG. 7. The pion-exchange three-body potential W_2.

$$T^{(+)}=2\pi i\delta(0)[2(A+B)\vec{q}\cdot\vec{k}+2D]. \qquad (19)$$

This amplitude contains only p and s waves, and is indeed dominated by the p-wave terms. As is well known, s-wave πN scattering is small, so we henceforth neglect D, the s-wave part of the amplitude. From this one derives the three-body potential

$$W_2=\frac{-f_r^{\,2}}{\mu^2}\,\vec{\tau}_1\cdot\vec{\tau}_3\vec{\sigma}_1\cdot\vec{k}\frac{H(k^2)}{k^2+\mu^2}2(A+B)\vec{q}\cdot\vec{k}\frac{H(q^2)}{q^2+\mu^2}\vec{\sigma}_3\cdot\vec{q}, \qquad (20)$$

or in coordinate space

$$W_2(\vec{r}_{12}\vec{r}_{23})=\frac{-f_r^{\,2}}{(4\pi)^2\mu^2}\,\vec{\tau}_1\cdot\vec{\tau}_3\vec{\sigma}_1\cdot\nabla_{12}H(\vec{\nabla}_{12}^2)2(A+B)\times\vec{\nabla}_{12}\cdot\vec{\nabla}_{23}\vec{\sigma}_3\cdot\vec{\nabla}_{23}H(\nabla_{23}^2)\,e\frac{-\mu(r_{12}+r_{23})}{r_{12}+r_{23}}. \qquad (21)$$

Here μ is the pion mass, f_r is the pseudovector πN coupling constant ($f_r^{\,2}/4\pi=0.08$), and $H(k^2)$ takes into account the momentum dependence introduced by the extrapolation technique. For the purposes of illustration in the following we set $H(k^2)=1$, to keep the algebra simple. The numerical results we quote use the extrapolation method of Brown and Green.

Notice that, while we have neglected the spin-isospin dependence of the correlations in the wave function, we have included the full spin-isospin structure of the three-body potential.

5. TERM OF FIRST ORDER IN W

To first order in W, we simply replace Ω by Ψ, and we have

$$E_W^{(1)} = 3\sum{}'' \langle \Psi'_{123} | W_2 | \Psi_{123} \rangle , \tag{22}$$

each W_i contributing equally. Since our correlated three-body wave function has the same spin-isospin structure as the uncorrelated wave function, the analysis of Refs. 4 to 7 can be applied to show that

(i) the direct term $\langle \Psi_{123} | W_2 | \Psi_{123} \rangle$ vanishes,

(ii) the only single-exchange term which is nonzero is $\langle \Psi_{321} | W_2 | \Psi_{123} \rangle$, and

(iii) while double-exchange terms like $\langle \Psi_{321} | W_2 | \Psi_{123} \rangle$ are nonzero, they are much smaller than the single-exchange term. We therefore wish to evaluate

$$E_W^{(1)} = -3 \sum_{123}{}' \langle \Psi_{321} | W_2 | \Psi_{123} \rangle . \tag{23}$$

Using the fact that we are neglecting the momenta of the unoccupied states, and that Ψ_{123} is therefore independent of these momenta, we can sum over $\vec{k}_1, \vec{k}_2, \vec{k}_3$, obtaining a factor $\frac{1}{6}(N/4)^3$, and we have

$$E_W^{(1)} = -\frac{1}{6}\left(\frac{N}{4}\right)^3 3 \sum_{\substack{\text{spin}\\ \text{isospin}}} \langle \Psi_{321} | W_2 | \Psi_{123} \rangle . \tag{24}$$

The explicit forms of Ψ and W, discussed above, show that the space dependence of each is entirely in the interparticle distances r_{12}, r_{23}, and r_{31}, so we can immediately do the angle integration using

$$\int d\vec{r}_1 \int d\vec{r}_2 \int d\vec{r}_3 = \mathcal{V} 8\pi^2 \int_0^\infty r_{12}\, dr_{12} \int_0^\infty r_{23}\, dr_{23} \times \int_{|r_{12}-r_{23}|}^{r_{12}+r_{23}} r_{31}\, dr_{31} . \tag{25}$$

