Dorenbusch, Bull. Am. Phys. Soc. 13, 174 (1968); private communication.

[4]J. J. Schwartz, Phys. Rev. 175, 1453 (1968).

[5]H. Ohnuma, W. Makofske, and D. Dehnhard, private communication.

[6]R. H. Cornett, private communication.

[7]R. A. Wallen, H. Ohnuma, and N. M. Hintz, Bull. Am. Phys. Soc. 14, 601 (1969); private communication.

[8]P. H. Debenham, Internal Report, John H. Williams Laboratory of Nuclear Physics, University of Minnesota, 1968 (unpublished); P. H. Debenham, D. Dehnhard, and R. W. Goodwin, Nucl. Instr. Methods 67, 238 (1969); H. Enge, *ibid.* 28, 126 (1964).

[9]R. A. Ristinen and A. W. Sunyar, Phys. Rev. 153, 1209 (1967); J. C. Glass and J. K. Kliwer, Nucl. Phys. A115, 234 (1968).

[10]J. H. Bjerregaard, P. F. Dahl, O. Hansen, and G. Sidenius, Nucl. Phys. 51, 641 (1964).

[11]J. A. Nolen, Jr., J. P. Schiffer, N. Williams, and D. von Ehrenstein, Phys. Rev. Letters 18, 1140 (1967).

[12]P. D. Kunz, private communication.

[13]J. L. Yntema and H. Ohnuma, private communication.

[14]J. C. Hafele, E. R. Flynn, and A. G. Blair, Phys. Rev. 155, 1238 (1967).

[15]R. H. Bassel, Phys. Rev. 149, 791 (1966).

[16]E. Kashy and T. W. Conlon, Phys. Rev. 135, B389 (1964); R. J. Peterson, Phys. Rev. 170, 1003 (1968); J. L. Yntema, Phys. Rev. 186, 1144 (1969).

[17]P. M. Endt and C. Van der Leun, Nucl. Phys. A105, 1 (1967).

[18]B. F. Bayman, private communication.

[19]H. Hefele *et al.*, *Jahresbericht* (Max-Planck-Institut für Kernphysik, Heidelberg, Germany, 1968), p. 34; R. Santo, private communication.

[20]F. D. Becchetti, D. Dehnhard, and T. G. Dzubay, to be published.

[21]R. K. Bansal and J. B. French, Phys. Letters 11, 145 (1964).

[22]J. J. Simpson, W. R. Dixon, and R. S. Storey, Phys. Letters 30B, 478 (1969).

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Angular Distribution of Particles Evaporated in Nuclear Reactions*

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A method for the evaluation of the average angular distribution, over a range of energy, of particles evaporated in succession from a compound nucleus is presented. The method is applied to the analysis of the angular distribution in the reaction $Fe^{56}(\alpha,p)Co^{59}$ and the reaction $Cu^{63}(C^{12},p)Se^{74}$. The analysis indicated that nuclei at high excitation have rigid moments of inertia.

I. INTRODUCTION

In the present paper a new method for the calculation of the angular distribution of particles evaporated in reactions proceeding via a compound nucleus is discussed. The method is particularly applicable to the study of angular distributions for reactions in which states in the continuum decay into a wide energy range of final highly excited states. The method suggested in this paper offers an alternative to the method developed by Douglas and Macdonald.[1] Douglas and Macdonald generalize the theory of angular distribution between discrete levels to make it applicable to transitions to the levels in the continuum. The method described in this paper consists of a straightforward application of ideas related to the concepts of the compound nucleus and reciprocity to the study of angular distributions. The present method has the advantage of yielding simple forms for the angular distribution which do not require the knowledge of W and Z Racah coefficients. This simplicity stems from the fact that one calculates the average angular distribution over a wide range of final states. The present method has the additional advantage of being applicable to the evaluation of the angular distribution in reactions in which many particles are evaporated in succession.

It is shown that the calculated angular distribution is very sensitive to the value of the spin cutoff parameter. Therefore, a comparison between the measured value and calculated value of the angular distribution offers a very convenient tool for the determination of the spin cutoff parameter. In particular, the analysis of angular distribution based on the present method suggests that nuclei at high excitation have rigid moments of inertia.

