Off-Shell Continuations of the Two-Particle Transition Matrix with Bound States*

K. L. Kowalski,† J. E. Monahan,‡ C. M. Shakin,† and R. M. Thaler†
Case Western Reserve University, Cleveland, Ohio 44106
and Argonne National Laboratory, Argonne, Illinois 60439
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The general framework of a recent proposal for constructing classes of two-particle off-shell transition matrices consistent with given bound-state binding energies and a given set of phase shifts for all energies is presented. This establishes certain properties of the underlying potential which previously needed to be assumed. The large arbitrariness in the choice of bound-state wave functions and the lack of correlation, in general, between the choice of bound-state wave functions and the values of the binding energies are demonstrated. A specific method for actually executing the construction program is proposed which should be useful when there exists reason to prefer a particular bound-state wave function.

I. INTRODUCTION

Baranger *et al.*[1] have made some interesting discoveries in connection with the two-particle transition operator, at least in the case without bound states. First, they exhibited the conditions that the half-shell transition matrices must satisfy in order that the completely off-shell matrix, which is defined via the Low equation,[2-5] be a solution of a Lippmann-Schwinger equation with a Hermitian potential as a generating function.[6] Second, they demonstrated a definite degree of arbitrariness in the half-shell matrices consistent with these conditions and a specified on-shell behavior.[7] Finally, they proposed a definite calculational scheme for completely determining the half-shell matrix elements from the arbitrarily specified (symmetric) portion. All of this was formulated without the explicit introduction of a potential.

Recently, Haftel[8] has considered the extension of these results to the circumstance where bound states are present. One is then left with the formalism for generating classes of transition matrices consistent with a prescribed on-shell behavior and a given bound-state binding energy (or energies in the case of more than one bound state).[9]

The results of Haftel are correct. However, the development of Ref. 8 appears to involve "true" and "model" potentials in a somewhat confusing manner even though their introduction is supposedly heuristic. Specifically, Haftel assumes that the "true" and the "model" potential have the same bound-state wave function(s) and eigenvalue(s). Since the whole purpose of the technique is to avoid any explicit mention of potentials, this should be a consequence of the procedure and not an initial constraint upon it.

In Secs. II and III of the present work we will establish Haftel's results in a manner which avoids any explicit introduction of potentials and which demonstrates the over-all consistency of the formalism.[6] This will explicate, for example, the truly enormous arbitrariness in the choice of bound-state wave functions and the complete lack of correlation between the choice of bound-state wave function and the value of the binding energy.[9] This will also provide a framework for evaluating the consistency of proposals such as that of Amado[10] for actually exploiting this formalism. Finally, in Sec. IV we propose a specific method for carrying out the computational program implied by Haftel's work.

II. OFF-SHELL FORMALISM WITH A BOUND STATE

As noted by Baranger *et al.*[1] the crucial constraints upon the two-particle transition matrices follow from demanding the orthonormality and completeness of the scattering states. What this implies, as we will show in detail below, is that the scattering, both on and off shell, is generated by a Hermitian Hamiltonian with scattering states which satisfy a Schrödinger equation. This is far more than could be inferred merely from the assumptions of time-reversal invariance, off-shell unitarity, and analyticity.[11]

In the case with bound states we will require that the scattering states form a complete orthonormal set on the orthogonal complement $\mathcal{H}_Q$ of the space $\mathcal{H}_P$ spanned by the bound states.[12] Specifically, we demand that

$$\int dk'' \langle k' | \psi_{k''}^{(0)} \rangle \langle \psi_{k''}^{(0)} | k \rangle = \delta(k' - k) - g(k')g(k) \tag{2.1a}$$

and

$$\int dk'' \langle \psi_{k'}^{(0)} | k'' \rangle \langle k'' | \psi_{k}^{(0)} \rangle = \delta(k' - k), \tag{2.1b}$$

where

$$g(k) \equiv \langle k | \psi_b \rangle$$

is the bound-state wave function in momentum space and

$$\langle \psi_b | \psi_b \rangle = 1 .$$

The limits of integration in Eqs. (2.1) and on all other integrals in the remainder of the paper go from 0 to ∞. The states $|\psi_k^{(0)}\rangle$ are related to the usual states $|\psi_k^{(+)}\rangle$ (corresponding to outgoing-wave boundary conditions) by[1]

$$|\psi_k^{(0)}\rangle = e^{-i\delta(k)} |\psi_k^{(+)}\rangle ,$$

where $\delta(k)$ is the phase shift for the partial wave in question.

