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Three-Fluid Hydrodynamical Model of Nuclei*

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A three-fluid model of nuclei is introduced, the three fluids being the protons, the neutrons of the same orbitals as protons, and the excess neutrons, to account for the fact that the excess neutrons interact less strongly with the protons than do the neutrons which occupy the same space-spin states as the protons. Calculations of proton and neutron density distributions, of isotope shifts, and of isospin impurities have been carried out. The giant-dipole phenomenon is also studied in the present model. It is found that considerable improvement is achieved in the results for the proton density distribution and for the isospin impurities as compared with the two-fluid model. The other results are found to be consistent with previous calculations.

1. INTRODUCTION

There have been several attempts, based on hydrodynamical models, to explain some of the collective properties of nuclei, e.g., giant-dipole-resonance (GDR) phenomena, neutron and proton distributions in nuclei, the isospin mixing in the ground states of nuclei, and fission, etc. In all cases, the nucleus was considered to be composed of a proton fluid and a neutron fluid (except for fission where only one fluid is considered). Goldhaber and Teller[1] proposed three different models in an attempt to account for the cross sections and the A dependence of the GDR energies. In the first model it was assumed that a displacement of the average position of the protons from the equilibrium position requires a force which is proportional to the displacement but is independent of A. This would be a good assumption if the nucleus consisted of α-particle clusters. In this model the GDR energy will be the same for all nuclei. In the second model the protons and neutrons on the

surface of the nuclei have fixed positions with respect to each other. The restoring forces per unit mass, in the event of a relative displacement of the neutron and proton fluids, is proportional to the gradients of the neutron and proton densities, and the GDR frequencies have an $A^{-1/3}$ dependence. The third of their models assumes that neutrons and protons are two interpenetrating incompressible fluids. In this model the two surfaces do not coincide. The total restoring force is proportional to the surface, i.e., R^2, and the GDR frequencies have an $A^{-1/6}$ dependence.

The drawback of the Goldhaber and Teller models is that they require the use of arbitrary parameters. Steinwedel and Jensen[2] improved upon the second of their models by eliminating the arbitrariness. They introduced a restoring force related to the symmetry-energy term in the expression for the binding energy per nucleon:

$$(\text{B.E.}/A)_{\text{sym}} = K(N-Z)^2, \quad \text{with } K \sim 25 \text{ MeV}. \tag{1.1}$$

Since the nuclear forces are of short range we can approximate the restoring-energy density as inferred from (1.1) by

$$\epsilon = K(\rho_n - \rho_p)^2/\rho_0, \tag{1.2}$$

where ρ_n and ρ_p are the neutron and proton densities, respectively. The A dependence of the GDR frequencies was found to be given by

$$\hbar\omega_0 = (4ZN/A^2)^{1/2}\, 60A^{-1/3}. \tag{1.3}$$

Experimentally, the $A^{1/3}$ law seems to hold for heavy nuclei ($A > 100$). For lighter nuclei, surface effects become important.

The two-fluid hydrodynamical model, with the restoring force and assumptions of Ref. 2, was used by Steinwedel and Danos[3] to estimate the variation of the equilibrium proton density inside a nucleus. Their results showed that the proton density near the nuclear surface rises as compared to the central proton density. It was recently found[4,5] that for ^{208}Pb this rise is of the order of 50% smaller than that which is required by the data from electron scattering and μ-mesonic x rays.

In a previous paper by the present authors[4] a three-fluid hydrodynamical model of nuclei was introduced to account for the rise in the proton density near the nuclear surface. The three fluids are the protons, the neutrons occupying the same orbitals as the protons, and the excess neutrons. We shall call the first kind of neutrons the "blocked" neutrons. The obvious reason for considering the nucleus to be made up of three fluids instead of two is the difference in the interaction of the protons with the two "kinds" of neutrons. The blocked neutrons interact much more strongly with the protons than the excess neutrons because of the large spatial overlap of protons and neutrons having the same space-spin quantum numbers. The blocked neutrons and the protons are coupled to isospin zero in the absence of any isospin violating interactions. The electromagnetic interactions break the symmetry between protons and neutrons, resulting in a state of the nucleus containing a mixture of isospins. The excess neutrons, on the other hand, are coupled to a unique isospin of $T_0 = \frac{1}{2}(N-Z)$. (See also Ref. 4.)

We present the model and its assumptions in Sec. 2. In Sec. 3 we use it to calculate the proton and neutron distributions in nuclei. It will be shown that the rms radius of protons for heavy nuclei is larger than that of neutrons and nuclear matter even though the neutron and nuclear-matter (neutrons plus protons) distributions extend beyond the proton distribution. In Sec. 3 we also calculate the isotope shift for the isotopes of lead.

For completeness, in Sec. 4 we summarize very briefly the isospin impurities in the nuclear ground states. This subject is treated in detail in Ref. 6.

Section 5 describes the GDR phenomena for neutron-excess nuclei. In the three-fluid model there are two independent modes of vibration in ^{208}Pb; one has its lowest dipole eigen energy at about 13.3 MeV and the other at about 4.4 MeV. The latter is found to be weaker by a factor of about 500–600. It is also shown that the present model gives an $A^{-1/3}$ dependence of the GDR energies. Throughout our presentation we shall take the example of ^{208}Pb.

2. MODEL

The three-fluid model is essentially a straightforward generalization of models of Steinwedel and Jensen[2] and Steinwedel and Danos.[3] We assume the nuclear fluid to be incompressible, and to be distributed uniformly inside a sphere of radius $R = R_0 A^{1/3}$. The three fluids, by themselves, are compressible. Formally

$$\begin{aligned} \rho_p + \rho_b + \rho_e = \rho_0 = \text{constant} \quad & \text{for } r \leq R, \\ = 0 \qquad\qquad\qquad\quad & \text{for } r > R. \end{aligned} \tag{2.1}$$

The subscripts p, b, and e stand for protons, the blocked neutrons, and the excess neutrons. The densities are normalized to yield

$$\int \rho_p \, dv = \int \rho_b dv = Z, \quad \int \rho_e \, dv = N - Z. \tag{2.2}$$

In analogy with expression (1.2) we introduce three restoring-energy densities corresponding to the

three pairs of fluids:

$$\epsilon_{pb}=K_{pb}(\rho_p-\rho_b)^2/\rho_0,$$
$$\epsilon_{be}=K_{be}(\rho_b-\rho_e)^2/\rho_0, \tag{2.3}$$
$$\epsilon_{ep}=K_{ep}(\rho_e-\rho_p)^2/\rho_0.$$

Only the terms quadratic in $\rho_i-\rho_j$ have been retained in (2.3), since constants and linear terms influence only the binding energy and the position of the energy minimum. They have no bearing on the strength of the restoring forces.

