Rev. 153, 1114 (1967).
[15]E. L. Lomon and M. McMillan, Ann. Phys. (N.Y.) 23, 439 (1963).
[16]The properties of nuclear matter as a function of the interior BC have recently been studied with the complete BC-model interaction. E. L. Lomon, private communication.
[17]D. D. Brayshaw and B. Buck, Phys. Rev. Letters 24, 733 (1970).
[18]T. Janssens, R. Hofstadter, E. B. Hughes, and M. R. Yearian, Phys. Rev. 142, 922 (1966).
[19]H. Collard, R. Hofstadter, E. B. Hughes, A. Johansson, and M. R. Yearian, Phys. Rev. 138, B57 (1965).
[20]C. de Vries, R. Hofstadter, A. Johansson, and R. Herman, Phys. Rev. 134, B848 (1948).
[21]J. S. McCarthy, I. Sick, R. R. Whitney, and M. R. Yearian, Phys. Rev. Letters 25, 884 (1970).
[22]V. A. Alessandrini and C. A. Garcia Canal, Nucl. Phys. A133, 590 (1969).
[23]T. Brady, E. Harms, L. Laroze, and J. S. Levinger, Phys. Rev. C 2 59 (1970).

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Multiple Scattering of Pions by Deuterons*

W. R. Gibbs
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico 87544
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The general theory of multiple scattering of pions from nuclei is expressed in a way which does not require the use of a series expansion. In an on-shell approximation this theory may be reduced to the solution of a set of A coupled integral equations. As a demonstration of the method the equations are specialized to the case of the deuteron and solved to give comparison with the experimental data. The deuteron tensor T_{20} is shown to be sensitive to the percentage of D state for deuterons scattered at 0°.

I. INTRODUCTION

In the energy region of 0–500-MeV pion kinetic energy there exists no completely acceptable theory of pion-nucleus scattering. The high-energy eikonal approximation due to Glauber[1,2] has been successful in explaining high-energy scattering data at small angles. In spite of a number of attempts to increase its range of validity,[3] it is doubtful if this basic approach would be of value for large angles in this energy range.

The Watson multiple-scattering series has been used with some success for pion-deuteron scattering by evaluating the double-scattering term in various approximations.[4] However, it is difficult to go beyond the second term and, furthermore, for large nuclei and near a resonance it is not certain that the series even converges.

Optical-model calculations[5] have been the most successful in this region, but their application is limited to large-A nuclei. Their use near a resonance also has questionable validity.

For these reasons the present paper attempts to develop a theory with the following aims:
(i) There should be no small-angle or forward-scattering approximations, as these are not appropriate for this energy regime.
(ii) There should be no truncation of the multiple-scattering series to avoid questions of convergence near a resonance.
(iii) There should be only on-shell information required (at least in the first-order theory) to make the calculation as simple as possible.

In order to develop such a theory we shall use the form of multiple-scattering theory used some years ago by Foldy and Brueckner[6] and more recently by Seki.[7] This method expresses the multiple-scattering amplitude from one of the nucleons as the simple-scattering amplitude plus a term which looks very much like a double-scattering amplitude. The difference between this second term and an actual double-scattering term is that one of the simple amplitudes has been replaced by the multiple-scattering amplitude. Thus, one has implicit equations for the multiple-scattering amplitudes. With the aid of some approximations these equations can be brought into a solvable form.

These general equations are developed in Sec. II and it will be seen that they may be written as coupled integral equations over angular variables. In Sec. III the equations are specialized to the case of the deuteron and solved in double scattering at high energies to compare with Glauber theory. Here also in Sec. III the case of π-deuteron scattering just below the (3, 3) resonance is calculated and compared with experiment.

II. THEORY

Let us consider the scattering from A fixed nucleons with position coordinates $\vec{r}_i (i=1,2,\ldots,A)$. The scattered part of the wave function can be represented as a superposition of scattered waves from each particle.

$$\Psi(\vec{k},\vec{r})=\sum_{i=1}^{A}\chi_i(\vec{k},\vec{r}-\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_i} . \tag{1}$$

Each χ_i is the result of many scatterings, and the label i denotes the *last* particle scattering. Note also that each χ_i depends on all of the coordinates of all of the nucleons. The translation of argument and phase factor come about because the functional form of the χ_i is expressed in terms of the relative pion-nucleon coordinate.

Now define an operator t_i such that

$$t_i e^{i\vec{k}\cdot\vec{r}}=\varphi(\vec{k},\vec{r}-\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_i} , \tag{2}$$

where $\varphi(\vec{k},\vec{r})$ is the scattered part of the π-nucleon wave function. One could construct an explicit representation of such an operator as follows: Let us suppose that we have an analytic form for the π-nucleon scattering wave function. A simple multiplicative form for the operator would then be

$$t=\varphi(\vec{k},\vec{r})e^{-i\vec{k}\cdot\vec{r}} . \tag{3}$$

This operator contains $\vec{k}$ as a parameter and thus is not acceptable. Since t is to operate on a plane wave, we need only replace $\vec{k}$ by an operator with $\vec{k}$ as its eigenvalue. Thus an explicit form for the operator t is

$$t=\lim_{\vec{r}'\to\vec{r}}\varphi(-i\vec{\nabla},\vec{r}')e^{-\vec{\nabla}\cdot\vec{r}} . \tag{4}$$

We may note also that since the operator may be constructed from the scattered wave function, we may define a similar operator for the χ_i:

$$T_i e^{i\vec{k}\cdot\vec{r}}=\chi_i(\vec{k},\vec{r}-\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_i} . \tag{5}$$

Since any wave may be written as a linear superposition of plane waves, these operators give the scattering from an arbitrary incoming wave.

We are now in a position to write down the self-consistency equations for multiple scattering. The scattering of the wave from the particle at position i will be due to the incoming plane wave plus the incoming waves scattered from all of the other particles.

$$\chi_i(k,\vec{r}-\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_i}=t_i[e^{i\vec{k}\cdot\vec{r}}+\sum_{j\neq i}\chi_j(\vec{k},\vec{r}-\vec{r}_j)e^{i\vec{k}\cdot\vec{r}_j}] . \tag{6}$$

In terms of operators assumed to be operating on plane waves,

$$T_i=t_i+t_i\sum_{j\neq i}T_j . \tag{7}$$

Since the function $\Psi(k,r)$ is given by the sum of the T_i, it is interesting to solve the above equation to obtain

$$T\equiv\sum T_i=[1-\sum t_i(1+t_i)^{-1}]-1 . \tag{8}$$

The expansion of the above expression has the same form as the Watson multiple-scattering series and may be identified with it.

