Application of the Marumori Boson Expansion to the Problem of Particle-Hole Excitation in Closed-Shell Nuclei*

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Marumori's boson-expansion method is applied to the problem of particle-hole excitation in closed-shell nuclei. It is shown that consistent (and in principle, straightforward) calculations yield correct results for the ground-state correlation energy, occupation probabilities, and the lowest-order random-phase approximation. A method introducing higher-order random-phase approximations based on the tranformation of Rowe's formulation in particle-hole space is described.

I. INTRODUCTION

Recently, da Providencia and Weneser[1] have discussed the overestimation of ground-state correlations by the conventional random-phase approximation (RPA).[2,3] After fairly elaborate discussion which brings in higher-order terms of the Beliaev-Zelevinsky boson expansions,[4] they suggest a procedure which yields the various physical quantities correctly in the lowest order. The modified formalism is, however, no longer recognizable as the conventional RPA, yielding *inter alia* slightly different equations of motion. Furthermore, the explicit generalization to higher order is by no means evident.

It is our purpose to show that by using Marumori's boson-expansion method,[5] one can reproduce the RPA equations, and at the same time obtain correct ground-state correlations. Moreover, the method can be used to define in a consistent way the higher-order RPA. The reason one can achieve all this is that the Marumori method transcribes physical quantities in the fermion space correctly into the boson space, taking account of the Pauli restrictions to all orders.

We shall consider a general shell-model Hamiltonian in the next section. Section III gives a brief review of Marumori's method, with a perturbation calculation showing that the transcribed Hamiltonian gives correct ground-state correlations. In Sec. IV, the RPA equations are derived from this Hamiltonian. Finally, we discuss in Sec. V how one can define a higher-order RPA in the boson space with Marumori's expansions.

II. HAMILTONIAN

The Hamiltonian is taken to have a general form

$$H=\sum_{ab}h_{ab}a_a^\dagger a_b+\tfrac{1}{4}\sum_{abcd}V_{abcd}a_a^\dagger a_b^\dagger a_d a_c\,, \tag{1}$$

where h_{ab} are matrix elements of a one-body operator, and V_{abcd} are antisymmetrized matrix elements of two-body interactions.

We now assume a Hartree-Fock-like decomposition which divides the single-particle states into an occupied set $a_\alpha^\dagger$, denoted by Greek subscripts, and an unoccupied set $a_m^\dagger$ with Latin subscripts. The Hamiltonian (1) then becomes

$$H=E_{\rm HF}+H_{11}+H_{22}+(H_{40}+{\rm H.c.})+H'_{22}+(H_{31}+{\rm H.c.})\,, \tag{2a}$$

where

$$E_{\rm HF}=\langle{\rm HF}|H|{\rm HF}\rangle=\sum_\alpha h_{\alpha\alpha}+\tfrac{1}{2}\sum_{\alpha\beta}V_{\alpha\beta\alpha\beta}\,, \tag{2b}$$

$$H_{11}=-\sum_\alpha\epsilon_\alpha a_\alpha a_\alpha^\dagger+\sum_m\epsilon_m a_m^\dagger a_m\,, \tag{2c}$$

$$H_{22}=\sum_{\alpha\beta mn}V_{m\beta\alpha n}a_m^\dagger a_\alpha a_\beta^\dagger a_n\,, \tag{2d}$$

$$H_{40}=\tfrac{1}{4}\sum_{\alpha\beta mn}V_{mn\alpha\beta}a_m^\dagger a_\alpha a_n^\dagger a_\beta\,, \tag{2e}$$

$$H'_{22}=\tfrac{1}{4}\sum_{mnpq}V_{mnpq}a_m^\dagger a_n^\dagger a_q a_p+\tfrac{1}{4}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}a_\delta a_\gamma a_\alpha^\dagger a_\beta^\dagger\,, \tag{2f}$$

and

$$H_{31}=\tfrac{1}{2}\sum_{mnp\alpha}V_{mn\alpha p}a_m^\dagger a_\alpha a_n^\dagger a_p+\tfrac{1}{2}\sum_{m\alpha\beta\gamma}V_{m\alpha\beta\gamma}a_m^\dagger a_\gamma a_\beta a_\alpha^\dagger\,. \tag{2g}$$

