Off-Energy-Shell Effects in the Triton with the Boundary-Condition Interaction*

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The boundary-condition model of the two-nucleon interaction due to Feshbach and Lomon is used without an external potential tail to determine the ground-state properties of the three-nucleon system for the purpose of studying the off-energy-shell properties of the two-body T matrix. Calculations are performed using an average singlet-triplet s-wave interaction giving a completely symmetric triton wave function. The T matrix is separable and depends on three energy-independent parameters – the boundary radius r_0 and the boundary condition f at $r_0+\epsilon$ which are determined from the two-nucleon phase shifts, and the boundary condition b at $r_0-\epsilon$ which is specifically an off-shell parameter that does not appear in the on-shell T matrix. With $f=0.11$ and $r_0=0.95$ F, the Faddeev equations are solved for the triton energy eigenvalues and spectator functions for several values of b. It is found that the binding energy varies from 3.9 to 19.4 MeV for b between 0.5 and 0.6, with a value of 8.5 MeV at $r=0.537$. The average kinetic energy and charge form factor are determined as a function of b, and qualitative agreement of the charge radius with experiment is obtained for $b=0.5$–0.6. The charge form factor, compared with experiment, does not drop off rapidly enough with q^2, indicating the presence of relatively too much large q component in the spectator functions. The changes to be expected from the inclusion of tensor coupling in the boundary parameters are discussed.

I. INTRODUCTION

A. Boundary Condition Model

The off-energy-shell behavior of the two-nucleon T matrix is important in the determination of the properties of systems of more than two nucleons. In recent years calculations for nuclear matter[1] and the three-body problem[2] have become testing grounds for the comparison of two-nucleon potentials that give essentially the same on-shell T matrix (the scattering phase shifts) but have different off-shell properties. This difference is exhibited in the behavior of the off-shell two-nucleon wave function at short distances and comes mainly from the differing properties of the potentials at short distances.

One way of describing the interaction at very short distances is by a boundary condition (BC). The BC model of Feshbach and Lomon[3] gives a high-precision fit to the two-nucleon scattering data. The effect of the two-nucleon interaction in the region of two-particle separation $r<r_0$, where r_0 is about one half the Compton wavelength of a pion, is represented by an energy-independent logarithmic-derivative BC at r_0 on each partial wave of the two-nucleon wave function. The potential in the exterior region $r>r_0$ consists of a combination of local potentials that account for one- and two-pion exchange and for the exchange of ρ, ω, and η mesons. Even with the exterior potential set equal to zero the BC alone gives an approximate fit to the two-nucleon data, provided that the radius r_0 is set equal to about one pion Compton wavelength so that the BC can substitute for some of the potential tail.[4]

For determining the two-nucleon wave function off the energy shell the BC model must be augmented by some additional assumptions about the potential in the core region $r<r_0$. On the energy shell the solutions to the Lippmann-Schwinger equation have the property of vanishing in $r<r_0$ regardless of what assumptions are made about the potential in this interior region. These are solutions of the Schrödinger equation confined to the interior region but they never couple to the exterior solutions and so do not affect the description of two-body scattering. Off the energy shell, however, the solution of the Lippmann-Schwinger equation does not vanish in $r<r_0$ but depends on the potential there. The on-shell description in terms of a BC f_l on the lth partial wave at $r_0+\epsilon$ ($\epsilon\to 0$) must be supplemented by a second energy-independent BC b_l at $r_0-\epsilon$.[5] In terms of the wave function $u_l(r)/r$ the off-shell BC's are

$$r_0[du_l(r)/dr]=f_l u_l(r), \quad r=r_0+\epsilon;$$
$$r_0[du_l(r)/dr]=b_l u_l(r), \quad r=r_0-\epsilon. \tag{1}$$

The BC's for the triplet states coupled by the tensor force are expressed in terms of 2×2 symmetric matrices $\mathfrak{f}$ and $\mathfrak{b}$ for each value of total angular momentum. A potential could be added in $r<r_0$ but it is assumed that for the energy range important in nuclei $\mathfrak{b}$ alone is sufficient to describe the off-shell effects of the interior region.