If we express χ_j as a Fourier transform we may write Eq. (6) as

$$\chi_i(\vec{k},\vec{r}-\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_i}=\varphi_i(\vec{k},\vec{r}-\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_i}+\frac{e^{i\vec{k}\cdot\vec{r}_j}}{(2\pi)^3}\sum_{j\neq i}\int d\vec{p}\,\varphi_i(\vec{p},\vec{r}-\vec{r}_i)e^{i\vec{p}\cdot(\vec{r}_i-\vec{r}_j)}\chi_j(\vec{k},\vec{p}) , \tag{9}$$

where

$$\chi_j(\vec{k},\vec{p})\equiv\int d\vec{r}\,e^{-i\vec{p}\cdot\vec{r}}\chi_j(\vec{k},\vec{r}) .$$

We can rewrite Eq. (9) as

$$\chi_i(k,\vec{r}-\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_i}=\varphi_i(\vec{k},\vec{r}-\vec{r}_i)e^{i\vec{k}\cdot\vec{r}_i}+\frac{e^{i\vec{k}\cdot\vec{r}_j}}{(2\pi)^3}\sum_{j\neq i}\int d\Omega_p\int p^2dp\,\varphi_i(\vec{p},\vec{r}-\vec{r}_i)e^{i\vec{p}\cdot(\vec{r}_i-\vec{r}_j)}\chi_j(\vec{k},\vec{p}) . \tag{10}$$

We may do the integral on p by contour integration. If $\cos(\vec{p},\vec{r}_i-\vec{r}_j)$ is positive (negative) the contour C_1 (C_2), shown in Fig. 1, is to be used. This prevents the exponential factor from causing the integrand to diverge for $\mathrm{Im}(p)$ large. It is clear that the only contributions which remain in the limit of large r are those coming from singularities in the complex plane located at $\mathrm{Re}(p)=k$, since otherwise the asymptotic form will be wrong. Since $\chi_j(\vec{k},\vec{p})$ is the only function in the integrand that has reference to $\vec{k}$, we may evaluate the integral by examining the analytic structure of χ_j as a function of p.

Let us first make a partial-wave expansion of $\chi(\vec{k},\vec{r})$ (by which we mean any one of the χ_i):

$$\chi(\vec{k},\vec{r})=2\pi\sum i^l Y_l^m(\hat{k})Y_l^{m*}(\hat{r})\Psi_l(k,r)f_l , \tag{11}$$

where

$$\Psi_l(k,r)\to h_l^{(+)}(kr) , \tag{12}$$

since $\chi(k,r)$ has only outgoing waves. Now we define an auxiliary function

$$\chi^a(\vec{k},\vec{r})\equiv 2\pi\sum i^l Y_l^m(\hat{k})Y_l^{m*}(\hat{r})h_l^{(+)}(kr)f_l , \tag{13}$$

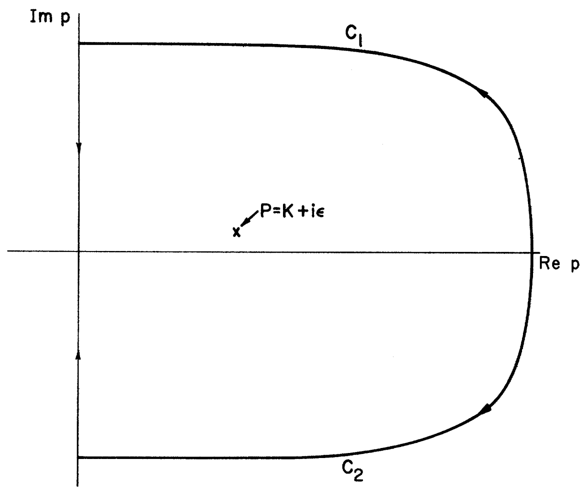

FIG. 1. Contours used in the integration in Eq. (10).

and rewrite Eq. (9) as

$$\chi(\vec{k},\vec{p})=\int d\vec{r}\,\chi^a(\vec{k},\vec{r})e^{i\vec{p}\cdot\vec{r}}+\int d\vec{r}\,[\chi(\vec{k},\vec{r})-\chi^a(\vec{k},\vec{r})]e^{i\vec{p}\cdot\vec{r}} \tag{14}$$

$$=\chi^a(\vec{k},\vec{p})+\chi^{NZ}(\vec{k},\vec{p})\ , \tag{15}$$

where χ^{NZ} is the contribution from the near-zone part of the wave function.

Evaluating $\chi^a(\vec{k},\vec{p})$ we find

$$\chi^a(\vec{k},\vec{p})=\frac{4\pi}{(p+k+i\epsilon)(p-k-i\epsilon)}\tilde{F}(\vec{k},\vec{p})\ , \tag{16}$$

where

$$\tilde{F}(\vec{k},\vec{p})=\frac{2\pi}{ik}\sum Y_l^m(\hat{k})Y_l^{m*}(\hat{p})f_l\left(\frac{p}{k}\right)^l\ . \tag{17}$$

For $|\vec{p}|=k$ we see that $\tilde{F}(\vec{k},\vec{p})=\hat{F}(\vec{k},\vec{p})$, the scattering amplitude of $\chi(\vec{k},\vec{r})$.

From the form of Eq. (16) it is easily seen that χ^a gives the contribution of one pole at $p=k+i\epsilon$ when $C1$ is used and zero when $C2$ is used. The evaluation of the contribution of this pole requires only knowledge of the on-shell π-nucleon scattering amplitude.

We must now consider the contributions from the singularities of χ^{NZ}. The relevant singularities are found along the line through the points $p=k+i\mu, p=k+2i\mu,\ldots$, etc., where μ is the inverse range of the basic force involved. The nature of the singularities depends on the form of the pion-nucleon interaction. In general this contribution can be represented by a branch cut from $p=k+i\mu$ to $p=k+i\infty$ with proper modification of the contour. This cut is actually the "left-hand cut."

While the study of the contribution of this cut to the integral is bound to be of interest, it is beyond the scope of this paper, since the aim is to use only on-shell information. For this reason the existence of this cut will be neglected and only the contribution of the pole of χ^a will be retained.

We may simplify the equations by writing

$$F_i(\vec{k},\vec{k}')=\hat{F}_i(\vec{k},\vec{k})e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_i}.$$

Combining Eqs. (16) and (10) with the above approximation we obtain

$$F_i(\vec{k},\vec{k}')=f_i(\vec{k},\vec{k}')e^{i\vec{r}_i\cdot(\vec{k}-\vec{k}')}+\frac{ik}{2\pi}e^{-i\vec{k}'\cdot\vec{r}_i}\sum_{j\neq i}\int d\Omega_p\,f_i(\vec{p},\vec{k}')e^{i\vec{p}\cdot\vec{r}_i}\,\theta(\vec{p}\cdot(\vec{r}_i-\vec{r}_j))F_j(\vec{k},\vec{p})\ , \tag{18}$$

$$F(\vec{k},\vec{k}')=\sum F_i(\vec{k},\vec{k}')\ . \tag{19}$$

After these equations are solved for the $F_i(\vec{k},\vec{k}')$ (which are functions of all of the nuclear coordinates) the expectation value on the ground state of the nucleus must be taken to get the elastic scattering amplitude.