In contrast to the two-fluid model where the constant K could be determined from the symmetry energy, the situation here is more complex. Physically, the energy densities ϵ are averages containing both kinetic and potential energy contributions. One could think of obtaining them from, say, nuclear-matter calculations. We shall, however, estimate them from other sources.

The constant K_{pb} seems to be the most certain one. According to Meyers and others,[7-9] the constant K_{pb} should depend upon A as $K=\alpha+\beta A^{-1/3}$ while being independent of $N-Z$. Using this hypothesis we can obtain K_{pb} for ^{208}Pb from the value of K valid for a nucleus with $A=164$. We use Brambletts data[10] and obtain $K_{pb}\simeq 25.2$ MeV.

Less can be said about K_{be} and K_{ep}. An estimate of their ratio can be made by noting that the ratio of the related two-body interaction matrix elements, i.e.,

$$\langle\phi_p\phi_e|v|\phi_p\phi_e\rangle/\langle\phi_b\phi_e|v|\phi_b\phi_e\rangle$$

is on the average of the order of 3. Even though the constants K_{ij} are not directly proportional to the two-body matrix elements, the ratio can be expected to be of a similar order of magnitude. Still, we will have to condider this ratio to be an almost free parameter. We shall see later that this ratio is not of critical importance; the results of our calculation are quite insensitive to it. We shall tentatively assume it to be 3.

3. DENSITY DISTRIBUTIONS AND ISOTOPE SHIFTS

To determine the equilibrium proton and neutron density distributions we need only the "static" Hamiltonian:

$$H=\int(\epsilon_{pb}+\epsilon_{ep}+\epsilon_{be})dv+\tfrac{1}{2}\int\frac{e\rho_p(r)\times e\rho_p(r')}{|\vec{r}-\vec{r}'|}\,dvdv'. \tag{3.1}$$

We insert (2.3) and eliminate one of the densities, say ρ_e, using the condition of constancy of total density, (2.1). Then we can apply the variational principle to (3.1) with the constraints (2.2) included with the help of Lagrange multipliers. This yields

$$2K_{pb}\frac{\rho_p-\rho_b}{\rho_0}-4K_{be}\frac{2\rho_b+\rho_p-\rho_0}{\rho_0}-2K_{ep}\frac{2\rho_p+\rho_b-\rho_0}{\rho_0}-\lambda_1=0 \tag{3.2}$$

and

$$2K_{pb}\frac{\rho_p-\rho_b}{\rho_0}+2K_{be}\frac{2\rho_b+\rho_p-\rho_0}{\rho_0}+4K_{ep}\frac{2\rho_p+\rho_b-\rho_0}{\rho_0}+e\psi_p+\lambda_2=0, \tag{3.3}$$

where λ_1 and λ_2 are the Lagrange multipliers and $\psi_p(r)$ is the electrostatic potential due to the proton distribution; i.e.,

$$\psi_p(r)=\int\frac{e\rho_p(r')}{|\vec{r}-\vec{r}'|}\,dv'. \tag{3.4}$$

To find λ_1 and λ_2 we integrate (3.2) and (3.3) over the nuclear volume obtaining

$$\lambda_1=2(2K_{be}+K_{ep})(A-3Z)/A \tag{3.5}$$

and

$$\lambda_2=2(K_{be}+2K_{ep})(A-3Z)/A-\epsilon_c. \tag{3.6}$$

The constant ϵ_c is given by

$$\epsilon_c=\frac{1}{\frac{4}{3}\pi R^3}\int e\psi_p(r)dv, \tag{3.7}$$

substituting λ_1 and λ_2 back into (3.2) and (3.3) we get the following expressions:

$$\rho_b(K_{pb}+4K_{be}+K_{eb})=\rho_p(K_{pb}-2K_{be}-2K_{ep})+(3Z/A)\rho_0(2K_{be}+K_{ep}) \tag{3.8}$$

and

$$\rho_p(K_{pb}+K_{be}+4K_{ep})=\rho_b(K_{pb}-2K_{be}-2K_{ep})+(3Z/A)\rho_0(K_{be}+2K_{ep})-[e\psi_p(r)-\epsilon_c]. \tag{3.9}$$

Elimination of ρ_b between (3.8) and (3.9) yields

$$8K_e\left(\frac{\rho_p}{\rho_0}-\frac{Z}{A}\right)+e\psi_p-\epsilon_c=0, \tag{3.10}$$

where

$$K_e=\frac{9}{4}\frac{K_{pb}K_{be}+K_{be}K_{ep}+K_{ep}K_{pb}}{K_{pb}+4K_{be}+K_{ep}}. \tag{3.11}$$

Applying the Laplacian to (3.10) and using $-\nabla^2\psi=4\pi e\rho_p$ we obtain

$$[8K_e/(4\pi e^2\rho_0)]\nabla^2\rho_p=\rho_p. \tag{3.12}$$

The solution of (3.12) is

$$\rho_p = \rho_{pc}(r/\lambda)^{-1}\sinh(r/\lambda) \quad \text{for } r \leq R\,,$$
$$= 0 \quad \text{for } r > R\,, \tag{3.13}$$
$$\lambda^2 = 8K_e/(4\pi e^2\rho_0)\,, \tag{3.14}$$

where

$$\rho_{pc} = Z\Big/\left[4\pi\int_0^R \frac{\sinh(r/\lambda)}{r/\lambda}\, r^2 dr\right] \tag{3.15}$$

is the central density.