One expects intuitively that the spin and polarization of the compound nucleus affects the angular distribution of the emitted particles. Ericson and Strutinsky[2] and Ericson[3,4] derived expressions for the angular distribution, in the classical limit, for particles emitted from compound nuclei. In most of their considerations the spin of the compound nucleus is polarized so that this spin is in a direction perpendicular to the beam of the incident particles.

The spin of the compound nucleus affects not only the angular distribution of the emitted particles but also their spectra as shown, among others, by Kammuri and Nakasima,[5] Pic-Pichak,[6] Broek,[7] and Thomas.[8]

The effect of the spin and polarization of the compound nucleus on the angular distribution of emitted γ rays has been studied both quantum mechanically and in the classical limit.[9-13] The angular distribution in which a few γ rays are emitted in succession has been discussed.[14]

In Sec. II the theory is developed. The method is applied to the analysis of experimental data in Sec. III. The results are discussed in Sec. IV.

II. THEORY

First the angular distribution for one emitted particle is evaluated. Later an expression for the angular distribution for the second emitted particle is derived. From this treatment it can be seen how this method can be generalized to obtain the angular distribution of any particle in a cascade.

Let Ψ_α, Φ_β, and U_γ be wave functions of the compound nucleus, the residual nucleus, and the outgoing particles, respectively, characterized by the sets of quantum numbers α, β, and γ, respectively. The wave function Ψ_α can be expanded as

$$\Psi_\alpha = \sum_{\beta,\gamma} a^\alpha_{\beta,\gamma} \Phi_\beta U_\gamma . \tag{1}$$

The angular distribution of the emitted particle $P(\theta,\phi)$ is given by[15]

$$P(\theta,\phi) = \int \psi_\alpha{}^* \psi_\alpha d\tau' . \tag{2}$$

In Eq. (2) the volume element $d\tau'$ includes all coordinates except the solid angle of the outgoing particle $d\Omega = \sin\theta\, d\theta d\phi$. The coefficient $a^\alpha_{\beta\gamma}$ can be written as

$$a^\alpha_{\beta,\gamma} = |a^\alpha_{\beta,\gamma}| e^{i\eta^\alpha_{\beta,\gamma}} . \tag{3}$$

The wave function of the emitted particle U_γ can be written as a product of a radial function R_γ and an angular function A_γ

$$U_\gamma = R_\gamma A_\gamma . \tag{4}$$

In the present paper interest is focused on the average angular distribution $\langle P(\theta,\phi)\rangle_{\rm av}$ over a range of energy E such that

$$\langle P(\theta,\phi)\rangle_{\rm av} = \frac{1}{E}\int_{E_0 - E/2}^{E_0 + E/2} P(\theta,\phi) dE . \tag{5}$$

In Eq. (5), the averaging is over the states of the compound nucleus. This yields

$$\langle P(\theta,\phi)\rangle_{\rm av} = \sum_{\beta,\gamma} |a^\alpha_{\beta,\gamma}|^2 |A_\gamma|^2 . \tag{6}$$

The simplified expression in Eq. (6) stems from normalization and from the relation

$$\frac{1}{E}\int e^{-i\eta^{\alpha'}_{\beta',\gamma'}} e^{i\eta^\alpha_{\beta,\gamma}} dE = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\delta_{\gamma,\gamma'} . \tag{7}$$

The above relation, Eq. (7), is due to the independence and randomness of the phase factors $\eta^\alpha_{\beta,\gamma}(E)$ for different channels. In particular, if $\alpha \neq \alpha'$, $\beta \neq \beta'$, or $\gamma \neq \gamma'$, the integrand $\exp\{i[\eta^\alpha_{\beta,\gamma}(E) - \eta^{\alpha'}_{\beta',\gamma'}(E)]\}$ oscillates very rapidly, and when integrated over a wide range of energy the value of the integral is negligible in comparison with 1; therefore the integral is approximated by zero. It is easy to see that for $\alpha = \alpha'$, $\beta = \beta'$, and $\gamma = \gamma'$ the integral is 1.