We quote here for later reference the second-order perturbation formula for the ground-state correlation energy:

$$E_{\rm corr}^{(2)}=\frac{1}{4}\sum_{\alpha\beta mn}\frac{V_{\alpha\beta mn}V_{mn\alpha\beta}}{\epsilon_\alpha+\epsilon_\beta-\epsilon_m-\epsilon_n}\,, \tag{3}$$

and for the occupation number in the ground state,

$$\langle a_p^\dagger a_p\rangle_{\rm corr}^{(2)}\equiv\langle\psi^{(1)}|a_p^\dagger a_p|\psi^{(1)}\rangle=\frac{1}{2}\sum_{\alpha\beta m}\frac{V_{\alpha\beta,mp}V_{mp,\alpha\beta}}{(\epsilon_\alpha+\epsilon_\beta-\epsilon_m-\epsilon_p)^2}\,, \tag{4}$$

with

$$|\psi^{(1)}\rangle=\sum_I\frac{1}{E_{\rm HF}-E_I}|I\rangle\langle I|H_{40}|{\rm HF}\rangle\,. \tag{5}$$

We shall need these results later to compare with those calculated from the boson Hamiltonian.

III. BOSON-EXPANSION METHOD

In the space of N nucleons, a complete orthonormal basis can be designated as $\{|Np\rangle_F\}$, where p labels the fermion states. Also we consider a boson space with n bosons, where a complete orthonormal basis will be $\{|nk)_B\}$. Here k labels the boson states.

In the method of Marumori, a subset $\{|np)_B\}$ of the entire n-boson space $\{|nk)_B\}$ is chosen as the image of the fermion space $\{|2n, p\rangle_F\}$. Once this choice is made, a transformation V can be introduced

$$V \equiv \sum_{np} |\tfrac{1}{2}N, p)_{B\,F}\langle Np| , \tag{6}$$

which transforms any fermion operator T into the boson space:

$$\begin{aligned} T_B &\equiv VTV^\dagger \\ &= \sum_{NN'pp'} |\tfrac{1}{2}Np)_{B\,F}\langle Np|T|N'p'\rangle_{F\,B}(\tfrac{1}{2}N', p'| \\ &\equiv \sum_{NN'pp'} T_{Np,N'p'}|\tfrac{1}{2}N, p)_{B\,B}(\tfrac{1}{2}N', p'| . \end{aligned} \tag{7}$$

We remark that V is not unitary, because

$$V^\dagger V = \sum_{Np} |Np\rangle_{F\,F}\langle Np| = 1 , \tag{8}$$

but

$$VV^\dagger = \sum_{Np} |\tfrac{1}{2}N, p)_{B\,B}(\tfrac{1}{2}N, p| , \tag{9}$$

where the right-hand side is not a sum over a complete set of states and consequently is not the identity operator.

If we write the boson states as

$$|\tfrac{1}{2}N, k)_B \equiv \tilde{O}^\dagger_{\frac{1}{2}N,k}|0)_B , \tag{10}$$

where $|0)_B$ is the boson vacuum state, and $\tilde{O}^\dagger_{\frac{1}{2}N,k}$ is a polynomial of boson creation operators, then (7) becomes

$$T_B = \sum_{NN'pp'} T_{Np,N'p'}\tilde{O}^\dagger_{\frac{1}{2}N,p}|0)_{B\,B}(0|\tilde{O}_{\frac{1}{2}N',p'} . \tag{11}$$

The projection operator $|0)_{B\,B}(0|$ can be solved in terms of boson operators from the completeness relation

$$\begin{aligned} 1 &= \sum_{Nk} |\tfrac{1}{2}N, k)_{B\,B}(\tfrac{1}{2}N, k| \\ &= \sum_{Nk} \tilde{O}^\dagger_{\frac{1}{2}N,k}|0)_{B\,B}(0|\tilde{O}_{\frac{1}{2}N,k} \end{aligned} \tag{12}$$

by the method of iteration. Substituting the result back into (11), one thus obtains the boson expansion T_B for any fermion operator T.

In the following we will proceed to construct boson expansions for the fermion operators $a^\dagger_m a_\alpha$, $a^\dagger_m a_{m'}$, and $a^\dagger_\alpha a_{\alpha'}$ for a system with an even number of nucleons.