Note that the first term on the right side gives the impulse (single scattering) approximation. If we replace F_j by f_j in the integrand we would get a double-scattering approximation, and in principle one can continue to iterate this equation. However, this process may not converge for large A in a resonance region. For the few-nucleon problem it is possible to solve Eqs. (19) as they stand. For a large number of nucleons statistical approximations may be made to make the equations more tractable.

III. THE DEUTERON

A. General Formulation

For this case there are two equations only and they may be reduced to one (three-dimensional) coordinate by removing the center of mass.

$$\vec{r}_1=\vec{R}+\tfrac{1}{2}\vec{r},\quad \vec{r}_2=\vec{R}-\tfrac{1}{2}\vec{r},\quad F_i(\vec{k},\vec{k}')=H_i(\vec{k},\vec{k}')e^{i\vec{R}\cdot(\vec{k}-\vec{k}')}\ .$$

Thus, we have

$$H_1(\vec{k},\vec{k}')=f_1(\vec{k},\vec{k}')e^{i\vec{r}\cdot(\vec{k}-\vec{k}')/2}+\frac{ik}{2\pi}e^{-i\vec{k}'\cdot\vec{r}/2}\int d\Omega_p f_1(\vec{p},\vec{k}')H_2(\vec{k},\vec{p})e^{i\vec{r}\cdot\vec{p}/2}\theta(\vec{p}\cdot\vec{r}),$$
$$H_2(\vec{k},\vec{k}')=f_2(\vec{k},\vec{k}')e^{-i\vec{r}\cdot(\vec{k}-\vec{k}')/2}+\frac{ik}{2\pi}e^{i\vec{k}'\cdot\vec{r}/2}\int d\Omega_p f_2(\vec{p},\vec{k}')H_1(\vec{k},\vec{p})e^{-i\vec{r}\cdot\vec{p}/2}\theta(-\vec{p}\cdot\vec{r}). \quad (20)$$

Note that one can always obtain the equation for H_2 from the one for H_1 by the substitution

$1\leftrightarrow 2,\quad \vec{r}\leftrightarrow -\vec{r}$.

B. High-Energy *p-d* Scattering

For this case Glauber's high-energy approximation has been found to give a good account of the data.[2] Thus, in this region it is only necessary to compare the present work with the Glauber theory. For the extremely forward-peaked nucleon-nucleon amplitudes used in the Glauber calculations only double scattering is important, and for this case Eqs. (20) give

$$H(\vec{k},\vec{k}')=H_1(\vec{k},\vec{k}')+H_2(\vec{k},\vec{k}')=f_1(\vec{k},\vec{k}')e^{i\vec{r}\cdot(\vec{k}-\vec{k}')/2}+f_2(\vec{k},\vec{k}')e^{-i\vec{r}\cdot(\vec{k}-\vec{k}')/2}$$
$$+\frac{ik}{2\pi}\int d\Omega_p[f_1(\vec{p},\vec{k}')f_2(\vec{k},\vec{p})e^{i\vec{r}\cdot(\vec{k}+\vec{k}')/2}e^{i\vec{r}\cdot\vec{p}}\theta(\vec{p}\cdot\vec{r})+f_2(\vec{p},\vec{k}')f_1(\vec{k},\vec{p})e^{i\vec{r}\cdot(\vec{k}+\vec{k}')/2}e^{-i\vec{r}\cdot\vec{p}}\theta(-\vec{p}\cdot\vec{r})]. \quad (21)$$

Defining $\kappa=(\vec{k}+\vec{k}')/2$ and setting $f_1=f_2$ the expectation value of the scattering amplitude becomes

$$F_D(\vec{k},\vec{k}')=f_I(\vec{k},\vec{k}')+\frac{ik}{2\pi}\int d\Omega_p f(\vec{p},\vec{k}')f(\vec{k},\vec{p})\,Q(\vec{p},\vec{\kappa}), \quad (22)$$

where

$$Q(\vec{p},\vec{\kappa})\equiv\langle e^{i\vec{r}\cdot\vec{p}-i\vec{r}\cdot\vec{\kappa}}\theta(\vec{r}\cdot\vec{p})+e^{-i\vec{r}\cdot\vec{p}+i\vec{r}\cdot\vec{\kappa}}\theta(-\vec{r}\cdot\vec{p})\rangle=\langle e^{i\vec{r}\cdot(\vec{p}-\vec{\kappa})}\rangle-2i\langle\sin[\vec{r}\cdot(\vec{p}-\vec{\kappa})]\theta(-\vec{r}\cdot\vec{p})\rangle,$$

and $f_I(\vec{k},\vec{k}')$ is the impulse-approximation amplitude.

With the inclusion of forward-scattering and small-angle approximations (see appendix A) this expression can be transformed to

$$f(\vec{k},\vec{k}')\approx f_I+\frac{i}{2\pi k}\int d^2q'f\left(\vec{q}'+\frac{\vec{q}}{2}\right)f\left(\frac{\vec{q}}{2}-\vec{q}'\right)S(\vec{q}'), \quad (23)$$

which is the Glauber expression. In order to compare the two theories at larger angles, a comparison of Eqs. (22) and (23) has been made with

$$f(q)=\frac{ik\sigma_N}{4\pi}e^{-\alpha^2q^2/2},\quad S(q)=e^{-q^2/8\beta^2},$$

which were used in some of the comparisons with data. The two curves are shown in Fig. 2. As can be seen the two are essentially identical for $-t \lesssim 0.6\ (\text{GeV}/c)^2$. Fits to the data have been made out to $-t\sim 1.5\ (\text{GeV}/c)^2$ at which the curves are almost the same.

We conclude that, in the region where Glauber theory is valid, there is essentially no difference between that theory and the present equations. Further, since we have made no small-angle approximations the present equations should be valid at larger angles as well.

C. Low-Energy *π-d* Scattering

For low-energy pions we may represent the π-nucleon scattering amplitude in terms of the phase shifts. In the actual calculations done in this section, only *s* and *p* waves will be used, although it is a simple matter to extend the calculations through a few more waves.