Furthermore, the space parts of Ψ_{123} and Ψ_{321} are identical, so we have

$$E_W^{(1)} = -\frac{3}{6}\left(\frac{N}{4}\right)^3 \frac{\mathcal{V} 8\pi^2}{\mathcal{V}^3} \int_0^\infty r_{12}\, dr_{12} \int_0^\infty r_{23}\, dr_{23} \int_{|r_{12}-r_{23}|}^{r_{12}+r_{23}} r_{31}\, dr_{31} |\psi_{123}(r_{12}, r_{23}, r_{31})|^2 \sum_{\substack{\text{spin}\\ \text{isospin}}} \langle \chi_{321} | W_2(r_{12}, r_{23}, r_{31}) | \chi_{123} \rangle . \tag{26}$$

Notice that this gives $E_W^{(1)}/N \propto \rho^2$ where ρ is the density, which is what one would expect for the density dependence of a three-body energy. The explicit form of ψ is given in Eq. (16), and that of W_2 in Eq. (21). We can now do the spin-isospin sum explicitly, and setting $H=1$ for simplicity, we obtain

$$\frac{E_W^{(1)}}{N} = \frac{3}{6}\frac{\rho^2}{4^3}\frac{(2A+2B)}{(4\pi)^2\mu^2} f_r^{\,2} \times 24 \times 8\pi^2 \int_0^\infty dx \int_0^\infty dy \int_{|x-y|}^{x+y} dz\, xyz \left|\psi_{123}\left(\frac{x}{\mu}, \frac{y}{\mu}, \frac{z}{\mu}\right)\right|^2 \frac{2}{3}\left[3(x^2+y^2-z^2)^2 T(x)T(y) + Y(x)Y(y)\right], \tag{27}$$

where

$$x = \mu r_{12}, \quad y = \mu r_{23}, \quad z = \mu r_{13},$$
$$2A + 2B \approx 1/\mu^3,$$
$$T(x) = h_2(ix) = \left(1 + \frac{3}{x} + \frac{3}{x^2}\right)\frac{e^{-x}}{x},$$
$$Y(x) = h_0(ix) = e^{-x}/x . \tag{28}$$

This is the expression for the first-order contribution of the three-body force W to the binding energy of nuclear matter, taking correlations induced by the two-body force correctly into account. Our discussion of the properties of ψ in Sec. 3 suggests that a reliable estimate of the integral in Eq. (27) may be made by replacing $|\psi|^2$ by unity when all r_{ij} exceed 0.9 fm, and by zero otherwise. Calculations using this cutoff wave function, which they introduced in the hope of simulating the effect of correlations, have been made by Bhaduri, Nogami, and Ross,[6] in the case when $H=1$. They find that $E_W^{(1)}/N$ is 0.81 MeV for a cutoff at 1 fm. Calculations with the exact wave function (16) are unlikely to yield a significantly different result.

The extension to the more realistic case involving form factors $H(k^2)$ and $H(q^2)$ simply introduces algebraic complications. The calculation, with a cutoff wave function, has been performed by LNR.[7]

This method of evaluating $E_W^{(1)}$ differs from the effective-mass method of Brown and Green,[4] who first sum over the nucleon which scatters the pion to obtain an effective two-body interaction between the other two. This effective potential, added to the usual one-pion-exchange potential (OPEP), gives an OPEP type of potential with a renormalized pion mass. In the presence of correlation it is no longer possible to sum over the "middle" nucleon independently, so that the effective-mass technique breaks down. However, it is, of course, possible to perform the triple sum in Eq. (23) by summing over "middle" nucleon first. As pointed out by LNR,[7] this procedure defines an effective two-particle potential by

$$\langle\Psi_{13}|V_{\rm eff}|\Psi_{13}\rangle = -\sum_{2}\langle\Psi_{321}|W_2|\Psi_{123}\rangle . \tag{29}$$

This effective potential is simply a trivial definition if we are only interested in the first-order term, but for the second-order term it is an essential step in obtaining a tractable calculation.