Equations (2.1) have several important consequences. The most obvious is that

$$\langle \psi_b | \psi_k^{(0)} \rangle = 0 . \tag{2.2}$$

Other consequences of Eqs. (2.1) are conveniently derived if we write the (real) momentum representatives $\langle k' | \psi_k^{(0)} \rangle$ of our scattering states in the form

$$\langle k' | \psi_k^{(0)} \rangle = \delta(k' - k) \cos\delta(k) + \mathrm{P} \frac{1}{k^2 - k'^2} \phi(k, k') , \tag{2.3}$$

where P denotes the principal-value singularity prescription, and the half-shell function $\phi(k, k')$ is constrained such that

$$\phi(k, k) = -\frac{2k}{\pi} \sin\delta(k) . \tag{2.4}$$

It then follows that necessary and sufficient conditions for the validity of Eqs. (2.1) are

$$\phi(k, k') \cos\delta(k) - \phi(k', k) \cos\delta(k') + \mathrm{P} \int dq \phi(q, k') \phi(q, k) \left(\frac{1}{q^2 - k^2} - \frac{1}{q^2 - k'^2} \right) = (k'^2 - k^2) g(k') g(k) \tag{2.5a}$$

and

$$\phi(k, k') \cos\delta(k') - \cos\delta(k) \phi(k', k) + \mathrm{P} \int dq \, \phi(k', q) \phi(k, q) \left(\frac{1}{q^2 - k^2} - \frac{1}{q^2 - k'^2} \right) = 0 , \tag{2.5b}$$

respectively.

To proceed any further we recall that our objective is to determine an off-shell transition matrix $\langle k' | T(z) | k \rangle$, where z is a (complex) parametric energy, which is consistent with an underlying Hamiltonian description of the scattering. We stipulate as our *definition* of $\langle k' | T(z) | k \rangle$:

$$\langle k' | T(z) | k \rangle \equiv \phi(k, k') \cos\delta(k) + \int dq \, \phi(q, k') \phi(q, k) \left(\frac{1}{z - q^2} - \mathrm{P} \frac{1}{k^2 - q^2} \right) + g(k') g(k) (k'^2 - \omega_b)(k^2 - z) \frac{1}{z - \omega_b} , \tag{2.6}$$

which form is obviously motivated by the usual formal expression

$$T(z) = V + V(z - H)^{-1} V , \tag{2.6'}$$

where H is the total Hamiltonian.[1,3,8] In Eq. (2.6) ω_b is a real number which we will eventually associate with the binding energy.

We note that Eq. (2.6) implies that

$$\langle k' | T(z^*) | k \rangle = \langle k' | T(z) | k \rangle^*$$

and

$$\langle k' | T(k^2 + i0) | k \rangle = \phi(k, k') e^{i\delta(k)} .$$

Also, if we define a potential by

$$\langle k' | V | k \rangle \equiv \lim_{|z| \to \infty} \langle k' | T(z) | k \rangle ,$$

we then find Eq. (2.6) that

$$\langle k' | V | k \rangle = \phi(k, k') \cos\delta(k) + \mathrm{P} \int dq \frac{\phi(q, k') \phi(q, k)}{q^2 - k^2} - (k'^2 - \omega_b) g(k') g(k) . \tag{2.7}$$

It is worth pointing out that our assumption of a constant limit for $\langle k' | T(z) | k \rangle$ as $|z|$ goes to infinity rules out from our consideration off-shell transition matrices of the type generated by singular-core interactions such as potentials containing hard cores. These kinds of interactions give rise to a behavior for $|z| \to \infty$ as a polynomial in z.