In order to compare our results with experiments we have to incorporate the nuclear surface into our model. To this end we recall that the nuclear shape is, in fact, not rigid but can undergo surface vibrations. Upon quantizing this motion the surface becomes diffuse owing to the zero-point motion of all the possible surface modes. This effect has been estimated by Pieper and Greiner,[11] who showed that the full experimental surface diffuseness can be essentially accounted for by this mechanism when starting with the sharp intrinsic density drop-off, as, for example, in (2.1). As the vibrational ground-state wave functions are Gaussians, we shall smear out the obtained density with a Gaussian, considering the diffuseness to be an adjustable parameter. Thus, we write

$$\rho_{pg}(r) = \frac{1}{\sigma\sqrt{\pi}}\int_{-\infty}^{\infty}\rho_p(r')e^{-(r-r')^2/\sigma^2}dr'\,. \tag{3.16}$$

It turns out that a unique set of parameters R, σ, and K_e can be chosen so that ρ_{pg} exhibits the experimentally observed skin thickness, rms radius, and rise $\mathfrak{R}$ in nuclear density at the surface. This last quantity can be defined as

$$\mathfrak{R} = \frac{\rho_{p\,\max} - \rho_{pc}}{\rho_{pc}}\,. \tag{3.17}$$

For ^{208}Pb, $\mathfrak{R}$ has been found[6] to be 0.10. Curve (a) in Fig. 1 shows the proton density distribution for ^{208}Pb as given by the expression

$$\rho_p(r) = \rho_{pc}\left(1 + \frac{wr^2}{c^2}\right)[1 + e^{(r-c)/z}]^{-1}\,. \tag{3.18}$$

The parameters $w = 0.32$, $c = 6.40$, and $z = 0.54$ were obtained by Heisenberg *et al.*[5] by fitting the data on elastic electron scattering and on μ-mesic x rays. Curve (b) is obtained by minimizing the quantity

$$\chi^2 = \frac{1}{N}\sum_i^N\left[\frac{\rho_{pg}(r_i) - \rho_{ex}(r_i)}{\rho_{ex}(r_i)}\right]^2 + \left(\frac{t_{pg} - t_{ex}}{t_{ex}}\right)^2 + \left(\frac{\mathfrak{R}_{pg} - \mathfrak{R}_{ex}}{\mathfrak{R}_{ex}}\right)^2 + \left(\frac{\langle r_{pg}^2\rangle^{1/2} - \langle r_{ex}^2\rangle^{1/2}}{\langle r_{ex}^2\rangle^{1/2}}\right)^2 \tag{3.19}$$

by varying K_e, R_0, and σ. In (3.19) the subscript pg refers to quantities calculated using the density given by (3.16) and ex refers to the experimental results as obtained from (3.18). The skin thickness t is the distance in which the density falls from 90 to 10% of the maximum value. Table I lists the set of best-fit parameters obtained in this manner. Experimental and theoretical values of $\mathfrak{R}$, t, and the rms radii $\langle r^2\rangle^{1/2}$ are also listed. The agreement between these values is quite good although visually there seems to be some discrepancy in the two curves. The inability of the two curves to match perfectly results from the difference between the two shapes. The density $\rho_{pg}(r)$ falls more rapidly at the surface than the charge distribution fitted to the electron scatter-

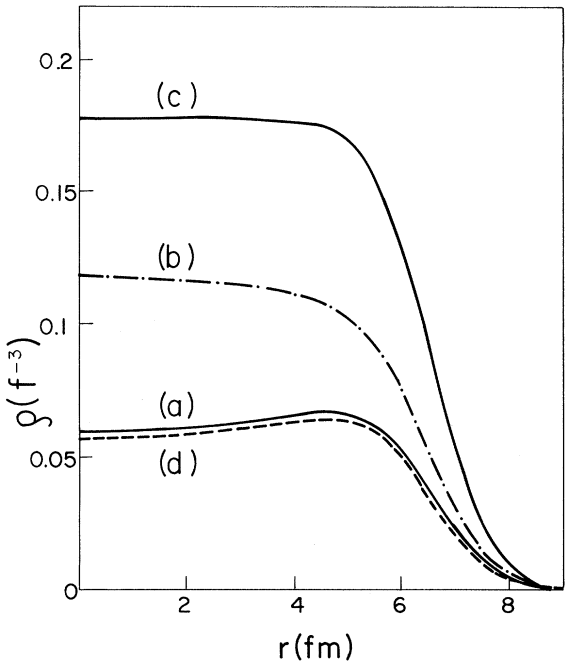

FIG. 1. Density distributions in ^{208}Pb. Curve (a) is the proton density resulting from fitting the data on electron scattering and μ-mesic x rays. (b), (c), and (d) are proton, neutron, and total (neutron+proton) density distributions calculated from the three-fluid model.

TABLE I. Rise $\mathfrak{R}$, skin thickness t, and rms radius of the proton distribution in ^{208}Pb in the three-fluid model and from the electron scattering data (Ref. 5). Best-fit parameters: $K = 10.4$ MeV, $R_0 = 1.300$.

	Three-fluid model	Ref. 5
Rise $\mathfrak{R}$	0.122	0.125
Skin thickness	2.14 fm	2.16 fm
$\langle r_p^2\rangle^{1/2}$	5.472 fm	5.481 fm

ing experiment because of the factor $1+wr^2/c^2$ in (3.18).

The value 10.4 MeV for K_e determined in this fashion provides us with a rather strong hold on the values of the K parameters. For example, with $K_{pb} \simeq 25$ MeV and $K_{ep}/K_{be} \simeq 3$ we have, using expression (3.11), $K_{ep} \simeq 4.8$ and $K_{be} \simeq 1.6$.

In the present model, the neutron density is obtained by taking $\rho_n = \rho_0 - \rho_p$, with ρ_p given by (20), and then smearing ρ_n out in the same manner as ρ_p in (23) with the same set of parameters K_e, R_0, and σ which gave the best fit for the proton density. The rms radii are:

	Protons	Neutrons	Matter
$\langle r^2\rangle^{1/2}$	5.472 fm	5.348 fm	5.397 fm.

Although the neutron and the matter densities extend beyond the proton density, the rms radii $\langle r_n^2\rangle^{1/2}$ and $\langle r_m^2\rangle^{1/2}$ are smaller than $\langle r_p^2\rangle^{1/2}$. This apparently arises because of the central depression in the proton density.

We must point out here that the shell-model calculations and also the Hartree-Fock calculations[12,13] fail to account for the experimentally observed central depression. This indicates that nuclear short-range correlations are better taken care of in a hydrodynamical model. The requirement of incompressibility of nuclear matter seems to accomplish this quite effectively; in fact, overeffectively. (See also Ref. 4.)