6. TERM OF SECOND ORDER IN W

From Eqs. (9) and (12) we can expand the wave function Ω in powers of W,

$$\Omega = \Psi - \frac{Q}{e}\left(1 - T\frac{Q}{e}\right)W\Psi + \cdots , \tag{30}$$

and thus find that the second-order contribution of W to the energy is

$$E_W^{(2)} = -\sum_{123}{}^{\prime\prime}\left\langle\Psi_{123}'\left|W\frac{Q}{e}\left(1 - T\frac{Q}{e}\right)W\right|\Psi_{123}\right\rangle . \tag{31}$$

This includes the effect of correlations in the initial, final, and intermediate states. Introducing a complete set of uncorrelated states $\Phi_{\alpha\beta\gamma}$, we can write $E_W^{(2)}$ as

$$E_W^{(2)} = -\sum_{123}{}^{\prime\prime}\sum_{\alpha\beta\gamma}\left\langle\Psi_{123}'\left|W\frac{Q}{e}\right|\Psi_{\alpha\beta\gamma}\right\rangle\langle\Phi_{\alpha\beta\gamma}|W|\Psi_{123}\rangle . \tag{32}$$

The correlated wave functions $\Psi_{\alpha\beta\gamma}$ which appear in the expansion have arbitrary momenta, and do not necessarily have the properties of the wave function for particles inside the Fermi sea discussed in Sec. 3. We could, of course, adapt Day's method to the high-momentum states.

However, when cut off by the correlations in Ψ_{123}, W is a purely long-range force which will not give rise to large momentum transfers.[16] The tensor nature of W which excites relative s states to relative d states increases the momentum transfer somewhat. But even then Brown and Green show that the momentum transfer induced by W is typically of the order of $\vec{q}^2 \approx 7\mu^2$, or $|\vec{q}| \approx 1.3k_F$. Their calculation was for W cut off only in one radial variable. A triple cutoff is likely to make the momentum transfers even smaller.

Thus, only states $\Phi_{\alpha\beta\gamma}$ just outside the Fermi sea will be excited by W to any significant extent, and for these states the correlated wave function $\Psi_{\alpha\beta\gamma}$ will not differ greatly from the Day solution. If we now go to the cutoff approximation for $\Psi_{\alpha\beta\gamma}$, Eq. (32) simplifies greatly because the cutoff is

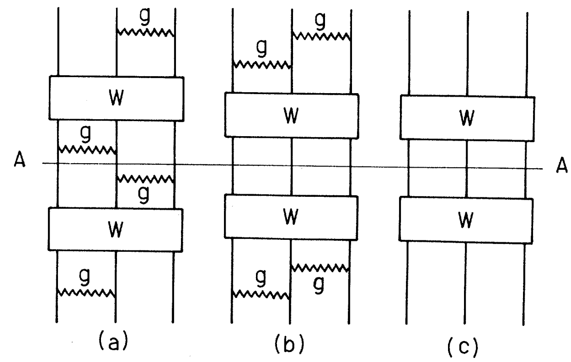

FIG. 8. Some diagrams of second order in W.

imposed by the correlated final state Ψ_{123}, and we can therefore set $\Psi_{\alpha\beta\gamma} = \Phi_{\alpha\beta\gamma}$ in the intermediate state and use closure.

Thus we have

$$E_W^{(2)} \approx -\sum_{123}{}^{\prime\prime}\left\langle\Psi_{123}'\left|W\frac{Q}{e}W\right|\Psi_{123}\right\rangle . \tag{33}$$

In other words, we can neglect the correlations in the intermediate states, and take them into account in the initial and final states in a cutoff approximation.

