$$\vec{M}_G = m_1 \vec{\nabla}_\alpha \langle \vec{p}_1', \vec{p}_2' | [1 + t(E')G_0(E')][V, e^{i\vec{\alpha}\cdot\vec{r}_1}][1 + G_0(E)t(E)] | \vec{p}_1, \vec{p}_2 \rangle |_{\vec{\alpha}=0} + O(K) \tag{19}$$

for the generalization of Ref. 1. For the total internal scattering amplitude $\vec{M}_I = \vec{M}_R + \vec{M}_G$, we conclude:

$$\begin{aligned} \vec{M}_I &= m_1 \vec{\nabla}_{p_1} \langle f | t(e) | i \rangle |_{e=\bar{E}} + O(K) \\ &= m_1 \vec{\nabla}_\alpha \langle \vec{p}_1', \vec{p}_2' | t(E') e^{i\vec{\alpha}\cdot\vec{r}_1} - e^{i\vec{\alpha}\cdot\vec{r}_1} t(E) | \vec{p}_1, \vec{p}_2 \rangle |_{\vec{\alpha}=0} + O(K) \end{aligned} \tag{20}$$

which belong to the approaches of Refs. 2 and 1, respectively.

Since the operator identity holds in a finite matrix space, these same derivations can be written for particles with spin.

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Effective Orthogonality of Channel Wave Functions and Coupled Equations for Direct Interactions

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An explicit proof is given of the *effective orthogonality* among the channel wave functions obtained from a set of coupled equations by showing that the nonorthogonal parts of the wave functions are physically redundant, thus providing a rigorous basis for the treatment of nuclear stripping and pickup reactions. An orthogonality-preserving procedure of the coupled-channel method is outlined.

The nonorthogonality among the rearrangement channel wave functions has been the subject of much discussion, specifically in connection with the nonuniqueness of solutions of the scattering integral equations[1] and subsequent formulation of the Faddeev equations,[2] the resonating-group-structure approach[3] (RGSA), the coupled-channel method[4] (CCM) with additional distortion potentials, and the evaluation of projection operators in Feshbach's reaction theory.[5]

Recently, however, a simple mathematical procedure has been developed[6,7] by which rearrangement channels can be projected out of the original scattering equations step by step. As an immediate consequence of the more general result obtained earlier[6] using this formalism, it is now possible to answer rigorously the nonorthogonality question of the stripping and pickup reactions. Partly to clarify some of the apparent confusions we explicitly spell out the meaning of the *effective orthogonality* and its proof; the main result is (4).

Following the notation of Ref. 6, we consider the simple two-channel reactions

$$1 + (c+2)_0 \rightleftharpoons (1+c)_0 + 2 \tag{1}$$

and denote the wave functions of the bound ground states of $(c+i)_0$, $i = 1, 2$, by $\psi_0^{(i)}$, and the channel projections by $P_i = \psi_0^{(i)} \rangle\langle \psi_0^{(i)}$ with $P_i^2 = P_i = P_i^\dagger$ and $Q_i = 1_i - P_i$. We can in general write the total wave function for the process (1) in the form[8]

$$\Psi \equiv P_1\Psi_1 + P_2\Psi_2 + \Psi(Q) = \Psi(P) + \Psi(Q), \tag{2}$$

where P and Q are the orthoprojectors with $P + Q = 1$. The operator P, for example, is in general not a simple linear combination of the P_i. In the discussion to follow, an explicit knowledge of P and Q is not required, although their effects are retained implicitly. Therefore, the appearance of these operators in (2) is purely notational.

The nonorthogonality between the two channels

involved is conveniently expressed in a slightly weakened form

$$[P_1, P_2] \neq 0 . \tag{3}$$

In formulating the theory for the process (1), Goldberger and Watson[9] have shown that the *asymptotic orthogonality* $P_1 P_2 \to 0$ as either of the two asymptotic channel regions is approached should be sufficient. However, in practice, the $\Psi(Q)$ part in (2) is often difficult to handle because of (3). We present below a stronger result for the channel wave functions which satisfy a set of coupled equations derived with (2). That is, we show that only the component of $P_i\Psi_i$ which is orthogonal to P_j, $j \neq i$, is physically relevant, so that $P_1\Psi_1$ and $P_2\Psi_2$ are *effectively orthogonal*. We denote this property by

$$(P_1\Psi_1, P_2\Psi_2) \sim 0 \tag{4a}$$

and similarly

$$(P_i\Psi_i, \Psi(Q)) \sim 0 . \tag{4b}$$

The effective orthogonality (4) does *not* necessarily imply that the $P_i\Psi_i$ are mutually orthogonal, as will be discussed more fully later. We will consider here the orthogonality property of the wave functions only in the sense of (4).

(1) We first consider the case with $\Psi(Q) = 0$; i.e.,

$$\Psi \approx \Psi^P \equiv P_1\Psi_1^P + P_2\Psi_2^P . \tag{5}$$

Writing $M_0 \equiv H - E$, and substituting (5) into the original scattering equation, we obtain a set of coupled equations

$$P_i M_0 P_i \Psi_i^P = -P_i M_0 P_j \Psi_j^P , \quad i \neq j , \tag{6}$$

where i and j assume the values 1 and 2. If we further define the homogeneous solutions and the Green's functions of (6) by

$$\begin{aligned} &P_i M_0 P_i \Psi_{i0}^P = 0 , \\ &P_i M_0 P_i G_{i0}^P = -P_i , \end{aligned} \tag{7}$$

then we can rewrite (6) in the form

$$P_i M_j P_i \Psi_i^P = -a_j P_i N_j , \tag{8}$$

where

$$\begin{aligned} &M_j = M_0 + M_0 G_{j0}^P M_0 , \\ &N_j = M_0 P_j \Psi_{j0}^P . \end{aligned} \tag{9}$$