In Table II we give the values of the "reduced" radius r_0 (the unit radius of a corresponding uniform distribution) defined by

$$r_0 = (\tfrac{5}{3}\langle r_p^2\rangle^{1/2})/A^{1/3} \tag{3.20}$$

for ^{206}Pb and ^{208}Pb. Column 1 gives the results of the present calculation and column 2 that of Anderson *et al.*[14] A charge distribution of the Fermi type

$$\rho(r) = \rho_0[1+e^{n(r-c)/c}]^{-1} \tag{3.21}$$

was assumed by Anderson *et al.*, with $n = 14.37$ and 13.95, respectively, for ^{206}Pb and ^{208}Pb to obtain the results shown in column 2.

The deviations from the $A^{1/3}$ law can be expressed in terms of the quantity

$$\gamma = \frac{3A}{\langle r^2\rangle^{1/2}} \frac{d}{dA}\langle r^2\rangle^{1/2}. \tag{3.22}$$

Generally, γ is found to be less than 1 for isotopes and greater than 1 for isotones. We computed γ for ^{208}Pb for the three-fluid model and found it to be 0.9 ± 0.01, while the data of Ref. 14 give $\gamma = 0.8$. The uncertainties in the former reflect the uncertainties with which the parameters of the model have been determined. As can be seen, the model is very inflexible with respect to this quantity. Thus, deviations resulting, say, from shell closure, cannot be incorporated without modification of the model.

TABLE II. Isotopic shift in isotopes of lead. First three rows are the reduced radii.

	Three-fluid model	Anderson *et al.*
^{206}Pb	1.1686	1.1987
^{208}Pb	1.1682	1.1979
^{210}Pb	1.1679	
γ	0.91	0.79

4. ISOSPIN IMPURITIES

A detailed description of this problem has been given elsewhere.[6] Since the three-fluid model plays an important role in that problem we briefly summarize here the relevant results.

The early calculations of the ground-state isospin impurities[15,16] used the independent-particle shell model. They gave values 1 to 2 orders of magnitude larger than one would expect from β-decay data.[17]

Bohr, Damgaard, and Mottelson[17] introduced the two-fluid hydrodynamical model to include the residual interactions. They derived the following expression for isospin impurities:

$$I(T_0) = [1/(T_0+1)] \times 5.55Z^{8/3} \times 10^{-7}. \tag{4.1}$$

The values calculated from (4.1) are still too high, especially for neutron excess nuclei. A shell-model calculation with a δ-function residual interaction, carried out by the present authors,[6] also reduces the impurities to about the same order of magnitude as (4.1).

In the three-fluid model a further reduction arises in the calculated isospin impurities as compared with the results of the two-fluid model. In the three-fluid model the impurities result from the loss of overlap between the distribution functions of the protons and the blocked neutrons. In the two-fluid model the neutrons have to move in a direction opposite to that of protons to preserve the nuclear-matter density. On the other hand, in the three-fluid model, the protons and the blocked neutrons can both move toward the surface without violating the incompressibility because the excess neutrons, i.e., the third fluid, can move in toward the nuclear center to take their place. The expression for the isospin impurity in the three-fluid model, derived in Ref. 6, is

$$I(T_0) = \frac{1}{2(T_0+1)} \int_{-\infty}^{\infty} (\sqrt{\rho_p} - \sqrt{\rho_p})^2 dv, \tag{4.2}$$

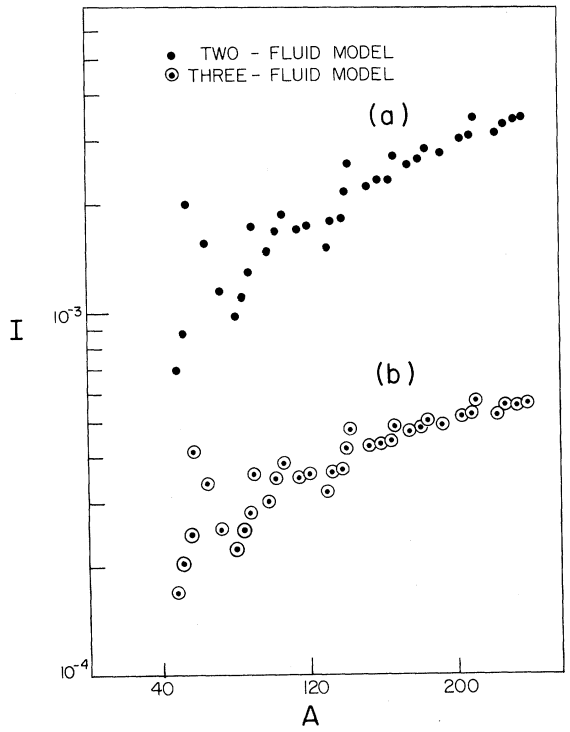

FIG. 2. Isospin impurities: (a) two-fluid model of Bohr, Damgaard, and Mottelson; (b) the three-fluid model.

ρ_p is given by (3.13) and ρ_b by (3.8). We must also take into account the fact that the monopole excitation of some of the proton orbitals does not result in isospin breaking, because the neutron level corresponding to the excited proton level is occupied. For heavy nuclei, this means multiplying impurities from (4.2) by a factor of $\sim\frac{3}{4}$. Isospin impurities calculated with the help of expressions (4.1) and (4.2) are plotted in Figs. 2(a) and 2(b). We note that, as expected, there is a significant reduction compared with the two-fluid model; i.e., by a factor of ~6.

5. GIANT DIPOLE RESONANCE

Until now we have been discussing what can be called a "static" three-fluid model, because no kinetic energies were involved. We were interested only in the quantities in equilibrium under the influence of opposing forces. In this section we shall include the oscillatory motion of the three fluids. We shall then derive an expression for the dipole eigen energies in this model. The relative strengths of the various modes of vibration will be determined and an attempt will be made to identify the mode corresponding to the experimentally observed GDR.