A. Expansions of Fermion Pairs

We define the fermion-pair operators

$$B^\dagger_{m\alpha} \equiv a^\dagger_m a_\alpha , \tag{13a}$$

$$B_{m\alpha} = (B^\dagger_{m\alpha})^\dagger = a^\dagger_\alpha a_m , \tag{13b}$$

$$N_{mm'} \equiv a^\dagger_m a_{m'} \tag{13c}$$

$$N^\dagger_{mm'} = a^\dagger_{m'} a_m = N_{m'm} , \tag{13d}$$

$$\bar{N}_{\alpha\alpha'} \equiv a_\alpha a^\dagger_{\alpha'} , \tag{13e}$$

and

$$\bar{N}^\dagger_{\alpha\alpha'} = N_{\alpha'\alpha} , \tag{13f}$$

which satisfy the commutation relations

$$[B^\dagger_{m_1\alpha_1}, B^\dagger_{m_2\alpha_2}] = 0 , \tag{14a}$$

$$[B_{m_1\alpha_1}, B^\dagger_{m_2\alpha_2}] = \delta_{m_1m_2}\delta_{\alpha_1\alpha_2} - \delta_{m_1m_2}\bar{N}_{\alpha_2\alpha_1} - \delta_{\alpha_1\alpha_2}N_{m_2m_1} , \tag{14b}$$

$$[N_{m_1m_2}, B^\dagger_{m_3\alpha_3}] = \delta_{m_2m_3}B^\dagger_{m_1\alpha_3} , \tag{14c}$$

$$[\bar{N}_{\alpha_1\alpha_2}, B^\dagger_{m_3\alpha_3}] = \delta_{\alpha_2\alpha_3}B^\dagger_{m_3\alpha_1} . \tag{14d}$$

Next, an orthonormal fermion basis is given as:

$$\begin{aligned} &|0\rangle \equiv |\mathrm{HF}\rangle , \\ &|m\alpha\rangle \equiv B^\dagger_{m\alpha}|0\rangle , \\ &|mm'\alpha\alpha'\rangle \equiv B^\dagger_{m\alpha}B^\dagger_{m'\alpha'}|0\rangle , \\ &\quad\cdots \\ &|m_1m_2\cdots m_N\alpha_1\alpha_2\cdots\alpha_N\rangle \equiv \prod_{i=1}^{N}(B^\dagger_{m_i\alpha_i})|0\rangle . \end{aligned} \tag{15}$$

In the boson space, the boson creation operator $\mathring{B}^\dagger_{m\alpha}$ and its Hermitian conjugate $\mathring{B}_{m\alpha}$ have the properties

$$[\mathring{B}_{m_1\alpha_1}, \mathring{B}^\dagger_{m_2\alpha_2}] = \delta_{m_1m_2}\delta_{\alpha_1\alpha_2} , \tag{16a}$$

$$[\mathring{B}^\dagger_{m_1\alpha_1}, \mathring{B}^\dagger_{m_2\alpha_2}] = 0 , \tag{16b}$$

and

$$\mathring{B}_{m\alpha}|0)_B = 0 , \tag{17}$$

where $|0)_B$ is the boson vacuum state. Among the boson states $|0)_B$, $|m\alpha)_B \equiv \mathring{B}^\dagger_{m\alpha}|0)_B$, $|mm'\alpha\alpha')_B \equiv \mathring{B}^\dagger_{m\alpha}\mathring{B}^\dagger_{m'\alpha'}|0)_B$, etc., the physical boson states are chosen according to Marumori's prescription[5] as:

$$|0)_P \equiv |0)_B ,$$
$$|m\alpha)_P \equiv \mathring{B}^\dagger_{m\alpha}|0)_B ,$$
$$|mm'\alpha\alpha')_P = \frac{1}{\sqrt{2}}(\mathring{B}^\dagger_{m\alpha}\mathring{B}^\dagger_{m'\alpha'} - \mathring{B}^\dagger_{m\alpha'}\mathring{B}^\dagger_{m'\alpha})|0)_B ,$$
$$|m_1m_2\cdots m_N\alpha_1\alpha_2\cdots\alpha_N)_P = \frac{1}{(N!)^{1/2}}\sum_{p_\alpha}(-)^{p_\alpha}p_\alpha(\mathring{B}^\dagger_{m_1\alpha_1}\mathring{B}^\dagger_{m_2\alpha_2}\cdots\mathring{B}^\dagger_{m_N\alpha_N})|0)_B , \tag{18}$$

where p_α is a permutation operator which permutes the indices α.