B. Description of Problem

In this paper we apply the BC interaction without an external potential to the three-nucleon bound state considering just an $l=0$ two-body interaction that is an average of the singlet and triplet forces. The boundary parameters are r_0, f, and b. The parameters f and r_0 are obtained from two-body data, but b must be obtained through a several- or many-body calculation. Thus we examine the triton binding energy and charge form factor as a function of the boundary parameter b in order to determine the range of b that best fits experiment. We consider the Faddeev equation[6] in terms of the two-body T matrices and follow methods that have been developed for separable potentials.[7,8] This study is preliminary. A more complete description, without the potential tail, requires the use of four interior boundary parameters for the singlet s state and the coupled s and d states, and ultimately an external potential must be included in order to get a true picture of the sensitivity of the three-body system to the interior boundary parameters.

II. TWO-BODY T MATRIX

The two-nucleon interaction is introduced into the Faddeev equations through the off-energy-shell two-nucleon scattering matrix $T(E)$. The BC T matrix may be obtained in a variety of ways, either by direct application of the boundary conditions b_l and f_l of (1) at $r_0-\epsilon$ and $r_0+\epsilon$ to the off-shell Schrödinger wave function or by solution of the integral equation $T=V+VG_0(E)T$ using an effective potential V.[5] In configuration space the effective potential for a single partial wave, neglecting tensor coupling in triplet states, is

$$V(r,r')=\frac{\hbar^2}{M}\frac{1}{4\pi r_0 rr'}[f_l\delta_+(r-r_0)\delta_+(r'-r_0)+r\delta_+(r-r_0)\delta_-'(r'-r_0)$$
$$+r'\delta_-'(r-r_0)\delta_+(r'-r_0)+rr'b_l^{-1}\delta_-'(r-r_0)\delta_-'(r'-r_0)]$$
$$=\frac{\hbar^2}{M}\frac{1}{4\pi r_0 rr'}(\delta_+(r-r_0),-r\delta_-'(r-r_0))\begin{pmatrix} f_l & -1\\ -1 & b_l^{-1}\end{pmatrix}\begin{pmatrix}\delta_+(r'-r_0)\\ -r\delta_-'(r'-r_0)\end{pmatrix}. \tag{2.1}$$

$\delta_+(r-r_0)$ is a δ function peaked at $r_+=r_0+\epsilon$, whereas $\delta_-(r-r_0)$ is peaked at $r_0-\epsilon$ and $\delta_-'(r-r_0)$ is its derivative. The momentum-space matrix element of this potential for the lth partial wave is

$$\langle p'|V|p\rangle=\int d^3r d^3r'\frac{\mathcal{J}_l(p'r)}{p'r}V(r,r')\frac{\mathcal{J}_l(pr')}{pr'}$$
$$=\frac{\hbar^2}{M}\frac{4\pi}{p'pr_0}J_l^{\dagger}(p'r_0)cJ_l(pr_0), \tag{2.2}$$

where $\mathcal{J}_l$ is the Riccati-Bessel function of order l and $\mathcal{J}_l'$ is its derivative. The matrices J_l and c are given by

$$J_l(pr_0)=\begin{pmatrix}\mathcal{J}_l(pr_+)\\ pr_-\mathcal{J}_l'(pr_-)\end{pmatrix},\quad c=\begin{pmatrix} f_l & -1\\ -1 & b_l^{-1}\end{pmatrix}. \tag{2.3}$$

M is the nucleon mass.

The solution of the integral equation for T with the potential (2.2) yields the result

$$\langle p'|T(E)|p\rangle=\frac{\hbar^2}{M}\frac{4\pi}{pp'r_0}J_l^{\dagger}(p'r_0)[c^{-1}-g_l(E)]^{-1}J_l(pr_0)$$
$$=\frac{\hbar^2}{M}\frac{4\pi}{pp'r_0}J_l^{\dagger}(p'r_0)c[I-g_l(E)c]^{-1}J_l(pr_0) \tag{2.4}$$

for the lth partial wave. The second form of $\langle p'|T|p\rangle$ in (2.4) is more general because c^{-1} does not exist if $b_l=f_l$. I is the unit matrix and g_l is a matrix derived from the free two-particle Green's function $G_0(E)$,

$$g_l(E)=\frac{2}{\pi r_0}\int_0^{\infty}dp\,J_l(pr_0)G_0(E)J_l^{\dagger}(pr_0)$$
$$=\begin{pmatrix}\mathcal{G}_l & \mathcal{G}_l'\\ \mathcal{G}_l' & \mathcal{G}_l''\end{pmatrix}, \tag{2.5}$$

where $G_0(E)=(E-p^2)^{-1}$.