We shall extend the formalism of Steinwedel and Jensen[2,18] to three fluids. The assumptions of Sec. 2 apply here also; i.e., the nucleus is considered to have a sharp boundary and only those modes of vibration are allowed which preserve the condition that $\rho_p+\rho_b+\rho_e=\rho_0=$ constant. In addition to these we assume that individual fluids are compressible, i.e.,

$$\rho_j+\mathrm{div}\rho_j\vec{v}_j=0\,,\quad j=p,\ b,\ e\,, \tag{5.1}$$

and that no surface vibrations are allowed (rigid surface):

$$\vec{r}\cdot\vec{v}_j=0\,. \tag{5.2}$$

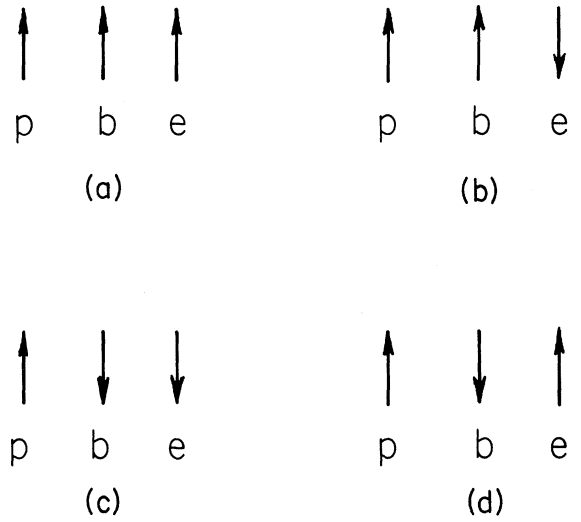

FIG. 3. Possible modes of vibration in a three-fluid model.

In the case of three fluids, four unique modes of vibration are possible. They are depicted in Fig. 3. The arrow heads pointing upward indicate motion of the fluid toward the surface, and the ones pointing downward indicate motion toward the center. Mode (a) involves center-of-mass motion and contibutes only to Thompson scattering. Out of the remaining three, only two are independent, because of the constant-density condition.

We shall set up the Lagrangian

$$L=T-U+\lambda(\rho_p+\rho_b+\rho_e-\rho_0)\,, \tag{5.3}$$

where λ is a Lagrange multiplier, and

$$T=\frac{m}{2}\int d^3r(\rho_p v_p^{\,2}+\rho_b v_b^{\,2}+\rho_e v_e^{\,2}) \tag{5.4}$$

and

$$U=\int d^3r(\epsilon_{pb}+\epsilon_{be}+\epsilon_{ep})\,. \tag{5.5}$$

Then Hamilton's principle states that

$$\int\delta L\,dt=0\,. \tag{5.6}$$

If we want to take into account friction damping and interaction with the electromagnetic field, we must include one more term in (5.6); i.e., instead of (5.6) we have

$$\int(\delta L+\delta A)dt=0\,, \tag{5.7}$$

where

$$\delta A=\int d^3r\Big[-m\Gamma_{pb}\frac{\rho_p\rho_b}{\rho_0}(\vec{v}_p-\vec{v}_b)\cdot(\delta\vec{r}_p-\delta\vec{r}_b) - m\Gamma_{be}\frac{\rho_b\rho_e}{\rho_0}(\vec{v}_b-\vec{v}_e)\cdot(\delta\vec{r}_b-\delta\vec{r}_e) - m\Gamma_{ep}\frac{\rho_e\rho_p}{\rho_0}(\vec{v}_e-\vec{v}_p)\cdot(\delta\vec{r}_e-\delta\vec{r}_p)+e\rho_p\vec{K}\cdot\delta\vec{r}_p\Big]. \tag{5.8}$$

Here the Γ's are the damping widths and the $\delta\vec{r}$'s are the displacements of infinitesimal volume elements of the fluids. The first three terms are what we may call the "friction" terms because of their proportionality to relative velocities, relative displacements, and the products of densities. The last term is the Lorentz force

$$K=\vec{E}+\vec{v}_p\times\vec{H}/c\,. \tag{5.9}$$

It accounts for the interaction with the external electromagnetic field which supplies energy to the system.

To obtain the equations of motion from (5.6), we assume that there is no rotational motion and make the usual linearization approximation of hydrodynamics; i.e., all second- or higher-order terms in velocities are gradients are neglected. Since the surface is rigid, all terms evaluated at the surface in performing partial integrations vanish. This implies that coupling between surface oscillations and the dipole vibration are not taken into account. This assumption simplifies the derivation tremendously. It can be easily incorporated after quantization of the motion.[19]

Before proceeding further, we express the variations in $\vec{v}_j$ and ρ_j in terms of $\delta\vec{r}_j$

$$\delta\vec{v}_j=(\partial/\partial t+\vec{v}_j\cdot\mathrm{grad})\,\delta\vec{r}_j\,, \tag{5.10}$$

$$\delta\rho_j=-\mathrm{div}(\rho_j\delta\vec{r}_j)\,. \tag{5.11}$$

Using these and the assumptions made above we arrive at

$$\int(\delta L+\delta A)dt=\int\int dt\,d^3r\Big[\sum_j^{p,\,b,\,e}\Big(-m\rho_j\frac{\partial\vec{v}_j}{\partial t}+2\rho_j\,\mathrm{grad}g_j - m\rho_j\vec{f}_j+\rho_j\,\mathrm{grad}\lambda\Big)\cdot\delta\vec{r}_j+e\rho_p\vec{K}\cdot\delta\vec{r}_p\Big]=0, \tag{5.12}$$

where

$$\begin{aligned}
g_p&=K_{pb}\frac{\rho_p-\rho_b}{\rho_0}-K_{ep}\frac{\rho_e-\rho_p}{\rho_0}\,,\\
g_b&=K_{be}\frac{\rho_b-\rho_e}{\rho_0}-K_{pb}\frac{\rho_p-\rho_b}{\rho_0}\,,\\
g_e&=K_{ep}\frac{\rho_e-\rho_b}{\rho_0}-K_{be}\frac{\rho_b-\rho_e}{\rho_0}\,,
\end{aligned} \tag{5.13}$$

$$\begin{aligned}
\vec{f}_p&=\Gamma_{pb}\frac{(v_p-v_b)\rho_b}{\rho_0}-\Gamma_{ep}\frac{(v_e-v_p)\rho_e}{\rho_0}\,,\\
\vec{f}_b&=\Gamma_{be}\frac{(\vec{v}_b-\vec{v}_e)\rho_e}{\rho_0}-\Gamma_{pb}\frac{(\vec{v}_p-\vec{v}_b)\rho_p}{\rho_0}\,,\\
\vec{f}_e&=\Gamma_{ep}\frac{(\vec{v}_e-\vec{v}_p)\rho_p}{\rho_0}-\Gamma_{be}\frac{(\vec{v}_b-\vec{v}_e)\rho_b}{\rho_0}\,.
\end{aligned} \tag{5.14}$$