The physical basis of this simplification from Eqs. (31) to (32) is clear. The hard cores act to keep the three nuclei apart, and at these distances W is a rather weak force so it does not alter the configuration very much. At these large separations the strong short-range nucleon-nucleon interaction has little effect, and cannot build additional correlations into the intermediate state. Diagrammatically, this says that in a typical diagram of second order in W, such as that of Fig. 8(a), the nucleons are sufficiently far apart at the level A so that the two-body interactions at this level are weak. Thus the only significant diagrams are those like Fig. 8(b) with no interactions at this level. Furthermore, the correlations built up by the two-body interactions before and after the three-body interactions are such that they may be taken into account approximately by evaluating the diagram of Fig. 8(c) with a cutoff when any r_{ij} is less than 0.9 fm.

It is clear from the diagrammatic interpretation of the argument leading to Eq. (33) that a similar result will be obtained for the higher-order terms – correlations may be ignored in all intermediate states and a cutoff introduced in the initial and final states. The long-range part of W is quite weak, so that it suffices to retain only the first- and second-order terms, and even the second-order term turns out to be about 10% of the first.

To calculate the second-order energy we choose correlated wave functions as intermediate states in Eq. (33) which then reads

$$E_W^{(2)} \approx -\sum_{123}'' \sum_{\alpha\beta\gamma} \langle \Psi'_{123} | W | \Psi_{\alpha\beta\gamma} \rangle \frac{Q}{e} \langle \Psi_{\alpha\beta\gamma} | W | \Psi_{123} \rangle . \tag{34}$$

The numerical evaluation of Eq. (34) would be time consuming, even in the cutoff approximation. However, we may estimate it using the effective potential and writing

$$E_W^{(2)} \approx -\sum_{13} \sum_{\alpha\gamma} \langle \Psi'_{13} | V_{\rm eff} | \Psi_{\alpha\gamma} \rangle \frac{Q}{e} \langle \Psi_{\alpha\gamma} | V_{\rm eff} | \Psi_{13} \rangle , \tag{35}$$

which is much simpler to compute.

The second-order energy has been obtained in this way by LNR and is found to be about 10% of the first-order term when the cutoff wave function is employed. We appeal to the smallness of the result to justify the rather cavalier approximation in going from Eqs. (34) to (35).

7. TERMS INVOLVING THREE-BODY AND LONG-RANGE TWO-BODY INTERACTIONS

Our analysis in the previous section took into account an arbitrary number of short-range two-body interactions of the type shown in Fig. 2 by introducing the correlations they build into the wave function. We have yet to discuss the effects of a number of long-range two-body interactions v_l in conjunction with W and short-range interactions g_s, for example the graphs of Fig. 9.

As one of us has shown previously,[16] v_l may be treated perturbatively. It may thus be introduced into the formalism of the previous section by the replacement.

$$W(1,2,3) \to W(1,2,3) + v_l(12) + v_l(23) + v_l(31) .$$

Then all our arguments in that section still apply, since, as shown in Ref. 16, v_l does not excite states far outside the Fermi sea and therefore does not upset the short-range correlations. Thus the important terms involving v_l are those of the type illustrated in Fig. 9(b). They have one long-range interaction v_l and one three-body interaction W. The contribution to the energy is

$$E_W = -\sum_{123}'' \left\langle \Psi'_{123} \left| \left(W \frac{Q}{e} v_l + v_l \frac{Q}{e} W \right) \right| \Psi_{123} \right\rangle , \tag{36}$$

where the intermediate states may be taken as correlated or uncorrelated as is convenient. E_{Wv} in the cutoff approximation is represented diagrammatically in Fig. 10, where W_2 is considered for definiteness. In diagram (a), particle 2 is a spectator in the long-range interaction and can be immediately summed over to give

$$E_W^{(a)} = -\sum_{13}' \left\langle \Psi_{13} \left| V_{\rm eff} \frac{Q}{e} v_l + v_l \frac{Q}{e} V_{\rm eff} \right| \Psi_{13} \right\rangle . \tag{37}$$

Diagram (b) is not so simply evaluated. However, when v_l is taken to be the OPEP potential the spin-isospin couplings force it to contribute only to exchange terms as indicated in the figure. As has been discussed in the literature,[17] such exchange terms are small for potentials of the OPEP type compared with the direct terms and we will not consider them further.