Since we wish to consider energies in the resonance region it will be necessary to solve Eqs. (20) without further approximation (i.e., we may not restrict to double scattering). To this end it would be useful to expand the $H_i(\vec{k},\vec{p})$ in spherical harmonics in the direction $\hat{p}$. If we first make a simple transformation this expansion will be greatly simplified:

$$G_1(\vec{k},\vec{k}')=H_1(\vec{k},\vec{k}')e^{i\vec{r}\cdot\vec{k}'/2},$$
$$G_2(\vec{k},\vec{k}')=H_2(\vec{k},\vec{k}')e^{-i\vec{r}\cdot\vec{k}'/2}.$$

Thus we have

$$G_1(\vec{k},\vec{k}')=f_1(\vec{k},\vec{k}')e^{i\vec{r}\cdot\vec{k}/2}+\frac{ik}{2\pi}\int d\Omega_p f_1(\vec{p},\vec{k}')G_2(\vec{k},\vec{p})e^{i\vec{r}\cdot\vec{p}}\theta(\vec{p}\cdot\vec{r}),$$
$$G_2(\vec{k},\vec{k}')=f_2(\vec{k},\vec{k}')e^{-i\vec{r}\cdot\vec{k}/2}+\frac{ik}{2\pi}\int d\Omega_p f_2(\vec{p},\vec{k}')G_1(\vec{k},\vec{p})e^{-i\vec{r}\cdot\vec{p}}\theta(-\vec{p}\cdot\vec{r}). \quad (24)$$

Note that the only dependence on $\vec{k}'$ comes from f_1 and f_2, so one needs only as many partial waves in the expansion of G_i as there are in the π-nucleon amplitude.

As an example of the method let us consider the simple case of only s-wave scattering the the π-nucleon interaction. In this case both f_1 and f_2 are independent of $\vec{k}'$ and thus so are G_1 and G_2.

$$G_1 = f_1 e^{i\vec{r}\cdot\vec{k}/2} + \frac{ik}{2\pi} f_1 G_2 \int d\Omega_p e^{i\vec{r}\cdot\vec{p}} \theta(\vec{p}\cdot\vec{r}) ,$$

$$G_2 = f_2 e^{-i\vec{r}\cdot\vec{k}/2} + \frac{ik}{2\pi} f_2 G_1 \int d\Omega_p e^{-i\vec{r}\cdot\vec{p}} \theta(-\vec{p}\cdot\vec{r}) , \tag{25}$$

$$\begin{aligned} g(\vec{r}) &\equiv \frac{1}{4\pi}\int d\Omega_p e^{i\vec{r}\cdot\vec{p}}\theta(\vec{p}\cdot\vec{r}) \\ &= \tfrac{1}{2} e^{ikr/2} j_0\left(\frac{kr}{2}\right) \\ &= g(r) . \end{aligned} \tag{26}$$

Thus

$$G_1 = \frac{f_1 e^{i\vec{r}\cdot\vec{k}/2} + 2ikf_1f_2e^{-i\vec{r}\cdot\vec{k}/2}g(r)}{1-(2ik)^2 f_1 f_2 g^2(r)}$$

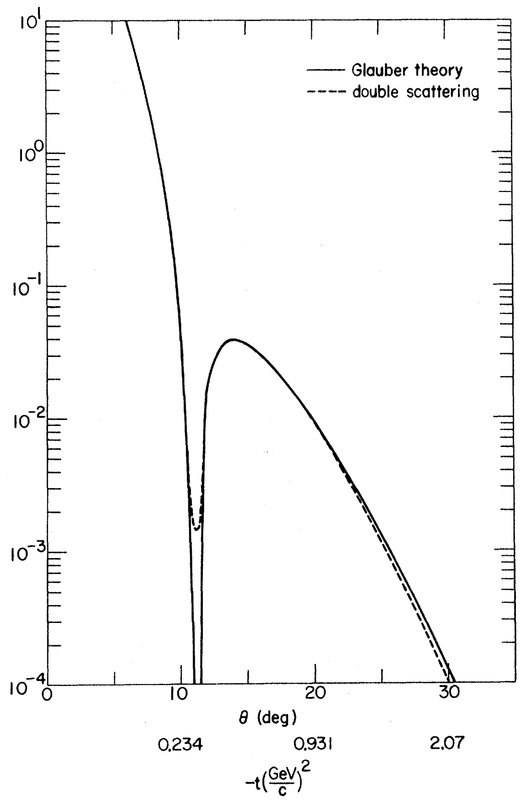

FIG. 2. Comparison of the present method with the Glauber theory for p-d scattering at 2.0 GeV. The deep dip around 12° is present because the D state of the deuteron was not included in these calculations.

and

$$H = e^{-i\vec{r}\cdot\vec{k}'/2}G_1 + e^{i\vec{r}\cdot\vec{k}'/2}G_2$$

$$= \frac{f_1 e^{i\vec{r}\cdot\vec{q}/2} + f_2 e^{-i\vec{r}\cdot\vec{q}/2} + 2ikf_1f_2g(r)(e^{i\vec{r}\cdot\vec{\kappa}} + e^{-i\vec{r}\cdot\vec{\kappa}})}{1-(2ik)^2 f_1 f_2 g^2(r)} .$$

For a spherical deuteron wave function the expectation of H is given by

$$\overline{H} = \left\langle \frac{(f_1+f_2)j_0(\frac{1}{2}qr) + 4ikf_1f_2g(r)j_0(\kappa r)}{1-(2ik)^2 f_1 f_2 g^2(r)} \right\rangle_r . \tag{27}$$

The first term in the numerator alone is just the single-scattering impulse approximation, the second term is double scattering, and the denominator gives a renormalization of both due to multiple scattering. This exercise has little physical relevance for the π-d problem and was only included to give the reader a feeling for the calculation.

A realistic calculation has been performed by including both s and p waves in the π-nucleon amplitudes in Eq. (24). We should expect this calculation to be reasonable for the kinetic energy of the π greater than about 50 MeV (below this Fermi momentum, which has been ignored throughout, becomes very important) and less than about 200 MeV (above which d waves become important). Since the theory outlined above is essentially a spin-independent theory and spin is important, the effects of spin must be included in an approximate way. The D state of the deuteron was included in the calculation but not always in an exact way. A list of the approximations made follows:

(i) The deuteron nonspin-flip amplitudes (T_{11}, T_{-1-1}, T_{00}) were computed by the method given above including the D state and assuming fixed nucleon spin (no spin flip).