The dependence of T on the energy variable E is contained in $g_l(E)$, which is an analytic function of E in the complex E plane cut along the real axis from zero to infinity. We concentrate on values of E along the negative real axis, since this is the range of importance in the three-body bound-state problem. Thus we consider g_l under the conditions

$$E<0,\quad \gamma=r_0\sqrt{-E}. \tag{2.6}$$

Then, in terms of the parameter γ,

$$\mathcal{G}_l=-\gamma^{-1}g_l(\gamma)\mathcal{K}_l(\gamma),$$
$$\mathcal{G}_l'=-g_l'(\gamma)\mathcal{K}_l(\gamma),$$
$$\mathcal{G}_l''=-\gamma g_l'(\gamma)\mathcal{K}_l'(\gamma). \tag{2.7}$$

The function $g_l(\gamma)/\gamma$ is the modified spherical Bessel function of the first kind and $\pi\mathcal{K}_l(\gamma)/2\gamma$ is the modified spherical Bessel function of the third kind.[9] For $l=0$, $g_0(\gamma)=\sinh\gamma$ and $\mathcal{K}_0(\gamma)=e^{-\gamma}$.

The evaluation of $T(E)$ in (2.4) using (2.5) and (2.7) results in the following expression for the

lth partial wave:

$$\langle p'|T(E)|p\rangle = \frac{\hbar^2}{M}\frac{4\pi}{p'pr_0}J_l{}^{\dagger}(p'r_0)\left[c - \frac{1}{f_l+\Gamma_l}\begin{pmatrix} f_l{}^2 & -f_l \\ -f_l & 1\end{pmatrix} - \frac{1}{b_l{}^{-1}-\Sigma_l}\begin{pmatrix} 1 & -b_l{}^{-1} \\ -b_l{}^{-1} & b_l{}^{-2}\end{pmatrix}\right]J_l(pr_0)$$
$$= (\hbar^2/M)(4\pi/p'pr_0)J_l{}^{\dagger}(p'r_0)D(\gamma)J_l(pr_0), \tag{2.8}$$

where $D(\gamma)$ is the expression in square brackets and

$$\Gamma_l = -\gamma\mathfrak{K}_l'(\gamma)/\mathfrak{K}_l(\gamma), \quad \Sigma_l = \gamma^{-1}g_l(\gamma)/g_l'(\gamma). \tag{2.9}$$

In particular, $\Gamma_0 = \gamma$ and $\Sigma_0 = \tanh\gamma/\gamma$.

It is apparent from (2.8) that the two-body matrix $T(E)$, for E real and negative, is the sum of a Born term, $(4\pi/p'pr_0)J_0{}^{\dagger}cJ$, and two pole terms. The first pole term has a pole at $f_l = -\Gamma_l$ and corresponds to a two-particle bound state of the exterior wave function for the lth partial wave—the deuteron for $l=0$. Neither the position nor the residue of this pole depends on b_l so that the properties of the deuteron are indeed independent of b_l. The second pole term (the b-pole) is the contribution coming from the off-energy-shell coupling of the exterior state to the complete set of solutions of the Schrödinger equation in the interior $r<r_0$ with logarithmic derivative b at r_0. The pole in γ, given by the solution of $b_l{}^{-1}-\Sigma_l=0$ ($\gamma\cot\gamma = b_0$ for $l=0$) occurs at the position of the interior bound state for $E<0$ and provides the dominant contribution to the b-pole term, whereas the other states ($E>0$) provide the background. The interior states do not couple to the two-body scattering states on the energy shell. Thus, for $E=p^2 \pm i\epsilon$, the contribution of the b-pole term to $T(E)$ vanishes, as does the b dependence of the Born term.