With the physical boson space thus defined, the expansion of $B^\dagger_{m\alpha}$ can be calculated to the second order as

$$VB^\dagger_{m\alpha}V^\dagger = \sum_{m'\alpha'}|m'\alpha')_P\langle m'\alpha'|B^\dagger_{m\alpha}|0\rangle_B(0| + \tfrac{1}{4}\sum_{m_1m_2\alpha_1\alpha_2}\sum_{m'\alpha'}[|m_1m_2\alpha_1\alpha_2)_P \times\langle m_1m_2\alpha_1\alpha_2|B^\dagger_{m\alpha}|m'\alpha'\rangle_p(m'\alpha'|] + O(5) , \tag{19}$$

where the factor $\frac{1}{4}$ occurs because we have summed over all possible values of m_1, m_2, α_1, and α_2. Using the relation

$$|0)_B\,{}_B(0| = 1 - \sum\mathring{B}^\dagger_{m\alpha}\mathring{B}_{m\alpha} + O(4) , \tag{20}$$

which is derived from (12), the expansion of $B^\dagger_{m\alpha}$ becomes

$$VB^\dagger_{m\alpha}V^\dagger = B^\dagger_{m\alpha} + \left(-1+\frac{1}{\sqrt{2}}\right)\sum_{m'\alpha'}\mathring{B}^\dagger_{m\alpha}\mathring{B}^\dagger_{m'\alpha'}\mathring{B}_{m'\alpha'} - \frac{1}{\sqrt{2}}\sum_{m'\alpha'}\mathring{B}^\dagger_{m\alpha'}\mathring{B}^\dagger_{m'\alpha}\mathring{B}_{m'\alpha'} + O(5) . \tag{21}$$

With similar calculations we obtain the expansions

$$VN_{mm'}V^\dagger = \sum\mathring{B}^\dagger_{m\alpha}\mathring{B}_{m'\alpha} + O(4) , \tag{22}$$

and

$$V\overline{N}_{\alpha\alpha'}V^\dagger = \sum_m\mathring{B}^\dagger_{m\alpha}\mathring{B}_{m\alpha'} + O(4) . \tag{23}$$

B. Expanded Hamiltonian

From (21)–(23) we can write down the expansion for the Hamiltonian (2) up to the second order in boson operators: (We set $E_{\rm HF} = 0$.):

$$H_B \equiv VHV^\dagger = (H_{11})_B + (H_{22})_B + [(H_{40})_B + \text{H.c.}] + (H'_{22})_B + [(H_{31})_B + \text{H.c.}] , \tag{24a}$$

with

$$(H_{11})_B \equiv VH_{11}V^\dagger = \sum_{m\alpha}(\epsilon_m - \epsilon_\alpha)\mathring{B}^\dagger_{m\alpha}\mathring{B}_{m\alpha} + O(4) , \tag{24b}$$

$$(H_{22})_B = VH_{22}V^\dagger = \sum_{\alpha\beta mn}V_{m\beta\alpha n}\mathring{B}^\dagger_{m\alpha}\mathring{B}_{n\beta} + O(4) , \tag{24c}$$

$$(H_{40})_B = VH_{40}V^\dagger = \tfrac{1}{4}\sum_{\alpha\beta mn}V_{mn\alpha\beta}(VB_{m\alpha}V^\dagger)(VB^\dagger_{n\beta}V^\dagger) = \tfrac{1}{4}\sum_{\alpha\beta mn}V_{mn\alpha\beta}\frac{1}{\sqrt{2}}(\mathring{B}^\dagger_{m\alpha}\mathring{B}^\dagger_{n\beta} - \mathring{B}^\dagger_{m\beta}\mathring{B}^\dagger_{n\alpha}) + O(4) , = \frac{1}{2\sqrt{2}}\sum_{\alpha\beta mn}V_{mn\alpha\beta}\mathring{B}^\dagger_{m\alpha}\mathring{B}^\dagger_{n\beta} + O(4) . \tag{24d}$$

$$(H'_{22})_B \equiv VH'_{22}V^\dagger = \tfrac{1}{4}\sum_{mnpq}V_{mnpq}(Va^\dagger_m a^\dagger_n a_q a_p V^\dagger) + \tfrac{1}{4}\sum_{\alpha\beta\gamma\delta}V_{\alpha\beta\gamma\delta}(Va_\delta a_\gamma a^\dagger_\alpha a^\dagger_\beta V^\dagger) = O(4) , \tag{24e}$$

and

$$(H_{31})_B \equiv VH_{31}V^\dagger = O(3) . \tag{24f}$$

The easiest way to verify the last two equations is to start from the definition (7).