(ii) Two contributions were considered to the spin-flip amplitudes. The first was due to the D state with fixed nucleon spins. The second contribution was due to the π-nucleon spin flip. Both of

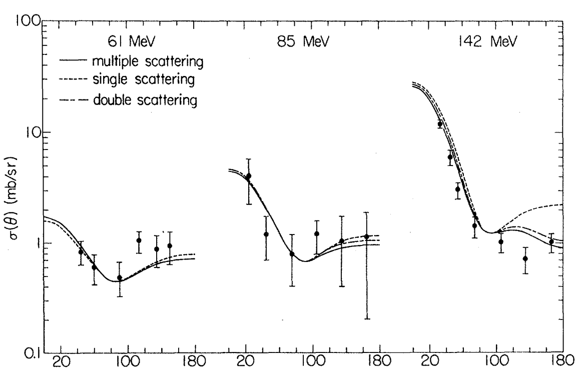

FIG. 3. Comparison of single, double, and multiple scattering with low-energy π-d data.

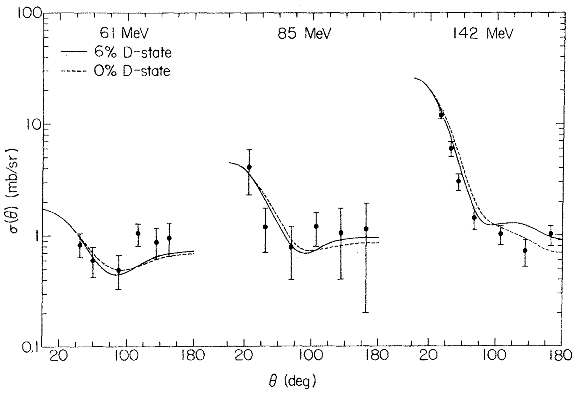

FIG. 4. The curves shown demonstrate the sensitivity to the amount of D state included in the deuteron.

these contributions were estimated by single-scattering impulse approximation.

The error due to these approximations is believed to be very small. Of course, this includes only the approximations just mentioned. The error due to the neglect of Fermi momentum and off-shell effects is a separate matter. Appendix B gives the details of the calculation.

Figure 3 shows the results of calculations made at 61, 85, and 142 MeV compared with data.[8-10] The dotted curve represents the single scattering, the dash-dot curve the double-scattering approxi-

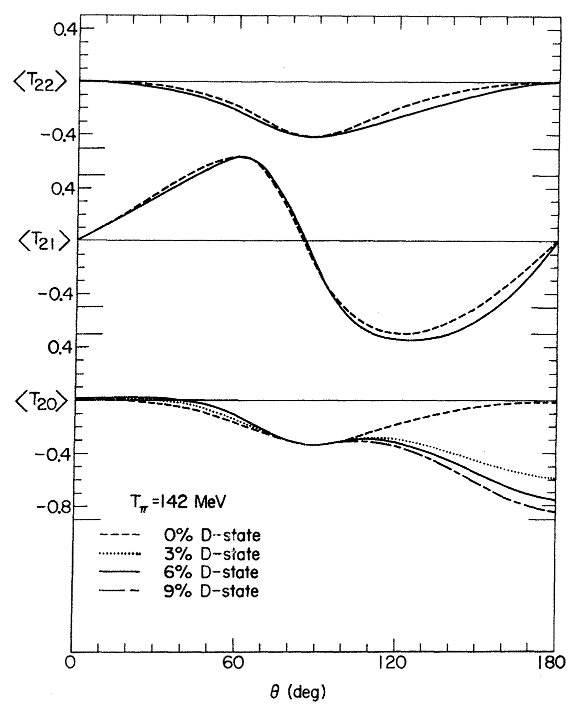

FIG. 5. The tensor polarization of the recoil deuteron. The tensor $-i\langle T_{11}\rangle$ is nonzero but its magnitude was very small and hence was not plotted.

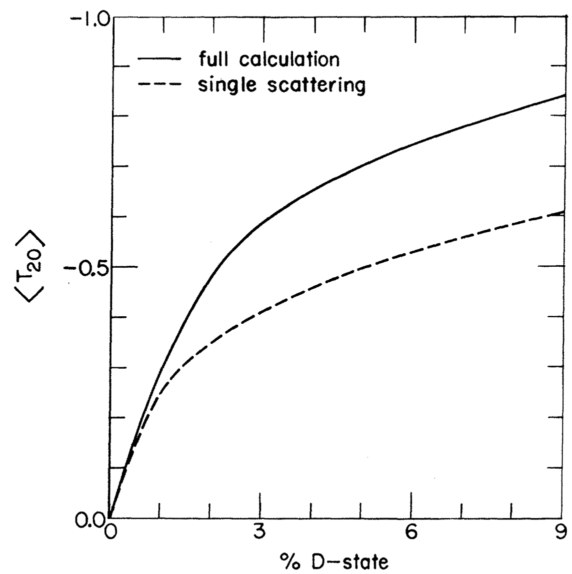

FIG. 6. The tensor $\langle T_{20}\rangle$ as a function of the percent D state in the deuteron.

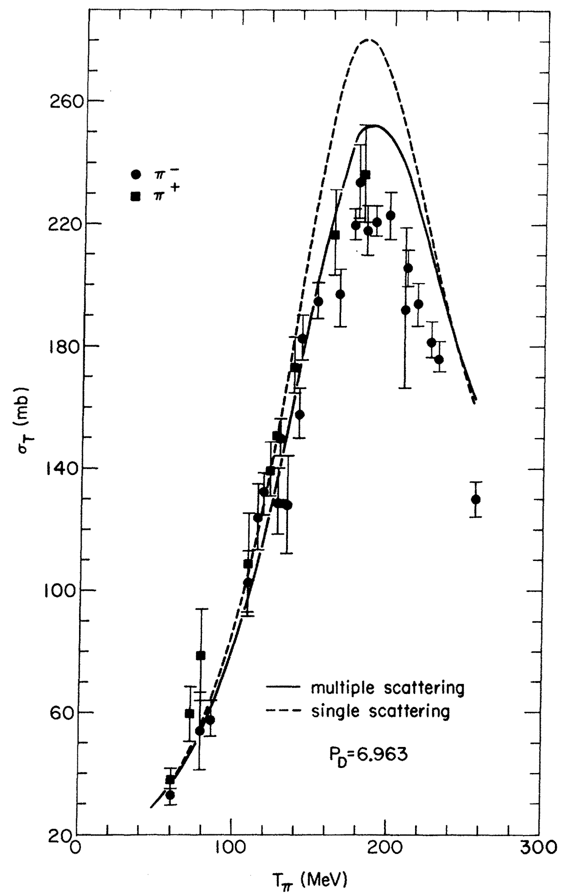

FIG. 7. Comparison of the present calculation with total cross-section data.

mation, and the full curve the complete calculation. One may see that multiple scattering must be included near the resonance if a meaningful comparison to data is to be made. These curves were computed assuming 6% D state in the deuteron.

Figure 4 shows the effect of variation of the amount of D state. It may be seen that the effect is not large but measurable.