We now return to explore the consequences of Eqs. (2.5). It is clear that the completeness condition (2.5a) is also a necessary and sufficient condition that the operator V defined by (2.7) is Hermitian, viz.,

$$\langle k' | V | k \rangle = \langle k | V | k' \rangle ,$$

and therefore acceptable as a potential. In connection with this remark we note that (2.5a) along with Eq. (2.6) implies that

$$\langle k | T(k^2 + i0) | k' \rangle = \phi(k, k') e^{i\delta(k)}$$

and consequently that $\langle k' | T(z) | k \rangle$ as defined by (2.6) will satisfy the correct discontinuity relations across the right-hand cut in the z plane (off-shell unitarity).

Next, if we employ definitions (2.7), the orthonormality condition (2.5b), and Eq. (2.2) [which

is, in effect, implied by Eqs. (2.5)], it follows that

$$\phi(k,k') = \langle k' | V | \psi_k^{(0)} \rangle \,. \tag{2.8}$$

Equation (2.8) along with Eq. (2.3) imply that

$$H | \psi_k^{(0)} \rangle = k^2 | \psi_k^{(0)} \rangle \,, \tag{2.9a}$$

where

$$H = H_0 + V \,.$$

Here H_0 denotes the kinetic energy operator and V is defined by Eq. (2.7). It should be recalled at this point that we are working in a definite state of orbital angular momentum and all quantities including H_0 and V refer to this space.

We note that the orthogonality condition (2.2) can be rewritten in the form

$$\cos\delta(k)\, g(k) + \mathrm{P} \int dq \frac{\phi(k,q)\, g(q)}{k^2 - q^2} = 0 \,. \tag{2.10}$$

Then it follows, using definition (2.7), that corresponding to Eq. (2.9a) we also have

$$H | \psi_b \rangle = \omega_b | \psi_b \rangle \,. \tag{2.9b}$$

It is important to note at this point the arbitrariness of the binding energy ω_b for a fixed $| \psi_b \rangle$; any change in ω_b is merely reflected in a change in the potential.

We have shown that Eqs. (2.5) ensure the consistency of the off-shell scattering prescription (2.6) with an underlying Hamiltonian picture. Moreover, Eqs. (2.7)–(2.9) allow us to rewrite Eq. (2.6) in the operator form (2.6′), which automatically implies the usual Lippmann-Schwinger equations

$$T(z) = V + V \frac{1}{z - H_0} T(z) = V + T(z) \frac{1}{z - H_0} V \,. \tag{2.6''}$$

Equations (2.6′) and (2.6″) are the usual starting points for replacing the two-particle interaction V, when it occurs in composite-particle problems, by the off-shell transition operator $T(z)$. This is the last step in the explicit demonstration of the over-all consistency of the formalism of Refs. 1 and 8.

III. CONSTRUCTION OF CLASSES OF TRANSITION MATRICES

The problem at hand is to determine, using minimal information, the functions $\phi(k,k')$ subject to the constraint (2.4) for some given set of $\delta(k)$ such that the resultant $| \psi_k^{(0)} \rangle$ satisfy Eqs. (2.1) for a given $g(k)$. Our ultimate solution is the same as that of Haftel.[8] However, we believe our development will reveal the general scope of this formalism with no extraneous assumptions.

Let us begin with any normalized state[12] $| \psi_b \rangle$,

$$\langle \psi_b | \psi_b \rangle = 1 \,.$$

Any such state can eventually be associated with the bound state. Let $\{ | \chi_k \rangle \}$ be any complete orthonormal (in a Dirac-δ-function sense) set of states which span the space $\mathcal{H}_Q$. Thus, we suppose that

$$\int dk | \chi_k \rangle \langle \chi_k | = 1 - P \equiv Q \,, \tag{3.1a}$$

and

$$\langle \chi_{k'} | \chi_k \rangle = \delta(k' - k) \,, \tag{3.1b}$$

where

$$P \equiv | \psi_b \rangle \langle \psi_b | \,.$$

As was the case with Eqs. (2.1), we find from Eqs. (3.1) that

$$\langle \psi_b | \chi_k \rangle = 0 \,. \tag{3.2}$$

We confine ourselves to only those $| \chi_k \rangle$ which can be written in the (real) form

$$\langle k' | \chi_k \rangle = \delta(k' - k) \cos\eta(k) + \mathrm{P} \frac{1}{k^2 - k'^2} \beta(k,k') \,, \tag{3.3}$$

where

$$\beta(k,k) = -\frac{2k}{\pi} \sin\eta(k)$$

and $\eta(k)$ (the "model" phase shift) is a real function of k.