Thus, the expanded Hamiltonian becomes

$$H_B = \sum_{m\alpha}(\epsilon_m - \epsilon_\alpha)\mathring{B}^\dagger_{m\alpha}\mathring{B}_{m\alpha} + \sum_{\alpha\beta mn}V_{m\beta\alpha n}\mathring{B}^\dagger_{m\alpha}\mathring{B}_{n\beta} + \frac{1}{2\sqrt{2}}\sum_{\alpha\beta mn}(V_{mn\alpha\beta}\mathring{B}^\dagger_{m\alpha}\mathring{B}^\dagger_{n\beta} + \text{H.c.}) + O(3) , \tag{25}$$

which is different from both the RPA Hamiltonian and the Beliaev-Zelevinsky expanded Hamiltonian [given in (2.9) and (4.6), respectively] in Ref. 1.

C. Perturbation Calculations with the Expanded Hamiltonian

With (25), one can calculate the correlation energy and occupation number in the ground state by the second-order perturbation method which gives

$$(E^{(2)}_{\rm corr})_B=\sum_I\left(0\left|\frac{1}{2\sqrt{2}}\sum V_{\alpha\beta mn}\mathring{B}_{n\beta}\mathring{B}_{m\alpha}\right|I\right)_P\frac{1}{E_{\rm HF}-E_I}\,{}_P\!\left(I\left|\frac{1}{2\sqrt{2}}\sum V_{m'n'\alpha'\beta'}\mathring{B}^\dagger_{m'\alpha'}\mathring{B}^\dagger_{n'\beta'}\right|0\right)$$

$$=\frac{1}{4}\sum_{\alpha\beta mn}\frac{V_{\alpha\beta mn}}{\epsilon_\alpha+\epsilon_\beta-\epsilon_m-\epsilon_n}\,\frac{1}{4}\,{}_P\!\left(mn\alpha\beta\left|\sum_{\alpha'\beta'm'n'}V_{m'n'\alpha'\beta'}\right|m'n'\alpha'\beta'\right)_P$$

$$=\frac{1}{4}\sum_{\alpha\beta mn}\frac{V_{\alpha\beta mn}V_{mn\alpha\beta}}{\epsilon_\alpha+\epsilon_\beta-\epsilon_m-\epsilon_n}, \tag{26}$$

and

$$(\langle a_p^\dagger a_p\rangle^{(2)}_{\rm corr})_B\equiv{}_B\!\left(\psi^{(1)}\left|\sum_\gamma\mathring{B}^\dagger_{p\gamma}\mathring{B}_{p\gamma}\right|\psi^{(1)}\right)_B=\frac{1}{2}\sum_{m\alpha\beta}\frac{V_{\alpha\beta mp}V_{mp\alpha\beta}}{(\epsilon_\alpha+\epsilon_\beta-\epsilon_m-\epsilon_p)^2}, \tag{27}$$

with

$$|\psi^{(1)})_B=\sum_I\frac{1}{E_{\rm HF}-E_I}|I)_P\times{}_P\!\left(I\left|\frac{1}{2\sqrt{2}}\sum_{mn\alpha\beta}V_{mn\alpha\beta}\mathring{B}^\dagger_{m\alpha}\mathring{B}^\dagger_{n\beta}\right|0\right)_B=\frac{1}{4}\sum_{mn\alpha\beta}\frac{V_{mn\alpha\beta}}{\epsilon_\alpha+\epsilon_\alpha-\epsilon_m-\epsilon_n}|mn\alpha\beta)_F\,. \tag{28}$$

A comparison between the above results and those given in (3) and (4) reveals no discrepancy of a factor 2 between them. As one can see, the reason is that the Marumori expansion treats the Pauli principle correctly by restricting the sum over intermediate states in (26) and (28) to a sum over physical boson states only. In the next section we shall show how the RPA equations can be derived from the expanded Hamiltonian (25).