The polarization tensors of the recoiling deuterons were calculated and are shown in Fig. 5. The tensor $-iT_{11}$ is not shown, because it has a very small magnitude. These curves suggest immediately that the most sensitive measure of the D-state probability is T_{20} at the pion angle of 180° or deuterons at 0°. Since all of the other tensors are zero at this point, the measurement should not suffer from having to unravel the effects of the various tensors. Figure 6 shows $T_{20}(180°)$ plotted vs the D-state percentage. Unfortunately the curve has flattened by the time 6% has been reached.

Although all of the calculations shown were done with Moravcsik's[11] best fit to the Gartenhouse wave function, calculations were also done with the Humberston and Wallace[12] deuteron calculated from the Hamada-Johnston potential. As long as the percentage of D state was the same, the comparable calculations were the same to within ~2%.

Calculations of the total cross section are shown in Fig. 7 compared with data.[13] It is seen that agreement is very good from the lowest energy to the peak of the (3, 3) resonance. The inclusion of Fermi motion[14] would lower the peak values somewhat more.

IV. CONCLUSIONS

We may conclude from the calculations in Secs. B and C that the Eqs. (20) provide an adequate description of π-deuteron scattering and in fact seem to do as well or better than any of the theories[15] previously proposed.

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APPENDIX A

We wish to examine how Eq. (22) may be simplified at forward angles. We first note that, for forward-peaking amplitudes, since the vector $\vec{p}$ is restricted to point along the direction of $\vec{\kappa}$ the second term in Q may be expected to be small. (It is interesting to note that, with usual p-nucleon amplitudes and deuteron wave functions this second term is purely imaginary and hence gives no contribution to the total cross section.) The first term in Q is just $S(\vec{p}-\vec{\kappa})$ so we have [taking $f(\vec{k},\vec{k}')=f(\vec{k}-\vec{k}')$],

$$F(\vec{k},\vec{k}')=f_I+\frac{ik}{2\pi}\int d\Omega_p\, f(\vec{p}-\vec{k}')f(\vec{k}-\vec{p})S(\vec{p}-\vec{\kappa})=f_I+\frac{i}{2\pi k}\int d\vec{p}\,\delta(p-k)f(\vec{p}-\vec{k}')f(\vec{k}-\vec{p})S(\vec{p}-\vec{\kappa}).$$

Now change variables to $q'=\vec{p}-\vec{\kappa}$. Take the direction of $\vec{\kappa}$ to be the z axis and note that $\kappa=k\cos\frac{1}{2}\theta$, where θ is the scattering angle. For small angles the magnitude of $\vec{q}'$ will be small for the important region of integration. Hence we may write

$$\begin{aligned} p&=(q'^2+\kappa^2+2\vec{q}'\cdot\vec{\kappa})^{1/2}\approx(k^2\cos^2\tfrac{1}{2}\theta+2q'k\cos\tfrac{1}{2}\theta\,\cos\theta')^{1/2},\\ &\approx k\cos\tfrac{1}{2}\theta\left(1+\frac{q'\cos\theta'}{k\cos\frac{1}{2}\theta}\right)\\ &\approx k\cos\tfrac{1}{2}\theta+q'\cos\theta'\\ &\approx k+q'\cos\theta'.\end{aligned}$$

Thus

$$\delta(p-k)\to\delta(q'\cos\theta').$$

Our expression for the amplitude becomes

$$F(k,k')\to f_I+\frac{i}{2\pi k}\int q'^2dq'd\varphi'\sin\theta'd\theta'\,\delta(q'\cos\theta')f(\vec{q}'+\tfrac{1}{2}\vec{q})f(\tfrac{1}{2}\vec{q}-\vec{q}')S(\vec{q}'),$$

where as usual $\vec{q}\equiv\vec{k}-\vec{k}'$. Carrying out the δ-function integration we obtain

$$F(\vec{k},\vec{k}')\to f_I+\frac{i}{2\pi k}\int d^2q'f\left(\vec{q}+\frac{\vec{q}}{2}\right)f\left(\frac{\vec{q}}{2}-\vec{q}'\right)S(\vec{q}'),$$

which is the Glauber expression for the amplitude.

APPENDIX B

Proceeding from Eqs. (24) we simplify as follows:

$$G_n(\vec{k},\vec{k}') \equiv \frac{2\pi}{ik} J_n(\vec{k},\vec{k}') , \tag{B1}$$

$$J_n(\vec{k},\vec{k}') \equiv \sum J_n^{lm}(\vec{k}) Y_l^m(\vec{k}') , \tag{B2}$$

$$\begin{aligned} f_n(\vec{k},\vec{k}') &= \frac{1}{2ik}\sum (2l+1) f_l^n P_l(\cos\theta) \\ &= \frac{2\pi}{ik}\sum f_l^n Y_l^{m*}(\vec{k}) Y_l^m(\vec{k}') , \end{aligned} \tag{B3}$$

$$J_n^{lm} \equiv f_l^n \left(\frac{2l+1}{4\pi}\right)^{1/2} K_n^{lm} . \tag{B4}$$

With these definitions Eqs. (24) become

$$\begin{aligned} K_1^{lm}(\vec{r}) &= \delta_{m0} e^{i\vec{k}\cdot\vec{r}/2} + e^{-i\vec{k}\cdot\vec{r}/2} \sum f_{l'}^2 g_{ll'}^{m0}(\vec{r}) + \sum_{l'm'} {}^2h_{ll'}^{mm'} f_{l'}^1 k_1^{l'm'}(\vec{r}) , \\ K_2^{lm}(\vec{r}) &= \delta_{m0} e^{-i\vec{k}\cdot\vec{r}/2} + e^{i\vec{k}\cdot\vec{r}/2} \sum f_{l'}^1 g_{ll'}^{m0}(-\vec{r}) + \sum_{l'm'} {}^1h_{ll'}^{mm'} f_{l'}^2 k_2^{l'm'}(\vec{r}) , \end{aligned} \tag{B5}$$

where

$$g_{ll'}^{mm'}(\vec{r}) \equiv \left(\frac{2l'+1}{2l+1}\right)^{1/2} \int d\Omega_p \theta(\vec{p}\cdot\vec{r}) e^{i\vec{p}\cdot\vec{r}} Y_{l'}^{m'}(\vec{p}) Y_l^{m*}(\vec{p}) , \tag{B6}$$

$$\begin{aligned} {}^1h_{ll'}^{mm'} &\equiv \sum_{l''m''} f_{l''}^1 g_{ll''}^{mm''}(-\vec{r}) g_{l''l'}^{m''m'}(\vec{r}) , \\ {}^2h_{ll'}^{mm'} &\equiv \sum_{l''m''} f_{l''}^2 g_{ll''}^{mm''}(\vec{r}) g_{l''l'}^{m''m'}(-\vec{r}) . \end{aligned} \tag{B7}$$