Now the discussion is generalized to a situation in which two particles are emitted in succession. The wave function ψ_α after the emission of two particles emitted in succession can be expanded as

$$\psi_\alpha = \sum_{\beta_1,\beta_2,\gamma_1,\gamma_2} a^\alpha_{\beta_1,\gamma_1} a^{\beta_1}_{\beta_2,\gamma_2} \Phi_{\beta_2} U_{\gamma_1} U_{\gamma_2} . \tag{8}$$

In Eq. (8), U_{γ_1} and U_{γ_2} are the wave function of the first and second emitted particle, respectively, and Φ_{β_2} is the wave function of the residual nucleus after the emission of two particles. The angular distribution $P(\theta,\phi)$ is given by

$$\langle P(\theta,\phi)\rangle_{\rm av} = \frac{\sigma_1\langle P_1(\theta,\phi)\rangle_{\rm av} + \sigma_2\langle P_2(\theta,\phi)\rangle_{\rm av}}{\sigma_1 + \sigma_2} . \tag{9}$$

In Eq. (9), $\langle P_1(\theta,\phi)\rangle_{\rm av}$ and $\langle P_2(\theta,\phi)\rangle_{\rm av}$ are the average angular distributions for the emission of the first and second particle, and σ_1 and σ_2 are the cross sections for the emission of the first and second particle.[16] The angular distributions are given by

$$P_1(\theta,\phi) = \int \psi_\alpha{}^* \psi_\alpha d\tau_1' , \tag{10a}$$

$$P_2(\theta,\phi) = \int \psi_\alpha{}^* \psi_\alpha d\tau_2' . \tag{10b}$$

In Eqs. (10a) and (10b) the volume elements $d\tau_1'$ and $d\tau_2'$ include all the coordinates except the angular coordinates of the first emitted particle and the second emitted particle, respectively. For the angular distribution of the first particle one obtains

$$\begin{aligned}\langle P_1(\theta,\phi)\rangle_{\rm av} &= \sum_{\beta_1,\beta_2,\gamma_1,\gamma_2} |a^\alpha_{\beta_1,\gamma_1}|^2 |a^{\beta_1}_{\beta_2,\gamma_2}|^2 |A_{\gamma_1}|^2 \\ &= \sum_{\beta_1\gamma_1} |a^\alpha_{\beta_1,\gamma_1}|^2 |A_{\gamma_1}|^2 .\end{aligned} \tag{11}$$

The angular distribution of the second particle becomes

$$\langle P_2(\theta,\phi)\rangle_{\rm av} = \sum_{\beta_1,\beta_2,\gamma_1,\gamma_2} |a^{\alpha}_{\beta_1,\gamma_1}|^2 |a^{\beta_1}_{\beta_2,\gamma_2}|^2 |A_{\gamma_2}|^2 . \tag{12}$$

The form for the angular distribution of the first particle in Eq. (11) is equivalent to the form derived in Eq. (6) when only one particle is emitted. This is consistent with causality, since the angular distribution of the first particle does not affect the angular distribution of the second particle. Equation (12) can readily be generalized for reactions in which more than two particles are emitted in succession.

The square of the absolute value of the coefficient $a^{\alpha}_{\beta,\gamma}$ represents the probability of the decay of a state of the compound nucleus characterized by quantum numbers $\alpha \equiv (E, J, M)$ to a state in which the residual nucleus is characterized by quantum numbers $\beta \equiv (E', J', M')$ and the emitted particle is characterized by a set of quantum numbers $\gamma \equiv (\epsilon, l, j, m)$. Here and in the following discussion $E' = E - B - \epsilon$, where B is the binding energy. These quantum numbers do not describe the states completely but are sufficient for the present discussion of angular distributions. The coefficients $|a^{\alpha}_{\beta,\gamma}|^2$ are written as[16]

$$|a^{E,J,M}_{E',J',M';\epsilon,l,j,m}|^2 = T(E,J,M;E',J',M';\epsilon,l,j,m) = \frac{\hbar S(E,J,M;E',J',M';\epsilon,l,j,m)}{\Gamma(E,J,M)} . \tag{13}$$

In Eq. (13), $S(E,J,M;E',J',M';\epsilon,l,j,m)$ is the decay rate (or transition probability per unit time per unit energy) for a transition from a state characterized by quantum numbers E, J, and M to a state characterized by quantum numbers E', J', and M' and emitting a particle characterized by quantum numbers ϵ, l, j, and m.