If our "true" scattering states $| \psi_k^{(0)} \rangle$ did satisfy Eqs. (2.1), corresponding to the situation outlined in connection with Eqs. (3.1) (i.e., the same $| \psi_b \rangle$), the operator U defined by

$$\langle k' | U | k \rangle \equiv \langle \chi_{k'} | \psi_k^{(0)} \rangle \tag{3.4}$$

would certainly be unitary. However, if we turn the argument around and demand that U be unitary, we find that this merely implies, assuming Eqs. (3.1) but not Eqs. (2.1), that

$$Q \left(\int dk | \psi_k^{(0)} \rangle \langle \psi_k^{(0)} | \right) Q = Q \tag{3.5a}$$

and

$$\langle \psi_{k'}^{(0)} | \psi_k^{(0)} \rangle = \langle \psi_{k'}^{(0)} | \psi_b \rangle \langle \psi_b | \psi_k^{(0)} \rangle + \delta(k' - k) \,. \tag{3.5b}$$

In order to obtain Eqs. (2.1) from Eqs. (3.5) we obviously need to impose Eq. (2.2). This is most conveniently done in the present context by demanding that

$$\langle k' | \psi_k^{(0)} \rangle = \int dk'' \langle k' | \chi_{k''} \rangle \langle \chi_{k''} | \psi_k^{(0)} \rangle \,. \tag{3.6}$$

Thus, assuming Eqs. (3.1), we obtain the desired relation, Eq. (2.1), by demanding the unitarity of U as defined by Eq. (3.4), and the validity of Eq. (3.6).

It should now be clear that the two potentials V and V_M which are *defined* by expressions like Eq. (2.7) in terms of $\phi(k,k')$ and $\beta(k,k')$, respectively, both generate via equations like (2.9b) the same eigenstate $|\psi_b\rangle$ with (any) prescribed eigenvalue ω_b. This remark obtains only after we have assumed the unitarity of U and Eq. (3.6). It is especially important to emphasize that it is quite unnecessary to know V_M beforehand.

Since, in general,

$$\langle \chi_{k'}|\psi_k^{(0)}\rangle = \int dk''\langle \chi_{k'}|k''\rangle\langle k''|\psi_k^{(0)}\rangle ,$$

we find, using Eqs. (2.3) and (3.3), that

$$\langle \chi_{k'}|\psi_k^{(0)}\rangle = \delta(k'-k)\cos[\delta(k)-\eta(k)] + \mathrm{P}\frac{1}{k^2-k'^2}\Phi(k,k') ,$$

where

$$\Phi(k,k') = \cos\eta(k')\,\phi(k,k') - \cos\delta(k)\beta(k',k) + \mathrm{P}\int dk''\left(\frac{1}{k'^2-k''^2}-\frac{1}{k^2-k''^2}\right)\beta(k',k'')\phi(k,k'') . \tag{3.7}$$

The only useful fact which follows from Eq. (3.7) is that

$$\Phi(k,k) = -\frac{2k}{\pi}\sin[\delta(k)-\eta(k)] . \tag{3.8}$$

From the definition (3.4) and the assumed unitarity of U, it follows from Ref. 1 that $\Phi(k,k')$ is determined by the knowledge of its symmetric part which, in turn, is subject to the constraint (3.8) but is otherwise arbitrary.

We note that Eq. (3.6) implies that

$$\phi(k,k') = \cos\eta(k')\,\Phi(k,k') + \cos[\delta(k)-\eta(k)]\beta(k,k') - \mathrm{P}\int dk''\left(\frac{1}{k'^2-k''^2}-\frac{1}{k^2-k''^2}\right)\beta(k'',k')\Phi(k,k'') \tag{3.9}$$

which is just a form of the two-potential formula.