IV. RANDOM-PHASE APPROXIMATION AS A BOSON APPROXIMATION

In Ref. 1, one starts with a boson Hamiltonian specially constructed so that its commutator with the boson operator $\mathring{B}_{m\alpha}$ gives the RPA equations. This seems to be artifical, though, because if we were to derive RPA equations in the fermion space, we would have considered the commutator between the Hamiltonian (2) and the *fermion operator* $B_{m\alpha}$. Therefore, what we should do in the boson space is to calculate the commutator of the expanded Hamiltonian H_B in (25) and the boson expansion of $B_{m\alpha}$.

Let $|G\rangle$ be the ground state of the system, $|I\rangle$ one of the excited states, and $|G)_B$, $|I)_B$ their images in the boson space; we have

$$\begin{aligned}{}_B(I|[VB_{p\gamma}V^\dagger,H_B]|G)_B&={}_B\!\left(I\left|\left[\left\{\mathring{B}_{p\gamma}+\left(-1+\frac{1}{\sqrt{2}}\right)\sum_{p'\gamma'}\mathring{B}^\dagger_{p'\gamma'}\mathring{B}_{p'\gamma'}\mathring{B}_{p\gamma}-\frac{1}{\sqrt{2}}\sum_{p'\gamma'}\mathring{B}^\dagger_{p'\gamma'}\mathring{B}_{p'\gamma}\mathring{B}_{p\gamma'}+O(5)\right\},H_B\right]\right|G\right)_B\\&=\left(I\left|\left\{(\epsilon_p-\epsilon_\gamma)\mathring{B}_{p\gamma}+\sum_{m\alpha}V_{\alpha p m\gamma}\mathring{B}_{m\alpha}+\frac{1}{\sqrt{2}}\sum_{m\alpha}V_{mp\alpha\gamma}\mathring{B}^\dagger_{m\alpha}+\frac{1}{\sqrt{2}}\left(-1+\frac{1}{\sqrt{2}}\right)\sum_{m\alpha}V_{mp\alpha\gamma}\mathring{B}^\dagger_{m\alpha}\right.\right.\\&\qquad\left.\left.+\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)\sum_{m\alpha}V_{mp\gamma\alpha}\mathring{B}^\dagger_{m\alpha}+O(2)\right\}\right|G\right)_B\\&={}_B(I|\{(\epsilon_p-\epsilon_\gamma)\mathring{B}_{p\gamma}+\sum_{m\alpha}V_{\alpha pm\gamma}\mathring{B}_{m\alpha}+\sum_{m\alpha}V_{mp\alpha\gamma}\mathring{B}^\dagger_{m\alpha}+O(2)\}|G)_B\,.\end{aligned} \tag{29}$$

The above equation is an exact result derived from the Hamiltonian (2). Now in the boson approximation, we consider only the terms linear in the boson operators. To the same approximation, the usual RPA amplitudes are given by

$$\begin{aligned}Y^*_{p\gamma}(I)&\equiv\langle I|B^\dagger_{p\gamma}|G\rangle\\&={}_B(I|\mathring{B}^\dagger_{p\gamma}+O(3)|G)_B\\&\cong{}_B(I|\mathring{B}^\dagger_{p\gamma}|G)_B\,,\end{aligned} \tag{30a}$$

and similarly

$$Z_{p\gamma}(I)\cong{}_B(I|\mathring{B}_{p\gamma}|G)_B\,. \tag{30b}$$

With these definitions we finally obtain from (29) the following equation:

$$\begin{aligned}-(E_I-E_G)Z_{p\gamma}(I)=(\epsilon_p-\epsilon_\gamma)Z_{p\gamma}(I)&+\sum_{m\alpha}V_{\alpha pm\gamma}Z_{m\alpha}(I)\\&+\sum_{m\alpha}V_{mp\alpha\gamma}Y^*_{m\alpha}(I)\,,\end{aligned} \tag{31}$$

which together with its conjugate derived from the commutator $[VB^\dagger_{m\alpha}V^\dagger,H_B]$ constitute the usual RPA equations.

We have shown that the Marumori method gives a boson Hamiltonian which can be used in both the perturbation calculation and the derivation of RPA equations. This is in fact not surprising at all, because it can be proved trivially from (6), (8), and (9) that every algebraic identity in the fermion space is preserved in the physical boson space. For instance, if H is rotationally invariant,

$$[J,H]=0 .$$

We must then have

$$[J_B,H_B]=0 \tag{32}$$

in the boson space, where

$$J_B \equiv VJV^\dagger$$

is the boson expansion of the total angular momentum operator J, and (32) will lead to a *set* of conditions when arranged in normal form.