The coefficients $T(E,J,M;E',J'M';\epsilon,l,j,m)$ can be reduced using the Wigner-Eckart theorem[17]

$$T(E,J,M;E',J',M';\epsilon,l,j,m) = T(E,J;E',J';\epsilon,l,j)\begin{pmatrix} J & j & J' \\ -M & m & M' \end{pmatrix}^2 . \tag{14}$$

In Eq. (14) $\left(\begin{smallmatrix} J & j & J' \\ -M & m & M' \end{smallmatrix}\right)$ is a 3-j symbol. Therefore, the average angular distribution takes a particularly simplified form

$$\langle P(\theta,\phi)\rangle_{\rm av} = \sum_{J',M',l,j,m} \int_{E_0-E/2}^{E_0+E/2} dE' T(E,J'E',J';\epsilon,l,j) \times \begin{pmatrix} J & j & J' \\ -M & m & M' \end{pmatrix}^2 |A_{ljm}(\theta,\phi)|^2 . \tag{15}$$

In Eq. (15) the averaging is over the states of the residual nucleus.

For particles with intrinsic spin zero the functions $A_{lm}(\theta,\phi)$ become the spherical harmonics $Y_{lm}(\theta,\phi)$. For particles with intrinsic spin $\frac{1}{2}$ these functions are linear combinations of products of spherical harmonics with spin eigenfunctions. The structure of the terms $T(E,J;E',J';\epsilon,l,j)$ is discussed in detail in Ref. 16; in particular, the decay rate $S(E,J;E',J';\epsilon,l,j)$ is given by

$$S(E,J;E',J';\epsilon,l,j) = \frac{\sigma(\epsilon,l,j;E',J';E,J)l_0^2}{8\hbar R^2\pi^2} \times \left\{\left[1-\frac{(l-\frac{1}{2})^2}{l_0^2}\right]^{1/2} - \left[1-\frac{(l+\frac{1}{2})^2}{l_0^2}\right]^{1/2}\right\}\frac{\rho(E',J')}{\rho(E,J)} . \tag{16}$$

In Eq. (16), R is the nuclear radius, and the critical angular momentum l_0 is given by

$$l_0 = R\sqrt{2m\epsilon}/\hbar . \tag{17}$$

The density of levels $\rho(E,J)$ appearing in Eq. (17) can be written as a product of a spin-dependent part and an energy-dependent part[2,4,18,19] so that

$$\rho(E,J) = \rho(E)(2J+1)e^{-J^2/2\sigma^2} . \tag{18}$$

The spin cutoff parameter σ is related to the nuclear temperature T and the nuclear moment of inertia $\mathscr{I}$ by

$$\sigma^2 = \mathscr{I}T/\hbar^2 . \tag{19}$$

The energy-dependent part of the density of levels $\rho(E)$ is taken from the work of Gilbert and Cameron.[20] The cross section for the inverse reaction $\sigma(\epsilon,l,j;E',J';E,J)$ is written as[15]

$$\sigma(\epsilon,l,j;E',J';E,J) = (2j+1)\pi\lambda\!\!\!^{-2}T_{l,j}(\epsilon) . \tag{20}$$

The transmission coefficients $T_{l,j}(\epsilon)$ are obtained from the optical-model potential.

So far, the angular distribution from a single level to a continuum of levels has been considered. However, in the reactions under consideration many levels of the compound nucleus are populated, so that the angular distribution has to be weighted by the probability of the formation of the compound nucleus in a state with an energy E, spin J, and spin projection M. This probability is proportional to the cross section of the formation of the compound nucleus in a state characterized by these quantum numbers. Let the bombarding particle have an energy ϵ_0, orbital angular momentum l_0, and total angular momentum j_0.

$$\sigma(\epsilon_0,l_0,j_0;E_0,J_0;E,J,M) = (2j+1)\pi\lambda\!\!\!^{-2}T_{l_0,j_0}(\epsilon_0)N(M) . \tag{21}$$

In Eq. (21), E_0, J_0 is the spin and energy of the target.

When either the target or the projectile have zero spin, and either the projectile or the target have a spin J_0, the function $N(M)$ takes the form

$$N(M) = 1/(2J_0+1) \quad \text{for } |M| \leq J_0, \tag{22a}$$

$$N(M) = 0 \quad \text{for } |M| > J_0 \tag{22b}$$

(the direction of the beam is taken as the axis of quantization). The spin of the compound nucleus may be much higher than J_0 because of the orbital motion of the projectile. This limited range of M values affects the angular distribution considerably.