Because of the introduction of the Lagrange multiplier λ, the three variations are now independent and the equations of motion are

$$m\rho_j\frac{\partial v_j}{\partial t}+2\rho_j\,\mathrm{grad}g_j+m\rho_j\vec{f}_j-\delta_{jp}\rho_p\vec{K}-\rho_j\,\mathrm{grad}\lambda=0\,. \tag{5.15}$$

We take the divergence of (5.1) and neglect nonlinear terms. Furthermore, since $\mathrm{div}\vec{E}=0$ for an electromagnetic field and $\mathrm{div}(\vec{v}_p\times\vec{H})/c$ is a second-order term, the equations of motion reduce to

$$m\frac{\ddot{\rho}_j}{\rho_j}-2\nabla^2g_j-m\,\mathrm{div}\vec{f}_j+\nabla^2\lambda=0\,. \tag{5.16}$$

Eliminating λ we obtain two independent equations of motion

$$m\Big(\frac{\ddot{\rho}_p}{\rho_p}-\frac{\ddot{\rho}_b}{\rho_b}\Big)-2\nabla^2(g_p-g_b)-m\,\mathrm{div}(\vec{f}_p-\vec{f}_b)=0\,, \tag{5.17a}$$

$$m\Big(\frac{\ddot{\rho}_b}{\rho_b}-\frac{\ddot{\rho}_e}{\rho_e}\Big)-2\nabla^2(g_b-g_e)-m\,\mathrm{div}(\vec{f}_b-\vec{f}_e)=0\,. \tag{5.17b}$$

To solve (5.17) we (i) express $\vec{f}_j$ in terms of ρ_j using the equation of continuity, (ii) eliminate ρ_e in (5.17) with the help of the condition of constant total density, (iii) approximate ρ_0/ρ_j by A/N_j outside the differentials. Here $N_p=N_b=Z$ and $N_e=N-Z$. The equations of motion become

$$m\frac{A}{Z}(\ddot{\rho}_p-\ddot{\rho}_b)-\nabla^2(\alpha_p\rho_p-\alpha_b\rho_b)+m(\gamma_p\rho_p-\gamma_b\rho_b)=0\,, \tag{5.18a}$$

$$m\frac{A}{N-Z}\Big(\ddot{\rho}_p+\frac{N}{Z}\ddot{\rho}_b\Big)-\nabla^2(\beta_p\rho_p+\beta_b\rho_b)+m(\delta_p\rho_p+\delta_b\rho_b)=0\,, \tag{5.18b}$$

where

$$\begin{aligned}\alpha_p &= 2(2K_{pb} - K_{be} + 2K_{ep}),\\ \alpha_b &= 2(2K_{pb} + 2K_{be} - K_{ep}),\\ \beta_p &= 2(-K_{pb} + 2K_{be} + 2K_{ep}),\\ \beta_b &= 2(K_{pb} + 4K_{be} + K_{eb}),\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}\gamma_p &= 2\Gamma_{pb} - \Gamma_{be} + N\Gamma_{ep}/Z,\\ \gamma_b &= 2\Gamma_{pb} + N\Gamma_{be}/Z - \Gamma_{ep},\\ \delta_p &= -\Gamma_{pb} + N\Gamma_{be}/Z + N\Gamma_{ep}/(N-Z),\\ \delta_b &= \Gamma_{pb} + N^2\Gamma_{be}/[Z(N-Z)] + Z\Gamma_{ep}/(N-Z).\end{aligned} \tag{5.20}$$

Let us now assume that the solutions of (5.18) are of the form

$$\begin{aligned}\rho_p(\vec{r}, t) &= \rho_{p0} + \eta_p(\vec{r})e^{-i\omega t},\\ \rho_b(\vec{r}, t) &= \rho_{b0} + \eta_b(\vec{r})e^{-i\omega t}.\end{aligned} \tag{5.21}$$

This leads to the time-independent equations

$$(\alpha_p\nabla^2 + k_p^2)\eta_p - (\alpha_b\nabla^2 + k_b^2)\eta_b = 0, \tag{5.22a}$$

$$(\beta_p\nabla^2 + l_p^2)\eta_p + (\beta_b\nabla^2 + l_b^2)\eta_b = 0, \tag{5.22b}$$

where

$$\begin{aligned}k_p^2 &= m\omega^2(A/Z + i\gamma_p/\omega),\\ k_b^2 &= m\omega^2(A/Z + i\gamma_b/\omega),\\ l_p^2 &= m\omega^2[A/(N-Z) + i\delta_p/\omega],\\ l_b^2 &= m\omega^2[AN/Z(N-Z) + i\delta_b/\omega].\end{aligned} \tag{5.23}$$

(5.22a) and (5.22b) can be reduced to a single fourth-order partial differential equation

$$[(\alpha_p\beta_b + \beta_p\alpha_b)\nabla^4 + (k_p^2\beta_b + l_b^2\alpha_p + k_b^2\beta_p + l_p^2\alpha_b)\nabla^2 + (l_p^2k_b^2 + l_b^2k_p^2)]\eta_p = 0. \tag{5.24}$$

That is,

$$(\nabla^2 + k_1^2)(\nabla^2 + k_2^2)\eta_p = 0 \tag{5.25}$$

with

$$\begin{Bmatrix}-k_1^2\\ -k_2^2\end{Bmatrix} = -\frac{k_p^2\beta_b + l_b^2\alpha_p + k_b^2\beta_p + l_p^2\alpha_b}{2(\alpha_p\beta_b + \beta_p\alpha_b)} \times\left\{1 \pm \left[1 - 4\frac{(\alpha_p\beta_b + \beta_p\alpha_b)(l_b^2k_p^2 + l_p^2k_b^2)}{(k_p^2\beta_b + l_b^2\alpha_p + k_b^2\beta_p + l_p^2\alpha_b)^2}\right]^{1/2}\right\}$$

$$= -m\omega^2\begin{Bmatrix}F_1(1 + i\Gamma_1/\omega)\\ F_2(1 + i\Gamma_2/\omega)\end{Bmatrix}. \tag{5.25a}$$