The parameters f_l and r_0 are determined by fitting the two-nucleon phase shifts. The $l=0$ phase shift is given by[4]

$$\delta_0(p) = \arctan(pr_0/f_0) - pr_0. \tag{2.10}$$

However, the parameters b_l can be determined only by examining some system with properties that depend on the off-energy-shell properties of the two-nucleon interaction, such as the three-nucleon system or nuclear matter.

III. TRITON WAVE FUNCTION

A. Integral Equations for Spectator Functions

The three-nucleon wave function Ψ for the $J=\frac{1}{2}$, $I=\frac{1}{2}$ bound state at energy $E=-\alpha^2$ is written as the sum of three components ψ_i,

$$\Psi = G_0(E)\sum_{i=1}^{3}\psi_i, \tag{3.1}$$

where $G_0(E)$ is the Green's function for three free particles. The ψ_i are required to satisfy the Faddeev equations[6, 10]

$$\psi_i = T_iG_0(E)\sum_{j\neq i}\psi_j \quad i,j=1,2,3. \tag{3.2}$$

T_i is the two-nucleon T matrix for particles j and k.

ψ_i is completely antisymmetric and is written in terms of the $I=\frac{1}{2}$ isospin functions of mixed symmetry, ξ' and ξ'', as

$$\psi_i = 2^{-1/2}[\psi_i''\xi' - \psi_i'\xi''],$$

where ψ_i' and ψ_i'' are space-spin functions of mixed symmetry. Furthermore ψ_i' and ψ_i'' can be expressed in terms of a space- and spin-dependent operator Ω_i acting on the $S=\frac{1}{2}$ spin functions of mixed symmetry, χ' and χ'',

$$\begin{pmatrix}\psi_i' \\ \psi_i''\end{pmatrix} = \Omega_i\begin{pmatrix}\chi' \\ \chi''\end{pmatrix}.$$

The net result is[11, 12]

$$\psi_i = 2^{-1/2}\Omega_i[\chi''\xi' - \chi'\xi'']. \tag{3.3}$$

We consider the equations for ψ_i restricted to s-wave two-nucleon interactions without tensor forces. In the three-particle center-of-mass system Ω_i is explicitly a function of the variables $\vec{p}_i$, the relative momentum of particles j and k, and $\vec{q}_i$, the momentum of particle i relative to the center of mass of j and k. The T matrix T_i is a sum of singlet (S) and triplet (T) s-wave T matrices, as in (2.9), multiplied by two-particle spin and isospin projection operators $P_\sigma^{0,1}(i)$ and $P_\tau^{0,1}(i)$,

$$\langle\vec{p}_i'\vec{q}_i'|T_i|\vec{p}_i\vec{q}_i\rangle = [\langle p_i'|T_{iS}(E-\tfrac{3}{4}q_i{}^2)|p_i\rangle P_\sigma^0(i)P_\tau^1(i) + \langle p_i'|T_{iT}(E-\tfrac{3}{4}q_i{}^2)|p_i\rangle P_\sigma^1(i)P_\tau^0(i)](2\pi)^3\delta^3(\vec{q}_i-\vec{q}_i'),$$
$$\langle p_i'|T_{iS,T}(E-\tfrac{3}{4}q_i{}^2)|p_i\rangle = \frac{4\pi}{p_i'p_ir_0}J_0{}^{\dagger}(p_i'r_0)D_{S,T}(\gamma)J_0(p_ir_0),$$
$$\gamma = r_0(\alpha^2+\tfrac{3}{4}q_i{}^2)^{1/2}, \tag{3.4}$$

where D is given in (2.8). D_S depends on the BC parameters f_S and b_S for $l=0$ singlet states and D_T on the parameters f_T and b_T for triplet states.