When both target and projectile have nonvanishing spins, the function $N(M)$ depends on the number of ways this M value can be obtained. The calculation is a little more elaborate, but such cases are not considered in the present paper.

III. COMPARISON WITH EXPERIMENT

For comparison with experiment, two reactions were chosen, one an α-induced reaction in which states with moderate spin of the compound nucleus are populated. The other reaction chosen is a heavy-ion-induced reaction in which high-spin states are populated.

For the reaction $Fe^{56}(\alpha, p)Co^{59}$ the angular distribution has recently been measured by Fluss *et al.*[21] A comparison between theory and experiment is shown in Fig. 1. In Fig. 1 there are three curves representing the calculated values of the angular distribution assuming a quarter rigid moment of inertia, half rigid moment of inertia, and rigid moment of inertia.

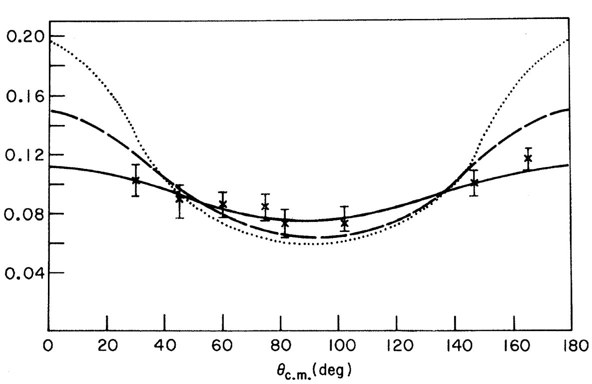

FIG. 1. The angular distribution in arbitrary units for emitted protons for the reaction $F^{56}(\alpha, p)Co^{59}$ in the center-of-mass system. The solid curve, dashed curve, and dotted curve represent calculated angular distributions using rigid-body, half-rigid-body, and quarter-rigid-body values of the moments of inertia, respectively. The curves are normalized so that the integral $\int P(\Omega)d\Omega$ is the same for all three curves.

For the heavy-ion-induced reaction $Cu^{13}(C^{12}, p)Se^{74}$ the angular distribution of the evaporated protons has been measured by D'Auria *et al.*[22] Again the angular distribution has been calculated for three different values of the moment of inertia. The results of a comparison between theory and experiment can be seen in Fig. 2. It can be seen that for both reactions the experimentally measured values of the angular distribution fall closest to the calculated values of the angular distribution assuming a rigid moment of inertia.

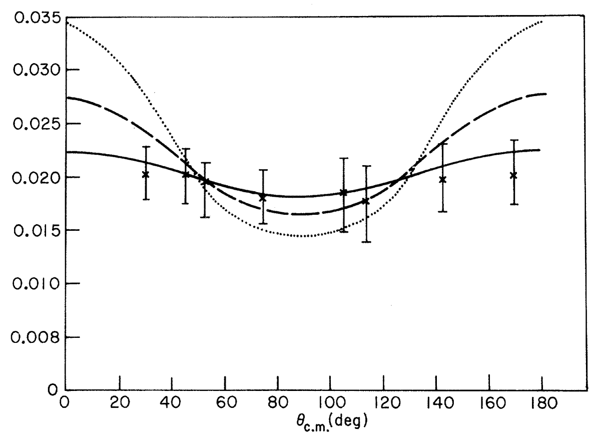

FIG. 2. The angular distribution in arbitrary units for emitted protons for the reaction $Co^{63}(C^{12}, p)Se^{74}$ in the center-of-mass system of coordinates. The solid curve, dashed curve, and dotted curve represent calculated angular distributions using rigid-body, half-rigid-body, and quarter-rigid-body values of the moments of inertia, respectively. The curves are normalized so that the integral $\int P(\Omega)d\Omega$ is the same for all three curves.

IV. DISCUSSION

Angular distributions of particles evaporated as a function of the moment of inertia were calculated. The angular distribution depends on the spin cutoff parameter σ, and this parameter is a function of the nuclear moment of inertia. All calculated results are based on an energy-dependent spin cutoff parameter. The moment of inertia is energy independent. However, the spin cutoff parameter depends on the product of the moment of inertia and the nuclear temperature. The variation of the nuclear temperature with energy is included in the present work.