Let us ignore the imaginary parts of k_1^2 and k_2^2 for the moment. Then for ^{208}Pb the substitution of the K parameters obtained in Sec. 3 yields

$$k_1^2 = 0.0238m\omega^2; \quad k_2^2 = 0.240m\omega^2. \tag{5.25b}$$

The solutions of (5.25) can be expanded in terms of spherical Bessel functions as

$$\eta_p(k_i, \vec{r}) = \sum_{l,m} C_{lm}^p(i)\, j_l(k_i r)Y_{lm}(\theta, \phi), \quad i = 1, 2. \tag{5.26}$$

The condition for the existence of a dipole resonance can be obtained by combining the boundary conditions $\vec{r}\cdot\vec{v}_j|_{r=R} = 0$ with (5.15). This yields

$$(-2\vec{r}\cdot\mathrm{grad}g_j + \delta_{jp}\vec{r}\cdot\vec{K} + \vec{r}\cdot\mathrm{grad}\lambda)|_{r=R} = 0. \tag{5.27}$$

On adding the three equations (5.27), the terms containing g's drop out on account of (5.13) and we obtain the following expression for gradλ:

$$\mathrm{grad}\lambda|_{r=R} = -\tfrac{1}{3}\vec{r}\cdot\vec{K}|_{r=R}. \tag{5.28}$$

Substitution of (5.28) in (5.27) results in

$$\vec{r}\cdot\mathrm{grad}g_p|_{r=R} = \tfrac{1}{3}\vec{r}\cdot\vec{K}|_{r=R}, \tag{5.29a}$$

$$\vec{r}\cdot\mathrm{grad}g_b|_{r=R} = -\tfrac{1}{6}\vec{r}\cdot\vec{K}|_{r=R}, \tag{5.29b}$$

$$\vec{r}\cdot\mathrm{grad}g_e|_{r=R} = -\tfrac{1}{6}\vec{r}\cdot\vec{K}|_{r=R}. \tag{5.29c}$$

Using the condition $\rho_p + \rho_b + \rho_e = \rho_0$ we can eliminate ρ_b and ρ_e from any two of the above equations to yield

$$(C/\rho_0)\vec{r}\cdot\mathrm{grad}\rho_b|_{r=R} = \vec{r}\cdot\vec{K}|_{r=R}, \tag{5.30}$$

where the constant C is given by

$$C = 6\left[\frac{(K_{pb} - K_{ep})(K_{pb} - K_{be})}{2K_{be} + K_{pb}} + (K_{pb} + 2K_{ep})\right] \times\frac{2K_{be} + K_{pb}}{K_{ep} + 4K_{be} + K_{pb}}. \tag{5.31}$$

For ^{208}Pb, with the K parameters of Sec. 3,

$$C = 83.5. \tag{5.32}$$

For the dipole resonance, (5.30) can be written as

$$\left.\frac{\partial\eta_p}{\partial r}\right|_{r=R} = \frac{eE}{(C/\rho_0)}\cos\theta. \tag{5.33}$$

$\vec{E}$ is the electric field and θ is the angle between $\vec{E}$ and $\vec{r}$. Thus from (5.26)

$$\eta_p(k_i, \vec{r}) = C_{10}^p(i)\, j_1(k_i r)\cos\theta \tag{5.34}$$

and dropping the subscript 10, we finally obtain

$$C^p(i) = \frac{eE}{(C/\rho_0)}\left[\left.\frac{\partial j_1(k_1R)}{\partial r}\right|_{r=R}\right]^{-1}, \tag{5.35a}$$

$$= \frac{eER}{(C/\rho_0)}[j_2(k_iR) - k_iRj_1(k_iR)]^{-1}. \tag{5.35b}$$

The poles of $C^p(i)$ will correspond to the dipole resonances of the system. These poles occur at

$$k_iR = 2.08,\ 5.95,\ 9.20,\ 12.42, \ldots. \tag{5.36}$$

It has been shown by Steinwedel and Jensen[2] that the lowest resonance exhausts about 86% of the

dipole sum rule. The lowest dipole resonances of ^{208}Pb corresponding to the k_j^2 given by (5.25b) occur at 13.3 and 4.44 MeV, respectively.

The giant dipole resonance in ^{208}Pb occurs experimentally at around 13.5 MeV. To identify the theoretical 13.3-MeV resonance with this resonance we shall have to show that almost all of the dipole strength is associated with this mode and almost none with the one at 4.44 MeV. To see if it is really so, we shall compute the integrated cross sections for the two modes.

It is obvious from (5.21) and from the condition $\rho_p+\rho_b+\rho_e=\rho_0$ that

$$\eta_p+\eta_b+\eta_e=0\,, \tag{5.37}$$

since the functions η_p, η_b, and η_e are all of the form (5.34) we have

$$C^p(i)+C^b(i)+C^e(i)=0\,. \tag{5.38}$$

Substituting η_p and η_b in (5.22a) we obtain the following relation between C^p and C^b

$$\frac{C^p(i)}{C^b(i)}=\frac{-\alpha_p k_i^2+k_b^2}{-\alpha_b k_i^2+k_p^2}\,. \tag{5.39}$$

The amplitudes in the two modes calculated using (5.38) and (5.39) are shown in Figs. 4(a) and 4(b) with the normalization $[C^p(i)]^2+[C^b(i)]^2+[C^e(i)]^2=1$. In the first mode, corresponding to 13.3 MeV, C^e and C^b have a phase opposite to that of C^b, whereas in the second mode ρ_b and ρ_p move together and ρ_e moves in the opposite direction. The lengths of the lines indicate the relative amplitudes. The relative amplitude for the protons is much larger in the first mode than in the second one.

We now calculate the ratio of the integrated absorption cross sections in the two modes. The classical integrated cross section in terms of the radiation width is given by[18,20]

$$\sigma_{\rm int}(i)=3\pi^2(C^2/\omega_i^2)\Gamma_{ri}\,. \tag{5.40}$$

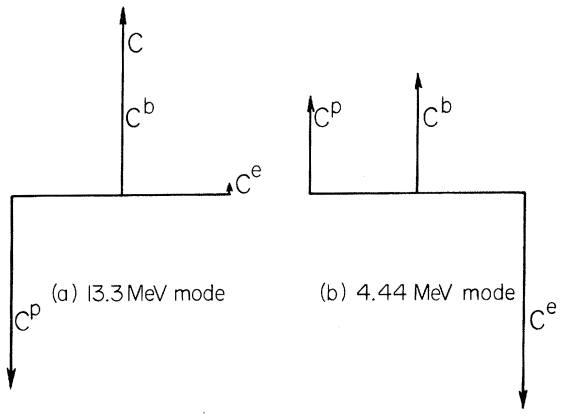

FIG. 4. Relative amplitudes of three fluids in the two giant-dipole-resonance modes.