In (8), the a_j are constants specifying the initial conditions and we have used throughout the standing-wave boundary conditions. The remarkable properties of M_j and N_j are that,[7] as follows immediately from (7) and (9),

$$\begin{aligned} &P_j M_j = M_j P_j = 0 , \quad && M_j = Q_j M_j Q_j , \\ &P_j N_j = 0 , && N_j = Q_j N_j . \end{aligned} \tag{10}$$

That is, both M_j and N_j are in the Q_j subspaces, which are orthogonal to P_j, so that the solutions $P_i\Psi_i^P$ of (8) should also be in the Q_j space. Therefore, the component of $P_i\Psi_i^P$ which is proportional to P_j will automatically be eliminated in (8) because of the property of M_j. Independent of whether or not the actual $P_i\Psi_i^P$ obtained from (8) satisfy $P_j P_i \Psi_i^P = 0$, only the Q_j part of $P_i\Psi_i^P$ is physically relevant. Thus, we have the effective orthogonality

$$P_j P_i \Psi_i^P \sim 0 , \tag{11}$$

or, simply, in a weaker form

$$(P_j\Psi_j^P, P_i\Psi_i^P) \sim 0 . \tag{11'}$$

(2) The orthogonality (11') is not quite the same as (4a) yet, although it is in itself a useful result, for example, in the RGSA which uses wave functions of the form (5). For convenience of discussion, we distinguish here between the RGSA and CCM by including in the latter effects of $\Psi(Q)$ in some approximations, such as the introduction of optical potentials.

In order to obtain (4a), we note that the steps involved in the proof of (11') starting from (5) are independent of the specific form of M_0 used. In fact, if we take the exact wave function Ψ in the form (2) and formally eliminate $\Psi(Q)$, we obtain another, *exact* set of coupled equations for $P_i\Psi_i$, analogous to (6),

$$P_i M_Q P_i \Psi_i = -P_i M_Q P_j \Psi_j , \tag{12}$$

where

$$\begin{aligned} &M_Q = M_0 + M_0 G^Q M_0 \\ &G^Q = -(Q M_0 Q)^{-1} . \end{aligned} \tag{13}$$

Thus, by replacing M_0 with M_Q and repeating the steps leading up to (11'), we have the desired orthogonality

$$(P_1\Psi_1, P_2\Psi_2) \sim 0 , \tag{4a}$$

which follows from $P_i P_j \Psi \sim 0$, as in (11).

In practice, the operator $M_0 G^Q M_0$ is usually impossible to evaluate accurately and (12) is useless. However, it is important to note that the exact form of M_Q is *not* necessary for the orthogonality (4a) to be valid. Any physically reasonable approximation for M_Q would be sufficient to maintain (4a). On the other hand, once a specific form for M_Q is chosen, the resulting equations of the form (12) should be solved *exactly without further approximations*. This is not often the case in the CCM analysis of nuclear reactions. The above discussion shows that, if the analysis is carried out correctly as indicated, the CCM should have an additional advantage over the distorted-wave Born

approximation in that (4a) is automatically satisfied. Furthermore, approximations are allowed *only* in the construction of M_Q, and thus further limits the number of parameters one can introduce in the usual CCM.[10]

(3) By reversing the procedure in the derivation of (12), we can first solve for the P components of Ψ formally and obtain an uncoupled equation for $\Psi(Q)$. We can then show, by exactly the same argument, that

$$(P_i\Psi, \Psi(Q)) \sim 0, \tag{4b}$$

which follows from $P_i\Psi(Q) = 0$; i.e., $\Psi(Q) = Q\Psi$. This result should also follow directly from the very definition of the operators P and Q, with the properties $PP_i = P_iP = P_i$ and $P_iQ = 0$. It is a nontrivial problem to evaluate the operators P and Q explicitly. In fact, we have shown in Ref. 6 that the actual forms for these operators are *not* needed in formulating the theory which incorporates the orthogonality property (4). When the operator $M_P \equiv M_0 + M_0G^PM_0$ analogous to M_Q of (13) is constructed, it can be shown that $P_iM_P = M_PP_i = 0$, so that $M_P = QM_PQ$. The function $\Psi(Q)$ can therefore be written without Q as $\Psi(Q) = -M_P^{-1}N_P$ where $N_P \equiv M_0\Psi^P$. This completes the proof of (4).

We have thus shown that if a set of coupled equations of the form (12), with M_Q replaced, perhaps, by its approximation is solved *exactly*, then the orthogonality relations (4) should automatically be satisfied. The orthogonality property is incorporated here into the operators M_j and the functions N_j, rather than orthogonalizing the wave functions $P_i\Psi_i$[11]; thus an explicit construction of the operators P and Q is not necessary. We further note that a separate adjustment of $P_iM_QP_i$ and $P_iM_QP_j$ in (12) will destroy (4a), for example, but an adjustment of M_Q *before* putting it into (12) will preserve the orthogonality. Obviously, the goodness of the solution $\Psi(P)$ depends directly on the proper choice of M_Q, but this has no effect on the property (4). Generalizations of the above considerations to cases with more than two coupled channels are straightforward. The present discussion also clarifies precisely the basis of the matrix formulation of the reaction theory given earlier.[12]

The recent result of an application[13] of the formalism of Ref. 6 to the positron-hydrogen scattering above the pickup threshold and positronium formation fully supports the properties (10) and, consequently, (4) as well. In this model (p, d) reactions, the $\Psi(Q)$ part was approximated variationally, taking into account the coupling of *all* the closed channels including those of breakup. A procedure along a similar line to the above has been discussed also by Ohmura *et al.*[14]

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