The comparison between theory and experiment shows that the calculated angular distribution is very sensitive to the value of the spin cutoff parameter. Therefore, the angular distribution allows, by comparison of theory and experiments, an accurate determination of nuclear moments of inertia.

In recent years a comparison between measured and calculated values of isomer ratios has been used to extract the values of the spin cutoff parameters. However, the uncertainty in the measured value of the isomer ratio on one hand and the lack of knowledge of quadrupole admixture on the other does not allow an exact determination of the spin cutoff parameter from isomer-ratio studies.[23,24]

A comparison between theory and experiment indicates that the value of the moment of inertia of nuclei at high excitations is very close to the value of the moment of inertia of a rigid body. This result is consistent with the earliest studies of the effect of the energy and spin on the value of the nuclear moments of inertia.[25-27]

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[1]A. C. Douglas and M. Macdonald, Nucl. Phys. 13, 382 (1959).

[2]T. Ericson and V. Strutinsky, Nucl. Phys. 8, 284 (1958).

[3]T. Ericson, Nucl. Phys. 17, 250 (1960).

[4]T. Ericson, in *Advances in Physics,* edited by N. F. Mott (Taylor and Francis Ltd., London, England, 1960), Vol. 9, p. 425.

[5]T. Kammuri and R. Nakasima, in *Proceedings of the Second Conference on Reactions Between Complex Nuclei,* edited by A. Zucker, F. T. Howard, and F. T. Halbert (John Wiley & Sons, Inc., New York, 1960).

[6]G. A. Pic Pichak, Zh. Eksperim. i Teor. Fiz. 38, 768 (1960) [transl.: Soviet Phys. – JETP 11, 557 (1960)].

[7]H. W. Broek, Phys. Rev. 124, 233 (1961).

[8]T. D. Thomas, Nucl. Phys. 53, 577 (1964).

[9]V. Strutinsky, Zh. Eksperim. i Teor. Fiz. 37, 861 (1959) [transl.: Soviet Phys. – JETP 10, 613 (1960)].

[10]V. V. Babikov, Zh. Eksperim. i Teor. Fiz. 42, 1643 (1967) [transl.: Soviet Phys. – JETP 15, 1143 (1962)].

[11]D. Sperber, Nuovo Cimento 36, 1164 (1964).

[12]D. Sperber, Phys. Rev. 150, 791 (1966).

[13]D. Sperber, Phys. Rev. 162, 1022 (1967).

[14]D. Sperber, Phys. Rev. 142, 578 (1966).

[15]A. Messiah, *Quantum Mechanics* [translated from the French by G. M. Temmer] (John Wiley & Sons, Inc., New York, 1968).

[16]D. Sperber, Phys. Rev. 178, 1688 (1969).

[17]E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (The Cambridge University Press, Cambridge, 1935).

[18]H. A. Bethe, Rev. Mod. Phys. 9, 69 (1937).

[19]C. Bloch, Phys. Rev. 93, 1094 (1953).

[20]A. Gilbert and A. G. Cameron, Can. J. Phys. 43, 1446 (1965).

[21]M. J. Fluss, J. M. Miller, J. M. D'Auria, N. Dudey, B. M. Forman, L. Kowalski, and R. C. Reedy, Phys. Rev. 187, 1449 (1969).

[22]J. M. D'Auria, M. J. Fluss, G. Herzog, L. Kowalski, J. M. Miller, and R. C. Reedy, Phys. Rev. 174, 1409 (1968).

[23]D. Sperber and J. W. Mandler, Nucl. Phys. A113, 689 (1968).

[24]G. Liggett and D. Sperber, Phys. Rev. (to be published).

[25]B. R. Mottelson and J. G. Valatin, Phys. Rev. Letters 5, 511 (1961).

[26]B. R. Mottelson, in *Proceedings of the International Conference on Nuclear Structure, Kingston, 1960,* edited by D. A. Bromley and E. W. Vogt (University of Toronto Press, Toronto, Canada, 1960).

[27]D. Sperber, Phys. Rev. 138, B1028 (1965).