Because each T matrix is separable of rank two, Ω_i contains two spectator functions, $H_1(q_i)$ and $H_2(q_i)$ for each state, that form a column vector

$H(q_i)$. Thus

$$\Omega_i = \Omega_{iS} P_\sigma^0 + \Omega_{iT} P_\sigma^1,$$

$$\Omega_{iS,T} = \frac{1}{p_i r_0}\left(\mathcal{J}_0(p_i r_+), p_i r_- \mathcal{J}_0'(p_i r_-)\right)\begin{pmatrix} H_{1S,T}(q_i) \\ H_{2S,T}(q_i)\end{pmatrix}$$

$$= (p_i r_0)^{-1} J_0^{\dagger}(p_i r_0) H_{S,T}(q_i). \tag{3.5}$$

To complete the expression for Ψ in (3.1) we need the expression

$$G_0^{-1}(E) = -[\alpha^2 + p_i^2 + \tfrac{3}{4}q_i^2], \quad i = 1, 2, 3.$$

Fredholm integral equations for $H_S(q)$ and $H_T(q)$ are obtained by inserting (3.3), (3.4), and (3.5) into (3.2). The coupled equations are:

$$H_S(q) = -\tfrac{1}{2} r_0^3 D_S(\gamma)\int_0^\infty q'^2 dq' K_0(q,q') H_S(q') - \tfrac{3}{2} r_0^3 D_T(\gamma)\int_0^\infty q'^2 dq' K_0(q,q') H_T(q'),$$

$$H_T(q) = -\tfrac{3}{2} r_0^3 D_S(\gamma)\int_0^\infty q'^2 dq' K_0(q,q') H_S(q') - \tfrac{1}{2} r_0^3 D_T(\gamma)\int_0^\infty q'^2 dq' K_0(q,q') H_T(q'). \tag{3.6}$$

The kernel $K_0(q,q')$ is a matrix of functions integrated over the cosine of the angle between $\vec{q}$ and $\vec{q}'$,

$$K_0(q,q') = \frac{1}{\pi r_0^2}\int_{-1}^{1} dz \frac{J_0(p_1' r_0)}{p_1' r_0} R(\vec{q}, \vec{q}') \frac{J_0^{\dagger}(p_2' r_0)}{p_2' r_0}, \tag{3.7}$$

where

$$R^{-1}(\vec{q}, \vec{q}') = \alpha^2 + q^2 + q'^2 + \vec{q}\cdot\vec{q}',$$

$$p_1' = |\vec{q}' + \tfrac{1}{2}\vec{q}|, \quad p_2' = |\tfrac{1}{2}\vec{q}' + \vec{q}|,$$

$$\vec{q}\cdot\vec{q}' = qq'z. \tag{3.8}$$

In addition, D_S and D_T are the γ-dependent parts of the two-body T matrices evaluated off the energy shell at $\gamma = r_0(\alpha^2 + \frac{3}{4}q^2)^{1/2}$. Expressed in terms of the dimensionless variables qr_0 and $q'r_0$, Eqs. (3.6) do not depend explicitly on r_0. Thus the boundary radius r_0 is a scaling factor both for the q dependence of the spectator functions and the energy eigenvalue $(\alpha r_0)^2$.

For purposes of numerical calculation we drop the distinction between singlet and triplet interaction ($S = T$) in (3.6). The number of equations drops from four to two, and there is just a single set of boundary parameters, b and f:

$$H(q) = -2 r_0^3 D(\gamma)\int_0^\infty q'^2 dq' K_0(q,q') H(q'),$$

with

$$D(\gamma) = \begin{pmatrix} f & -1 \\ -1 & b^{-1}\end{pmatrix} - \frac{1}{f+\gamma}\begin{pmatrix} f^2 & -f \\ -f & 1\end{pmatrix} - \frac{\gamma\cot\gamma}{\gamma\cot\gamma - b}\begin{pmatrix} b & -1 \\ -1 & b^{-1}\end{pmatrix}. \tag{3.9}$$