The computational procedure to be followed is now clear. First one chooses some normalizable vector $|\psi_b\rangle$.[9] Then one constructs a convenient set of comparison states $|\chi_k\rangle$ satisfying Eqs. (3.1). (We will illustrate such a set in the next section in terms of an arbitrary $|\psi_b\rangle$.) Given the sets of phase shifts $\delta(k)$ and $\eta(k)$ we can specify a symmetric part of $\Phi(k,k')$ for all k,k'. The construction described by Baranger *et al.*[1] allows one to determine the complete $\Phi(k,k')$. This along with our given $\beta(k,k')$ determines $\phi(k,k')$ via Eq. (3.9). The choice of binding energy ω_b is still arbitrary. Fixing this we then have the full off-shell transition matrix using Eq. (2.6).

An important conclusion to be drawn from the preceding discussion is that without further restrictions on the entire problem almost nothing can be learned about which bound-state wave functions are preferred by nature. Also, since the association of binding energy with $|\psi_b\rangle$ is entirely arbitrary, we may have a violation of our usual connection of ω_b with the asymptotic behavior of $\langle r|\psi_b\rangle$ as $r\to\infty$. This last appears to be a special instance of the point made in Ref. 4, relating to the apparent difficulty in translating requirements on the tail of the (underlying) potential into restrictions on the off-shell elements.

IV. CONSTRUCTION OF THE COMPARISON STATES

In this section we will propose a specific choice for the set of comparison states $\{|\chi_k\rangle\}$ introduced in Sec. III. This set will be constructed explicitly in terms of a given bound state[12] $|\psi_b\rangle$. Consequently we will present relatively simple analytic expressions for the quantities $\beta(k,k')$ and $\eta(k')$.

The procedure we will follow amounts to the construction of a model potential in the spirit of the work of Haftel.[8] However, it should be kept in mind that the general problem at hand is that of determining an orthonormal set complete on the orthogonal complement of the space defined by a bound-state wave function(s).[13]

Let us define a model Hamiltonian[12]

$$H_M = H_0 + V_M , \tag{4.1a}$$

where

$$V_M \equiv QH_0Q - H_0 + \omega_b P . \tag{4.1b}$$

Clearly V_M is (formally) Hermitian and is such that

$$H_M|\psi_b\rangle = \omega_b|\psi_b\rangle . \tag{4.2}$$

In order for V_M to be well defined, however, it is necessary to impose somewhat more stringent conditions on $|\psi_b\rangle$ than have been required up to this point. For example, upon writing Eq. (4.1b) in coordinate space it becomes clear that smoothness and asymptotic conditions need to be imposed on the first two derivatives of $\langle r|\psi_b\rangle$. We suppose that these conditions are satisfied as they should be for any reasonable bound-state wave function. Moreover, we assume that this suffices for H_M to generate a complete orthonormal set of states

of the sort we are seeking.

Since we are looking for eigenstates of H_M orthogonal to $|\psi_b\rangle$ we need consider only

$$(k^2 - H_0 - \bar{V}_M)|\chi_k\rangle = 0, \tag{4.3}$$

where

$$\bar{V}_M = V_M - \omega_b P .$$

Any solution of Eq. (4.3) automatically satisfies Eq. (3.2). The solution $|\chi_k^{(+)}\rangle$ of Eq. (4.3) corresponding to an outgoing-wave boundary condition satisfies the integral equation

$$|\chi_k^{(+)}\rangle = |k\rangle + G_0^{(+)}(k^2)\bar{V}_M|\chi_k^{(+)}\rangle , \tag{4.4}$$

where

$$G_0^{(+)}(k^2) = (k^2 - H_0 + i0)^{-1} .$$

Using Eq. (3.2) we can solve Eq. (4.4) to obtain

$$|\chi_k^{(+)}\rangle = |k\rangle - \frac{G_0^{(+)}(k^2)|\psi_b\rangle\langle\psi_b|k\rangle}{\langle\psi_b|G_0^{(+)}(k^2)|\psi_b\rangle} . \tag{4.5}$$