In the next section we shall discuss the extension of the boson-expansion approach to higher-order RPA.

V. EXTENSION TO HIGHER-ORDER RPA

To define higher-order RPA approximations in the boson space, we shall start with Rowe's method[6] of generating higher-order RPA approximations in the fermion space. One expresses an excited state $|I\rangle$ as

$$|I\rangle = O^\dagger(I)|G\rangle , \tag{33}$$

with

$$O^\dagger(I)=\sum_\alpha [Y(\alpha)\eta_\alpha^\dagger - Z^*(\alpha)\eta_\alpha] , \tag{34}$$

where $\{\eta_\alpha^\dagger\}$ is a complete set of n-particle–n-hole operators, $n=1, 2, \dots, N$. From the equation of motion

$$\tfrac{1}{2}\langle G|[O,[H,O^\dagger]]|G\rangle + \tfrac{1}{2}\langle G|[[O,H],O^\dagger]|G\rangle = \omega\langle G|[O,O^\dagger]|G\rangle , \tag{35}$$

one can derive the set of equations

$$\omega(I)\sum_\beta N_{\alpha\beta}Y_\beta(I) = \sum_\beta\{L_{\alpha\beta}Y_\beta(I)+M_{\alpha\beta}Z_\beta^*(I)\} , \tag{36a}$$

$$-\omega(I)\sum_\beta N_{\alpha\beta}Z_\beta^*(I) = \sum_\beta\{L_{\alpha\beta}^*Z_\beta^*(I)+M_{\alpha\beta}^*Y_\beta(I)\} , \tag{36b}$$

where

$$\omega(I)\equiv E_I - E_G , \tag{37a}$$

$$N_{\alpha\beta}\equiv\langle G|[\eta_\alpha,\eta_\beta^\dagger]|G\rangle = N_{\alpha\beta}^* , \tag{37b}$$

$$L_{\alpha\beta}\equiv\tfrac{1}{2}\langle G|[\eta_\alpha,[H,\eta_\beta^\dagger]]|G\rangle + \tfrac{1}{2}\langle G|[[\eta_\alpha,H],\eta_\beta^\dagger]|G\rangle = L_{\beta\alpha}^* , \tag{37c}$$

and

$$M_{\alpha\beta}\equiv -\tfrac{1}{2}\langle G|[\eta_\alpha,[H,\eta_\beta]]|G\rangle - \tfrac{1}{2}\langle G|[[\eta_\alpha,H],\eta_\beta]|G\rangle = M_{\beta\alpha} . \tag{37d}$$

As we have mentioned in the last section, every fermion identity is preserved by the Marumori transcription. Therefore all of the above equations are valid in the boson space. To define a higher-order RPA in terms of boson operators, we only have to evaluate the matrix elements (37) in the boson space consistently to a definite order in the boson operator. To illustrate this, we shall rederive the RPA equations from (37) and (36).

The RPA results from the restriction of $\eta_\alpha^\dagger$ to one-particle–hole pairs and the approximation of $|G\rangle$ by the Hartree-Fock ground state $|0\rangle$ in the calculation of matrix elements (37). Thus we have

$$\begin{aligned} N_{m\alpha,n\beta} &\cong \langle 0|[B_{m\alpha},B_{n\beta}^\dagger]|0\rangle \\ &= {}_B(0|V[B_{m\alpha},B_{n\beta}^\dagger]V^\dagger|0)_B \\ &= {}_B(0|[\mathring{B}_{m\alpha}+O(3),\mathring{B}_{n\beta}^\dagger+O(3)]|0)_B \\ &= \delta_{mn}\delta_{\alpha\beta} , \end{aligned} \tag{38a}$$

and

$$\begin{aligned} L_{m\alpha,n\beta} &= \tfrac{1}{2}\langle 0|[B_{m\alpha},[H,B_{n\beta}^\dagger]]|0\rangle + \tfrac{1}{2}\langle 0|[[B_{m\alpha},H],B_{n\beta}^\dagger]|0\rangle \\ &= \tfrac{1}{2}\,{}_B(0|[\mathring{B}_{m\alpha}+O(3),[H_B,VB_{n\beta}^\dagger V^\dagger]]|0)_B + \tfrac{1}{2}\,{}_B(0|[[VB_{m\alpha}V^\dagger,H_B],\mathring{B}_{n\beta}^\dagger+O(3)]|0)_B \\ &= \tfrac{1}{2}\,{}_B(0|\mathring{B}_{m\alpha}[H_B,VB_{n\beta}^\dagger V^\dagger]|0)_B + \tfrac{1}{2}\,{}_B(0|[VB_{m\alpha}V^\dagger,H_B]\mathring{B}_{n\beta}|0)_B . \end{aligned}$$