This approximation eliminates the spin dependence of the operator Ω_i in (3.3) and is equivalent to considering spinless nucleons. As a result the spatial wave function is completely symmetric and there is no admixture of the S' state of mixed symmetry. The three-nucleon wave function (3.1) under these circumstances is as follows:

$$\Psi = G_0(E)\sum_{i=1}^{3}\psi_i(p_i, q_i) = G_0(E)\sum_{i=1}^{3}\frac{1}{p_i r_0} J_0^{\dagger}(p_i r_0) H(q_i),$$

$$H(q) = \begin{pmatrix} H_1(q) \\ H_2(q)\end{pmatrix},$$

$$\vec{p}_2 = \tfrac{3}{4}\vec{q}_1 + \tfrac{1}{2}\vec{p}_1 = \vec{q}_3 + \tfrac{1}{2}\vec{q}_2,$$

$$\vec{p}_3 = -\tfrac{3}{4}\vec{q}_1 + \tfrac{1}{2}\vec{p}_1 = -\vec{q}_2 - \tfrac{1}{2}\vec{q}_3,$$

$$\vec{q}_2 = \tfrac{1}{2}\vec{q}_1 - \vec{p}_1, \quad \vec{q}_3 = \vec{p}_1 + \tfrac{1}{2}\vec{q}_1. \tag{3.10}$$

B. Triton Form Factor

When no distinction is made between the singlet and triplet two-nucleon forces, the spatial distributions of the like and odd nucleons in the triton, as defined by Schiff, are the same.[13,14] In that event the triton charge form factor, assuming additive contributions from each nucleon, is given by the expression

$$F_{ch}^{H^3}(q^2) = [2F_{ch}^{n}(q^2) + F_{ch}^{p}(q^2)]F(q^2), \tag{3.11}$$

where F_{ch}^{n} and F_{ch}^{p} are the neutron and proton form factors and $F(q^2)$ is the body form factor of the triton.

$F(q^2)$ is written in terms of the triton wave function (3.10) expressed as a function of a single set of momentum variables; for example, $\vec{q}_1$ and $\vec{p}_1$. Specifically,

$$F(q^2) = \frac{1}{\langle\Psi|\Psi\rangle}\int d^3p_1 d^3q_1 \Psi^{\dagger}(\vec{p}_1, \vec{q}_1 + \tfrac{1}{3}\vec{q})\Psi(\vec{p}_1, \vec{q}_1 - \tfrac{1}{3}\vec{q}), \tag{3.12}$$

where $\langle\Psi|\Psi\rangle$ is the normalization

$$\langle\Psi|\Psi\rangle = \int d^3p_1 d^3q_1 \Psi(\vec{p}_1\vec{q}_1)^{\dagger}\Psi(\vec{p}_1\vec{q}_1),$$

giving $F(0) = 1$.

C. Average Kinetic and Potential Energy

Since the two-body BC wave function $u(r)$ is discontinuous at r_0,[5] care must be exercised in calculating the expectation value of the two-body ki-

netic energy operator $-d^2/dr^2+l(l+1)/r^2$ to avoid introducing δ-function singularities from derivatives at the discontinuity. The correct expression for the two-body $l=0$ kinetic energy is

$$\int_0^{r_-} dr\, u(r)\left(\frac{-d^2u(r)}{dr^2}\right)+\int_{r_+}^{\infty} dr\, u(r)\left(\frac{-d^2u(r)}{dr^2}\right),$$

and this can be written as a single integral from 0 to ∞ provided that the definition of the kinetic energy operator is altered to remove a $\delta(r=0)$ term that remains after differentiation and integration. The modified expression for the kinetic energy is

$$\langle \text{K.E.}\rangle = \int_0^{\infty} dr\, u(r)\left\{\left[\frac{\vec{d}}{dr}-\vec{A}\delta(r-r_0)\right]\times\left[\frac{\vec{d}}{dr}-\delta(r-r_0)\vec{A}\right]\right\}u(r), \quad (3.13)$$

where the modified kinetic energy operator is the part of (3.13) in curly brackets with the operator $\vec{A}$ defined as $\vec{A}u(r)=u(r_+)-u(r_-)$.