E_W is now expressed in the usual second-order perturbation-theory form and can be readily computed. This has been done in the cutoff approximation by LNR, using the OPEP form for v_l, and they find that it gives the dominant contribution to the total energy E_W.

8. DISCUSSION

Our major result, that the effects of correlations may be approximated using a suitably cutoff wave function, has long been suspected. LNR[7]

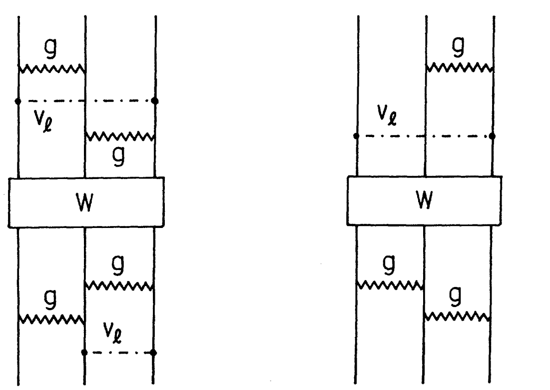

FIG. 9. Some diagrams involving the three-body force and the long-range part of the two-body force.

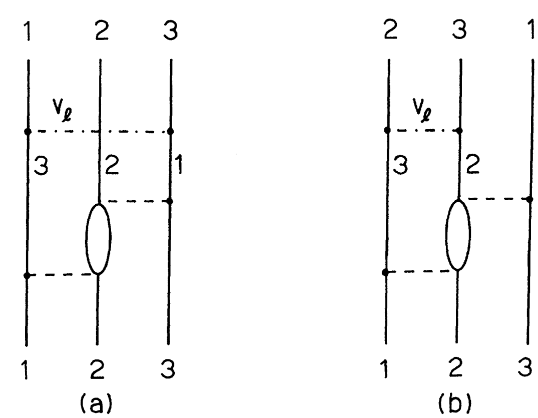

FIG. 10. Diagrams contributing to E_{Wv}.

TABLE I. Contribution of W to the energy of nuclear matter as calculated by Loiseau, Nogami, and Ross (Ref. 7). Energies are given in MeV.

Cutoff	0.8 fm	1 fm
$E_W^{(1)}$	0.78	0.54
$E_W^{(2)}$	−0.06	−0.03
E_{WV}	−1.86	−1.03
E_W (total)	−1.14	−0.52

have recently evaluated the first- and second-order terms with a triple cutoff of the type we recommend. In Table I, we quote their results for cutoffs of 0.8 and 1 fm, using the off-pion-mass-shell extrapolation of Brown and Green. It will be seen that the results are sensitive to the cutoff, and, as shown in Ref. 7, they are also dependent on the extrapolation procedure used. They are also dependent on the replacement of v_t by the OPEP form, which may be questioned at separation of 1 fm.

Because of these uncertainties, most of which are independent of the influence of correlations, we did not proceed to an evaluation of their effects more exactly. Rather, we have justified the method used on an *ad hoc* basis by LNR. We show that the cutoff they use should be the same in all three interparticle distances, and that the larger cutoffs, at about 0.9 fm, are to be preferred.

On this basis, remembering that the correlations damp out contributions from heavier-meson exchange, we conclude that three-body forces contribute about 1 MeV to the binding energy of nuclear matter. This confirms the popular superstition that the effects of three-body forces are small.

Our results may be regarded as the three-body analog of the Moszkowski-Scott separation method.[15] Moszkowski and Scott showed that one could take the effect of two-body correlations approximately into account in the first-order expression for the two-body energy of nuclear matter by taking the matrix element of the potential between plane-wave states cut off when $r<d$. We find that the effect of short-range two-body correlations on the contribution of the three-body potential W to the binding energy of nuclear matter may be taken into account by calculating the various terms in an expansion in powers of W using plane-wave states cut off when any interparticle distance is less than 0.9 fm.