TABLE III. Dipole eigen energies and ratio of the dipole strengths as functions of K_{ep}/K_{be}. The quantities K_{pb} and K_e are kept fixed at 25.2 and 10.4, respectively.

$\frac{K_{ep}}{K_{be}}$	Dipole energies (MeV) Mode 1	Mode 2	$\frac{\sigma_{\rm int}(\text{Mode 1})}{\sigma_{\rm int}(\text{Mode 2})}$
1	13.4	4.87	426
2	13.3	4.57	515
3	13.3	4.44	576
4	13.3	4.37	620

ω_i is the resonance frequency and Γ_{ri} is the radiation width of mode i. This latter quantity is given by

$$\Gamma_{ri}=-\frac{dW_i}{dt}\frac{1}{W_i}\,, \tag{5.41}$$

where W_i is the stored energy. The power radiated at resonance is

$$\frac{dW_i}{dt}=\rho\omega_i^4[C^b(i)\times d]^2\,, \tag{5.42}$$

where P is a radiation constant, $C^b(i)\times d$ is the dipole moment in mode i, and

$$d=\int dv\, j_1(kr)er\cos\theta\,. \tag{5.43}$$

Using (2.3) and (5.34), the stored energy W_i can be written as

$$W_i=\tfrac{1}{2}G_i[C^p(i)]^2N\,, \tag{5.44}$$

where

$$G_i=K_{pb}\left[1-\frac{C^b(i)}{C^p(i)}\right]^2+K_{be}\left[\frac{C^b(i)}{C^p(i)}-\frac{C^e(i)}{C^p(i)}\right]^2 +K_{ep}\left[\frac{C^e(i)}{C^p(i)}-1\right]^2 \tag{5.45}$$

and

$$N=\int dv[j_1(kr)\cos\theta]^2/\rho_0^2\,. \tag{5.46}$$

Therefore the expression (5.41) reduces to

$$\Gamma_{ri}=\text{constant}\times\omega_i^4/G_i\,, \tag{5.47}$$

where the constant does not depend upon the mode. Thus the ratio of the two cross sections is

$$\sigma_{\rm int}(1)/\sigma_{\rm int}(2)=\omega_1^2G_2/\omega_2^2G_1\,. \tag{5.48}$$

For the values of the parameters K used in Sec. 3 this ratio is found to be 576. That is, the 13-MeV mode is found to be about 600 times stronger than the 4-MeV mode.

As noted earlier the value of 3 assigned to K_{pe}/K_{be} is rather uncertain. Therefore we have computed the dipole eigen energies and the strength ratios of the two modes for values of this ratio, ranging between 1 and 4. The lower limit was

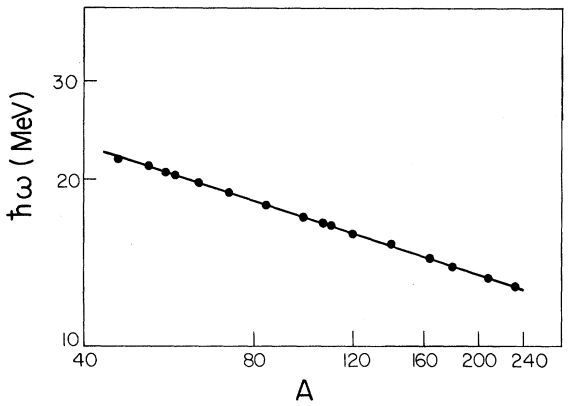

FIG. 5. A dependence of the giant-dipole-resonance energies in the three-fluid model.

imposed by the fact that the proton-neutron interaction is greater than the neutron-neutron interaction. The results are contained in Table III. Here the values of K_{pb} and K_e were kept fixed at 25.2 and 10.4, respectively. We note that the eigenvalue at 13 MeV is almost independent of variations in K_{ep}/K_{be}. Also, the eigenvalue at about 4 MeV changes only by about 10% in this range. The ratio of the integrated cross sections is seen to vary from 400 to 600. Thus most of the strength remains in the 13-MeV mode.

The expressions (5.25a), (5.35b), and (5.36) do not provide a transparent form for the A dependence of the eigenfrequencies. But, using the set of parameters of Sec. 3 we can calculate the GDR energies for the entire range of neutron excess nuclei. This was done for a number of nuclei between ^{48}Ti and ^{238}U. The results are shown in Fig. 5. We have plotted $\log_{10}E$ against $\log_{10}A$. The slope of this line is precisely −0.3330, implying an $A^{-1/3}$ dependence. Experimentally the $A^{-1/3}$ dependence is found to apply well for $A > 100$ nuclei. For lighter nuclei, the collective model becomes inaccurate. Therefore, the $A^{-1/3}$ dependence obtained from this model is not expected to give good agreement with experiment for the light nuclei. It should be pointed out that the K's have been assumed to be independent of A. This is not true, but their variation with A is small. Deviations from the $A^{-1/3}$ dependence should be expected if the A dependence of the K's is taken into account.

CONCLUSIONS

A three-fluid hydrodynamical model for neutron-excess nuclei was introduced to take into account the difference in the interaction of excess neutrons and of the remainder of the neutrons (the blocked neutrons) with protons. Equilibrium density distributions, isotope shifts, giant dipole resonances, and isospin impurities were studied within this formalism. The results show that the three-fluid model is a definite improvement over the two-fluid model in calculations of density distributions and isospin impurities. While in these cases the results are qualitatively consistent with previous calculations, a qualitative change arises with respect to the giant dipole resonances. Namely, in the three-fluid model there exist two independent modes instead of the single dipole mode in the two-fluid model. Numerically, for ^{208}Pb, one lies at 13.3 MeV and the other at about 4.4 MeV. The former is stronger (by more than 2 orders of magnitude) and is identified with the observed giant dipole resonance. The giant-dipole-resonance energies are found to obey an $A^{-1/3}$ law.

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