Transformed into momentum space for the center-of-mass system of three nucleons the modified kinetic energy operator becomes

$$\langle \vec{p}_1\vec{q}_1|\text{K.E.}|\vec{p}_1'\vec{q}_1'\rangle = (2\pi)^6\delta^3(\vec{p}_1-\vec{p}_1')\delta^3(\vec{q}_1-\vec{q}_1')(p_1^2+\tfrac{3}{4}q_1^2) + \text{correction}, \quad (3.14)$$

with

$$\text{Correction}=\sum_i \frac{4\pi}{p_ip_i'r_0}(\Delta \sin p_ir_0,\ p_ir_0\cos p_ir_0)\times\begin{pmatrix}\delta(r=0) & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\Delta \sin p_i'r_0\\ p_i'r_0\cos p_i'r_0\end{pmatrix}, \quad (3.15)$$

where $\Delta \sin p_ir_0=\sin p_ir_+-\sin p_ir_-$. The first term is the usual sum of single-particle kinetic energies, whereas the second term is a sum of two-particle operators, given here for s waves. When the kinetic energy is calculated using the wave function $\Psi(p_1,q_1)$ in (3.10) the two parts of the kinetic energy operator introduce terms proportional to $\delta(r=0)$ that cancel, as in the two-nucleon problem. The limit $r_+,r_-\to r_0$ is taken only after all integrations have been performed. Some integrations involving $\Delta \sin p_ir_0$ are just step-function discontinuities and give nonzero contributions.

The net result for the kinetic energy, using the wave function (3.10) and the operator (3.14), is the expression

$$\langle \text{K.E.}\rangle = \frac{\hbar^2}{Mr_0^2}\frac{3(2\pi)^3}{\langle\Psi|\Psi\rangle}\frac{1}{(b-f)}\times\left[\int_0^{\infty} q^2dq\, H_1^2(q)-bf\int_0^{\infty} q^2dq\, H_2^2(q)\right]-\frac{\hbar^2\alpha^2}{M}, \quad (3.16)$$

where $\langle\Psi|\Psi\rangle$ is the normalization as used in (3.12). The average potential energy, obtained from P.E. $=E-$ K.E., is then

$$\langle \text{P.E.}\rangle = -\frac{\hbar^2}{Mr_0^2}\frac{3(2\pi)^3}{\langle\Psi|\Psi\rangle}\frac{1}{(b-f)}\times\left[\int_0^{\infty} q^2dq\, H_1^2(q)-bf\int_0^{\infty} q^2dq\, H_2^2(q)\right]. \quad (3.17)$$

An immediate consequence of (3.17) is that the potential energy is positive definite for the simultaneous choice $f>0$ and $b<0$. Thus, there are no three-nucleon bound states for this choice of the parameters f and b:

$f>0$, $b<0$: no three-nucleon bound state. (3.18)

IV. RESULTS OF CALCULATION

A. Binding Energy and Wave Function

The coupled integral equations (3.9) were solved as a function of the BC off-shell parameter b for the binding energy $\hbar^2\alpha^2/M$ and the spectator functions $H_1(q)$ and $H_2(q)$ using $f=0.11$ and $r_0=0.95$ F. The phase shift obtained from (2.10) with these values of f and r_0 is an approximate fit to the characteristics of both the singlet and triplet s-

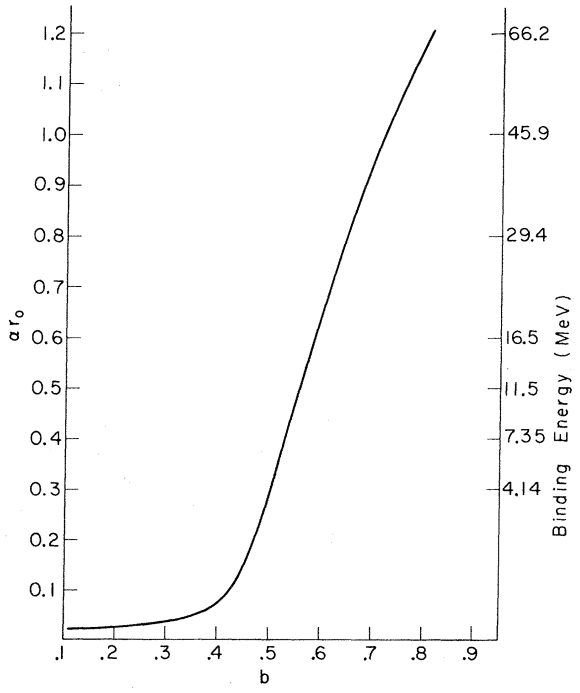

FIG. 1. αr_0 as a function of the boundary-condition parameter b. The left-hand scale shows the triton binding energy $(\hbar^2/M)\alpha^2$ for a boundary radius r_0 equal to 0.95 F.

wave phase shifts, and these BC parameters have been used previously in nuclear-matter calculations.[5,15] $f=0.11$ does not produce a two-nucleon bound state ($f<0$ for an s-wave bound state) but does give a virtual bound state at zero energy. The region $f<1$ corresponds to an attractive interaction for small relative momenta, but for $f>1$ the interaction is repulsive at all momenta.