From Eq. (4.5) we infer that

$$\cot\eta(k) = \frac{1}{(\pi/2k)g(k)^2}\,\mathrm{P}\int dk'\,\frac{g(k')^2}{k^2 - k'^2} \tag{4.6}$$

and that

$$|\chi_k\rangle \equiv e^{-i\eta(k)}|\chi_k^{(+)}\rangle = \cos\eta(k)\left[|k\rangle - G_0^P(k^2)\frac{|\psi_b\rangle\langle\psi_b|k\rangle}{\langle\psi_b|G_0^P(k^2)|\psi_b\rangle}\right], \tag{4.7}$$

where

$$G_0^P(k^2) = \mathrm{P}(k^2 - H_0)^{-1} .$$

Comparing Eqs. (3.3) and (4.7) we see that

$$\beta(k,k') = -\frac{g(k)g(k')\cos\eta(k)}{\langle\psi_b|G_0^P(k^2)|\psi_b\rangle} . \tag{4.8}$$

Equations (4.6) and (4.8) provide the necessary input for the calculational scheme outlined in the last section. We observe that all quantities here depend only upon the assumed bound-state wave function.

This last feature is the primary virtue of our method of constructing the comparison states. The phases $\eta(k)$ which are defined in this manner have no special properties. In contrast to this Amado[10] has advocated constructing the comparison states from some model potential which generates the "true" phases and the binding energy of one's choice. The present procedure would appear to be most useful in a context where one has some reason to prefer a particular bound-state wave function. Such a preference could arise in the nucleon-nucleon problem as a result of fits to deuteron form-factor data, for example. On the other hand, Amado's choice would appear to be most advantageous when one has at hand a phenomenological potential which may be viewed as defining the "true" on-shell scattering amplitudes.

Suppose that we decided to employ the two-potential technique to construct the "true" scattering states $|\psi_k^{(+)}\rangle$ in terms of our *ad hoc* states $|\chi_k\rangle$ as was done by Haftel[8]; the $|\psi_k^{(+)}\rangle$ satisfy

$$(k^2 - H_0 - V)|\psi_k^{(+)}\rangle = 0 ,$$

where V is the "true" interaction. One can do this in two ways by writing

$$V = V_M + V_1 \tag{4.9a}$$

or

$$V = \bar{V}_M + V_2 . \tag{4.9b}$$

With the first choice [Eq. (4.9a)], we have

$$|\psi_k^{(+)}\rangle = |\chi_k^{(+)}\rangle + G_M^{(+)}(k^2)V_1|\psi_k^{(+)}\rangle , \tag{4.10a}$$

where

$$G_M^{(+)}(k^2) = (k^2 - H_0 - V_M + i0)^{-1} .$$

With the decomposition [Eq. (4.9b)], we have

$$|\psi_k^{(+)}\rangle = |\chi_k^{(+)}\rangle + \bar{G}_M^{(+)}(k^2)V_2|\psi_k^{(+)}\rangle , \tag{4.10b}$$

where

$$\bar{G}_M^{(+)}(k^2) = (k^2 - H_0 - \bar{V}_M + i0)^{-1} .$$

The problem with Eqs. (4.10) is that $|\psi_k^{(+)}\rangle$ is not necessarily orthogonal to $|\psi_b\rangle$, since the latter is presumably an arbitrary normalizable vector. One can ensure Eq. (2.2) if and only if

$$(H_0 + V)|\psi_b\rangle = 0 . \tag{4.11}$$

It was just this sort of *a priori* assumption that the treatment of Secs. II and III completely avoids.