We may now see that the dependence of K on φ is trivial if we take $\vec{k}\parallel\hat{z}$. First we may use $g_{ll'}^{mm'}(-\vec{r}) = (-1)^{l+l'} g_{ll'}^{mm'}(\vec{r})$ to eliminate all $-\vec{r}$ arguments in the equations, then note

$$g_{ll'}^{mm'}(\vec{r}) = e^{i(m'-m)\varphi} g_{ll'}^{mm'}(r,\theta,\varphi=0)$$

and (B8)

$${}^nh_{ll'}^{mm'}(\vec{r}) = e^{i(m'-m)\varphi}\, {}^nh_{ll'}^{mm'}(r,\theta,\varphi=0) .$$

Since we have taken $\vec{k}\parallel\hat{z}$, $\vec{k}\cdot\vec{r}$ is independent of φ, so the equation for K becomes

$$K_1^{lm}(\vec{r}) = \delta_{m0} e^{-i\vec{k}\cdot\vec{r}/2} + e^{i\vec{k}\cdot\vec{r}/2} \sum_l f_{l'}^1 e^{-im\varphi} g_{ll'}^{m0}(r,\theta,\varphi=0)(-1)^{l+l'} + {}^2h_{ll'}^{mm'}(r,\theta,\varphi=0) e^{i(m'-m)\varphi} K_2^{l'm'} . \tag{B9}$$

If we define

$$K_n^{lm}(\vec{r}) = \hat{K}_n^{lm} e^{-im\varphi} , \tag{B10}$$

the equations become independent of φ and may be solved for $\varphi = 0$ to get $\hat{K}_n^{lm}(r,\theta)$. The scattering amplitude is then the expectation on the deuteron wave function of the following function:

$$\begin{aligned} \mathcal{K}(r,\theta,\varphi) &= \frac{2\pi}{ik}\sum\left(\frac{2l+1}{4\pi}\right)^{1/2} Y_l^m(\vec{k}') e^{-im\varphi}\left[f_l^2 e^{i\vec{k}'\cdot\vec{r}/2}\hat{K}_2^{lm}(r,\theta) + f_l^1 e^{-i\vec{k}'\cdot\vec{r}/2}\hat{K}_1^{lm}(r,\theta)\right] \\ &= \frac{2\pi}{ik}\sum (2l+1) i^\lambda C_{l\lambda 00}^{L0} C_{l\lambda m\mu}^{LM} j_\lambda(\tfrac{1}{2}kr) Y_L^M(\vec{k}') e^{-im\varphi} Y_\lambda^{\mu*}(\vec{r})\left[f_l \hat{K}_2^{lm}(r,\theta) + (-1)^\lambda f_l^1 K_1^{lm}(r,\theta)\right]\left[\frac{(2\lambda+1)}{(2L+1)}\right]^{1/2} . \end{aligned} \tag{B11}$$

$$\varphi^\sigma = \varphi_0^\sigma + \varphi_2^\sigma = \sum_{m_1 m_2} C_{\frac{1}{2}\frac{1}{2}m_1 m_2}^{1\sigma} Y_0^0 \varphi_0 |\tfrac{1}{2}m_1\rangle + \sum C_{\frac{1}{2}\frac{1}{2}m_1 m_2} C_{12m\mu}^{1\sigma} Y_2^\mu \varphi_2 |\tfrac{1}{2}m_1\rangle |\tfrac{1}{2}m_2\rangle ; \tag{B12}$$

then

$$f_{ss'}^{\sigma\sigma'} \equiv \langle \varphi_s^\sigma | \mathcal{H} | \varphi_{s'}^{\sigma'} \rangle$$
$$= \frac{e^{i(\sigma'-\sigma)\varphi_{\vec{k}}}}{2ik} \sum_{(\sigma'-\sigma=m+\mu)} (2l+1)(2\lambda+1) i^\lambda C_{l\lambda 00}^{L0} C_{l\lambda m\mu}^{LM} P_L^M(\theta_{\vec{k}}) \tilde{J}_{s\sigma s'\sigma'}^{\lambda l m}$$
$$\times \xi(\sigma'-\sigma)\xi(\mu)\xi(\sigma')\xi(\sigma)\left[\frac{(L-|M|)!(\lambda-|\mu|)!(s-|\sigma|)!(s'-|\sigma'|)!}{(L+|M|)!(\lambda+|\mu|)!(s+|\sigma|)!(s'+|\sigma'|)!}\right]^{1/2}[(2s+1)(2s'+1)]^{1/2}, \quad \text{(B13)}$$

where

$$\xi(M) = \begin{cases} (-1)^M & \text{if } M > 0, \\ 1 & \text{if } M < 0, \end{cases}$$

and

$$\tilde{J}_{s\sigma s'\sigma'}^{\lambda l m} = \tfrac{1}{2}\int_0^\pi \sin\theta d\theta P_\lambda^{|\mu|}(\theta) P_s^{|\sigma|}(\theta) P_{s'}^{|\sigma'|}(\theta) J_{ss'}^{\lambda l m}(\theta),$$

where

$$J_{ss'}^{\lambda l m}(\theta) = \int_0^\infty r^2 dr\, \varphi_s \varphi_{s'} [f_l^2 \hat{K}_2^{lm}(r,\theta) + (-1)^\lambda f_l^1 \hat{K}_1^{lm}(r,\theta)].$$

Now define

$$\hat{J}_{s\sigma s'\sigma'}^{\lambda l m} = [(2s+1)(2s'+1)]^{1/2}\xi(\sigma')\xi(\sigma)\xi(\sigma'-\sigma)\left[\frac{(s-|\sigma|)!(s'-|\sigma'|)!}{(s+|\sigma|)!(s'+|\sigma'|)!}\right]^{1/2}(2l+1)(2\lambda+1)i^\lambda\left[\frac{(\lambda-|\mu|)!}{(\lambda+|\mu|)!}\right]^{1/2}\xi(\mu)\tilde{J}_{s\sigma s'\sigma'}^{\lambda l m}$$

and

$$\hat{P}_L^M(\theta) = P_L^M(\theta)\left[\frac{(L-|M|)!}{(L+|M|)!}\right]^{1/2},$$

so that

$$f_{ss'}^{\sigma\sigma'} = \frac{e^{i(\sigma'-\sigma)\varphi}}{2ik}\sum_{\substack{\lambda L\\ lm}} C_{l\lambda 00}^{L0} C_{l\lambda m \sigma'-\sigma-m}^{L\sigma'-\sigma} \hat{P}_L^{|\sigma'-\sigma|} \hat{J}_{s\sigma s'\sigma'}^{\lambda l m}. \quad \text{(B14)}$$