With the literature, our work completes the estimation of *all* two- and three-body contributions to the binding energy of nuclear matter. To recapitulate, we have shown how to take terms involving both two- and three-body interactions into account. We then found that a significant effect of the two-body forces is to damp out terms in the pion-exchange three-body force of higher order than the first, and three-body forces involving heavier mesons. This leaves the contribution from the extensively studied pion-exchange three-body force to be added to the binding energy from two-body forces.

The contribution of two-body forces alone to the two- and three-body energy in nuclear matter is currently estimated to be about 13 MeV.[18] With the contribution of 1 MeV obtained above on the basis of the calculations of LNR, we get a total binding energy of about 14 MeV from two- and three-body effects.

Day[19] estimates that four-body clusters contribute an extra energy of about 1 MeV also. This gives a current total of 15 MeV. We seem to have exhausted the physical mechanisms which may contribute to the binding energy, and it is therefore satisfying that our result is so close to the experimental values of 16 MeV—especially so when one recalls the many approximations involved in arriving at this result.

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Neutron-Proton Scattering Cross Section at a Few Electron Volts and Charge Independence*

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We have measured the neutron-proton total cross section in hydrogen gas at 0°C between 0.3- and 400-eV lab energy to help improve the data for computing the n-p singlet effective range r_s. To obtain the free n-p scattering cross section at "zero" energy, we subtract the capture-cross-section, effective-range, and molecular-binding contributions from the total cross section. We used the asymptotic "MTV" theory of molecular binding due to Messiah, which limits our final analysis to the range 6.02 to 329 eV. From these data we obtain 20.436 ± 0.023 b for the free n-p scattering cross section. Using this number, the coherent-scattering-length measurements by Koester, a model due to Lomon and Feshbach for the triplet and singlet shape parameters, and the n-p cross section in the MeV range due to Engelke *et al.*, we obtain $r_s = 2.56 \pm 0.05$ F, 0.09 F higher than the former value. Adopting an energy scale due to Davis and Barschall for the higher-energy data raises r_s to 2.74 F, which is in close agreement with the value 2.70 F predicted by the charge-independence hypothesis. A measurement of the neutron-carbon cross section, formerly related to this problem, is also discussed.

I. INTRODUCTION

The data available in 1962[1] on the neutron-proton interaction indicated that the n-p singlet effective range r_s was 2.46 ± 0.11 F (1 F $= 10^{-15}$ m), compared with $r_{pp} = 2.71 \pm 0.011$ F for the proton-proton (singlet only) effective range. This is a greater violation of charge independence in nucleon-nucleon scattering than the one-boson-exchange studies current then or now[2] could account for, even considering the known $\pi^{\pm}$-π^0 mass difference. Two possible problems with the data were noted. First, data at 0.4926 and 3.205 MeV taken by Engelke *et al.*[3] were the chief source of a low value of r_s. Eliminating these points places r_s at 2.64 ± 0.12 F. But Engelke's data are too precise to be cast aside easily. Second, the low-energy data, particularly the n-p incoherent scattering cross section σ_0, were by then dominating the uncertainty in r_s. Noyes,[4] who investigated the problem in detail, suggested that new measurements of σ_0 and its companion quantity a_H, the hydrogen coherent scattering length, were needed before the high-energy data were worth remeasuring. Breit and colleagues,[5] made a very searching study and reached a similar conclusion.

The incoherent cross section was last measured precisely by Melkonian[6] in 1949. His work was afflicted by rate-dependent efficiency in his BF_3 neutron proportional counters. He used a fairly large counter bank with a consequently large correction for detector geometry. He determined his target density by weighings, with no mention of the effect of elastic distortion in his target vessel. We now have the superior He^3 neutron proportional counter at our disposal, and more intense neutron beams to permit using smaller detector banks. We also have improved data on the hydrogen equation of state, and excellent instruments and standards for assessing pressure, temperature, and length unavailable in 1949. Finally, the Harvard University cyclotron is a copious source of neutrons of high to eV energy. This suggested that we might undertake a remeasurement of σ_0 ourselves.