Actually, Eqs. (4.10) are quite useless for the construction procedure of Sec. III. However, they may be of considerable interest in other contexts[13] where one really does know V beforehand and $|\psi_b\rangle$ is fixed and satisfies Eq. (4.11). Then, for example, Eqs. (4.10) could be exploited in a perturbative manner. In this regard it is interesting to point out that if Eq. (4.11), and therefore Eq. (2.2), is valid, both of Eqs. (4.10) reduce to

$$|\psi_k^{(+)}\rangle = |\chi_k^{(+)}\rangle + \tilde{G}_M^{(+)}(k^2)V|\psi_k^{(+)}\rangle , \tag{4.12}$$

where

$$\tilde{G}_M^{(+)}(k^2) = \int dk'\,\frac{|\chi_{k'}\rangle\langle\chi_{k'}|}{k^2 - k'^2 + i0} .$$

In this case it is easily seen that Eq. (4.12) has the same content as Eqs. (2.3) and (3.9).

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†Present address: Case Western Reserve University, Cleveland, Ohio.

‡Argonne National Laboratory, Argonne, Illinois.

[1]M. Baranger, B. Giraud, S. K. Mukhopadhyay, and P. U. Sauer, Nucl. Phys. A138, 1 (1969). We will usually follow the notations and normalization conventions of this reference.

[2]The observation that the full off-shell matrix is determined entirely by the half-shell matrices via the Low equation was first made by H. P. Noyes, in *Proceedings of the International Conference on Polarized Targets and Ion Sources, Saclay, 1966* (La Documentation Francaise, Paris, France, 1967), p. 309; see also, in *Progress in Nuclear Physics,* edited by D. M. Brink and J. H. Mulvey (Pergamon Press Ltd., London, England, 1969), Vol. 10, p. 355. In this respect we should clarify a persistent confusion with regard to off-shell extensions. Obviously there exist an infinity of "transition" operators which give rise to the same on-shell scattering. However, once one has settled upon a particular formalism (for the description of some sort of composite system) in which two-particle scattering occurs under nonisolated circumstances, then a definite prescription for the virtual (off-shell) scattering can be specified. In this context the expression $T(z) = V + V(z - H)^{-1}V$ has proven to be the most useful. Any ambiguities in off-shell extensions then amount to different choices of spectrally and phase-shift equivalent Hamiltonians [see F. Coester, S. Cohen, B. Day, and C. M. Vincent, Phys. Rev. C 1, 769 (1970)]. For examples of ambiguities of the first type which arise for the three-particle scattering operators see T. A. Osborn and K. L. Kowalski, to be published.

[3]Noyes, Ref. 2.

[4]Coester, Cohen, Day, and Vincent, Ref. 2.

[5]Osborn and Kowalski, Ref. 2.

[6]We should remark that some of these points which relate to the consistency of the formalism are not proven explicitly in Ref. 1. They are established in the present paper in the general case with bound states.

[7]This remark may appear inconsistent with the discussion in Ref. 2. The philosophy of Ref. 1 is to construct on-shell equivalent transition matrices consistent with a particular type of off-shell extension from the knowledge of the half-shell matrices. The nonuniqueness referred to here is involved with the degree of arbitrariness in the specification of the half-shell matrices which is consistent with all the constraints we have discussed. This is the precise counterpart of the liberty in choosing unitary transformation which generate spectrally and phase-equivalent Hamiltonians (see Coester, Cohen, Day, and Vincent, Ref. 2).

[8]M. I. Haftel, Phys. Rev. Letters 25, 120 (1970).

[9]In a typical physical situation, such as the nucleon-nucleon problem, additional moment constraints will also be placed on the bound-state wave function(s).

[10]R. D. Amado, Phys. Rev. C 2, 2439 (1970).

[11]See K. L. Kowalski, Phys. Rev. 144, 1239 (1966) and T. R. Mongan, *ibid.* 184, 1888 (1969).

[12]For the sake of simplicity we will suppose that there is only one bound state, which we denote by $|\psi_b\rangle$. We will, of course, confine ourselves to a state of fixed orbital angular momentum. If there are several bound states in a given state of angular momentum these must, of course, be taken to be mutually orthogonal.

[13]Similar considerations can be found in N. Auerbach, J. Hüfner, A. K. Kerman, and C. M. Shakin, to be published; W. L. Wang and C. M. Shakin, Phys. Letters 32B, 421 (1970); and M. M. Stingl and M. W. Kirson, Nucl. Phys. 137, 283 (1969).