Since only nonspin-flip terms are considered in multiple scattering,

$$f_{ss'}^{\sigma\sigma} = \frac{1}{2ik}\sum_{\substack{\lambda L\\ lm}} C_{l\lambda 00}^{L0} C_{l\lambda m-m}^{L0} P_L^0 \hat{J}_{s\sigma s'\sigma}^{\lambda l m}. \quad \text{(B15)}$$

The deuteron amplitudes are given by

$$F_{\sigma\sigma'} = f_{00}^{\sigma\sigma'} + f_{02}^{\sigma\sigma'} + f_{20}^{\sigma\sigma'} + f_{22}^{\sigma\sigma'}. \quad \text{(B16)}$$

Expressing the spherical tensors describing the polarization of the recoil deuterons[16] in terms of these amplitudes we find

$$\begin{aligned} i\langle T_{11}\rangle &= \sqrt{6}\,\mathrm{Im}(F_{11}F_{01}^* + F_{10}F_{00}^* + F_{-11}F_{01}^*), \\ \langle T_{20}\rangle &= \sqrt{2}\,(|F_{11}|^2 + |F_{-11}|^2 - |F_{01}|^2 - |F_{00}|^2), \\ \langle T_{21}\rangle &= \sqrt{6}\,\mathrm{Re}(F_{11}F_{01}^* + F_{10}F_{00}^* + F_{1-1}F_{0-1}^*), \\ \langle T_{22}\rangle &= 2\sqrt{3}\,(2\,\mathrm{Re}F_{11}F_{-11}^* - |F_{10}|^2). \end{aligned} \quad \text{(B17)}$$

The π-nucleon parameter used for the nonspin-flip case were for the proton

$$f_0 = f_{13}, \quad f_1 = \tfrac{1}{3}f_{13} + \tfrac{2}{3}f_{33},$$

and for the neutron

$$f_0 = \tfrac{2}{3}f_{11} + \tfrac{1}{3}f_{13}, \quad f_1 = \tfrac{2}{9}f_{11} + \tfrac{1}{9}f_{13} + \tfrac{4}{9}f_{31} + \tfrac{2}{9}f_{33}.$$

The π-nucleon phase shifts were taken from the work of McKinley[17] with all data included.

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[1]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al.* (Interscience Publishers, Inc., New York, 1959), Vol. I, p. 315.

[2]V. Franco and R. J. Glauber, Phys. Rev. 142, 1195 (1966); V. Franco and E. Coleman, Phys. Rev. Letters 17, 827 (1966); R. J. Glauber and V. Franco, Phys. Rev. 156, 1685 (1967); C. Michael and C. Wilkin, Nucl. Phys. B11, 99 (1969).

[3]V. S. Bhasin and V. S. Varma, Phys. Rev. 184, 1338 (1969); J. Pumplin, *ibid.* 173, 1651 (1968); G. Alberi and L. Bertocchi, Phys. Letters 28B, 186 (1968); D. R. Harrington, Phys. Rev. 184, 1745 (1969).

[4]C. Carlson, Phys. Rev. C 2, 1224 (1970); S. D. Drell and L. Verlet, Phys. Rev. 99, 849 (1955); R. M. Rockmore, *ibid.* 105, 256 (1957); A. Ramakrishnan, V. Devanathan, and K. Venkatesan, Nucl. Phys. 29, 680 (1962); B. H. Bransden and R. G. Moorhouse, *ibid.* 6, 310 (1958); H. N. Pendleton, Phys. Rev. 131, 1833 (1963).

[5]E. H. Auerbach, D. M. Fleming, and M. M. Sternheim, Phys. Rev. 162, 1683 (1967); L. S. Kisslinger, *ibid.* 135, 761 (1955).

[6]L. L. Foldy, Phys. Rev. 67, 107 (1945); K. A. Brueckner, *ibid.* 89, 834 (1953); 90, 715 (1953).

[7]R. Seki, to be published; Phys. Rev. 178, 2316 (1969).

[8]A. M. Sachs, H. Winick, and B. A. Wooten, Phys. Rev. 109, 1733 (1958).

[9]K. C. Rogers and L. M. Lederman, Phys. Rev. 105, 247 (1957).

[10]E. G. Pewitt, T. H. Fields, G. B. Yodh, J. G. Fethovich, and M. Derrick, Phys. Rev. 131, 1826 (1963).

[11]M. J. Moravcsik, Nucl. Phys. 7, 113 (1958).

[12]J. W. Humberston and J. B. C. Wallace, Nucl. Phys. A141, 362 (1970).

[13]J. Ashkin, J. P. Blaser, F. Feiner, J. G. Gorman, and M. O. Stern, Phys. Rev. 96, 1104 (1954); H. L. Anderson, E. Fermi, D. E. Nagle, and G. B. Yodh, *ibid.* 86, 413 (1952); K. C. Rogers and L. M. Lederman, *ibid.* 105, 247 (1957); A. E. Ignatenko, A. I. Mukhin, E. B. Ozerov, and B. M. Pontecorvo, Dokl. Akad. Nauk SSSR 103, 209 (1955); P. S. Isaacs, A. M. Sachs, and J. Steinberger, Phys. Rev. 85, 803 (1952); D. E. Nagle, *ibid.* 97, 480 (1955); E. G. Pewitt, T. H. Fields, G. B. Yodh, J. G. Fethovich, and M. Derrick, Ref. 10.

[14]G. Fäldt and T. E. O. Ericson, Nucl. Phys. B8, 1 (1968).

[15]Carlson, Ref. 4; Drell and Verlet, Ref. 4; Rockmore, Ref. 4; Ramakrishnan, Devanathan, and Venkatesan, Ref. 4; Bransden and Moorhouse, Ref. 4; Pendleton, Ref. 4; G. Fäldt, Nucl. Phys. B10, 577 (1969); S. Fernbach, F. A. Green, and K. M. Watson, Phys. Rev. 82, 980 (1951); *ibid.* 84, 1084 (1952); D. Schiff and J. Tran Thahn Van, Nucl. Phys. B3, 671 (1967); J. J. Brehm and J. Sucher, Ann. Phys. (N.Y.) 25, 1 (1963); F. Ando and D. Feldman, Progr. Theoret. Phys. (Kyoto) Suppl. 37, 524 (1966); J. N. Chahoud and G. Russo, Nuovo Cimento 45, 493 (1966).

[16]W. Lakin, Phys. Rev. 98, 139 (1955).

[17]J. M. McKinley, Rev. Mod. Phys. 35, 788 (1963).