Using (29), we get

$$L_{m\alpha,n\beta} = (\epsilon_m - \epsilon_\alpha)\delta_{mn}\delta_{\alpha\beta} + V_{m\beta\alpha n} . \tag{38b}$$

Finally, we have

$$\begin{aligned} M_{m\alpha,n\beta} &= -\tfrac{1}{2}\langle 0|[B_{m\alpha},[H,B_{n\beta}]]|0\rangle - \tfrac{1}{2}\langle 0|[[B_{m\alpha},H],B_{n\beta}]|0\rangle \\ &= -\tfrac{1}{2}\,{}_B(0|\mathring{B}_{m\alpha}[H_B,VB_{n\beta}V^\dagger]|0)_B + \tfrac{1}{2}\,{}_B(0|\mathring{B}_{n\beta}[VB_{m\alpha}V^\dagger,H_B]|0)_B \\ &= \tfrac{1}{2}(V_{nm\beta\alpha}+V_{mn\alpha\beta}) \\ &= V_{mn\alpha\beta} . \end{aligned} \tag{38c}$$

Substituting (38) into (36b), we have

$$-(E_I-E_G)Z_{m\alpha}(I) = (\epsilon_m-\epsilon_\alpha)Z_{m\alpha}(I) + \sum_{n\beta}V_{\beta mn\alpha}Z_{m\alpha}(I) + \sum_{n\beta}V_{nm\beta\alpha}Y_{m\alpha}^*(I) , \tag{39}$$

which is identical to (31).

It is now quite clear that in order to obtain higher-order equations, we must include more operators in the set $\eta_\alpha^\dagger$ and then calculate the matrix element (37) to a definite order in the boson operators. For example, the next step beyond the RPA would be to take $\eta_\alpha^\dagger$ as a set of one- and two-particle–hole operators, and then evaluate (37) in the boson space, keeping only terms containing no more than three boson operators. In this way, our method does have the advantage of simplifying the calculation of matrix elements by doing it in the boson space so that only boson terms to a certain order need to be considered.

VI. DISCUSSION

We have used Marumori's method to obtain the boson expansion for a general Hamiltonian. With this expansion, one can:

(1) Derive the usual RPA equations. Furthermore, it provides a basis for the consistent definition of higher-order RPA.

(2) Evaluate the ground-state correlations using perturbation theory. The results are guaranteed to be the same as that obtained in the fermion space because the Marumori transformation V preserves all matrix elements.

(3) Simplify the shell-model calculation by truncating the expansion at a certain order. This requires the assumption that terms of a certain order in the boson expansion are smaller than terms of the preceding order. This assumption cannot be justified from the operator algebra (14), because it does not contain a small expansion parameter. Instead we have to consider the algebra of the angular-momentum-coupled operators

$$B_M^{(J)\dagger}(p\gamma) \equiv \sum_{m_p m_c} \begin{bmatrix} j_p & j_c & J \\ m_p & -m_c & M \end{bmatrix} (-)^{j_c - m_c} a^\dagger_{j_p m_p} a_{j_c m_c},$$

etc., where [ ] is a Clebsch-Gordan coefficient. In addition, we have to choose a physical boson space different from the one defined in (18) so that the boson expansions will be convergent in a certain subspace of the physical boson space.[7] Since boson expansions with different choices of physical boson spaces are related by unitary transformations, all the results obtained before are still valid in the convergent subspace.

We have thus found that the difficulties discussed in Ref. 1 can be solved by using the Marumori expansions. We have also indicated how to proceed further and define a higher-order RPA-type calculation for the ground-state correlations based on the boson-expansion method.

We remark finally that after this work was done, we received a preprint[8] which describes how the Marumori expansion used in this paper can be derived starting from the Beliaev-Zelevinsky approach.

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