The solution of (3.9) was carried out by iteration using 16-point Gauss quadratures for the angular integrations in the kernel and 40-point Gauss quadratures for the integrations over q'. The numerical calculations were performed on the IBM 360/65 computer at the Massachusetts Institute of Technology Information Processing Center.

Eigenfunctions H_1 and H_2 and eigenvalues αr_0 were found for $b \geq f$ with α being a monotonically increasing function of b. Although a search was made for solutions with $f=0.11$ and $b<0$, none were found, verifying the conclusion of (3.18). A plot of αr_0 vs b for b between 0.11 and 0.8 is shown in Fig. 1. The value of αr_0 starts to increase slowly with b but then rises rapidly between $b=0.4$ and 0.6. For larger b the rise continues but at almost constant rate, with the slope falling off gently to about 1.5 at $b=1$ and 1.1 at $b=10$. For $r_0=0.95$ the graph of Fig. 1 includes binding energies ranging from 0 to 60 MeV, more than covering the region of physical interest.

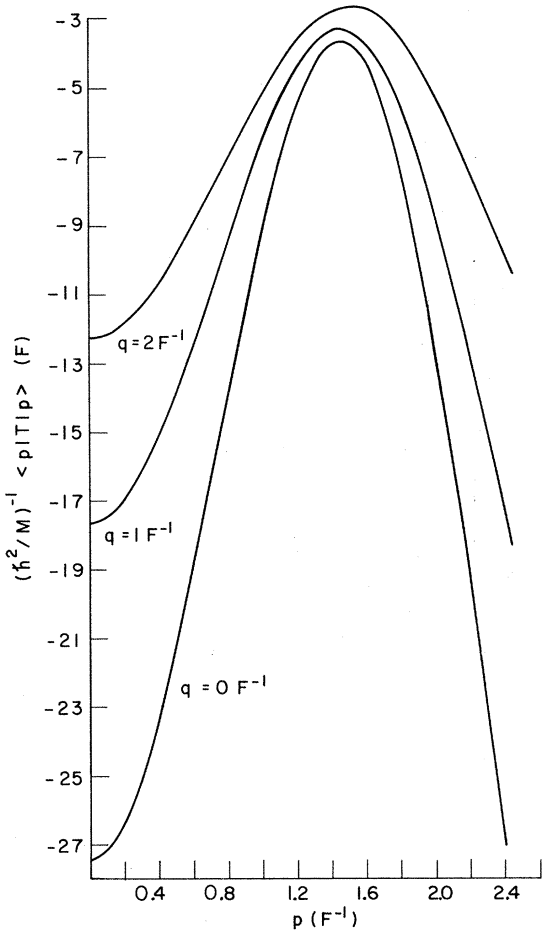

FIG. 2. Diagonal element of off-energy-shell two-nucleon T matrix $\langle p|T(\alpha^2-\frac{3}{4}q^2)|p\rangle$ as a function of p for various values of q, the momentum of the spectator nucleon. The boundary-condition parameters are $b=0.537$, $f=0.11$, and $r_0=0.95$ F corresponding to $\alpha r_0=0.43$ and a binding energy of 8.5 MeV.

The continued rise in αr_0 vs b for large b is due to the influence of the b-pole term in $D(\gamma)$ (3.9). This term becomes increasingly important as b increases through 1 because the denominator $b-\gamma\cot\gamma$ can be made to vanish for some γ, whereas this cannot occur for $b<1$. As a result there is a pronounced change in the character of the two-body off-shell wave function and T matrix for $q \lesssim 2$ F^{-1} when b passes through 1. The diagonal element of the T matrix for $b<1$ is exemplified by the plot of Fig. 2 for $\alpha r_0=0.43$, $b=0.537$, $q=0, 1, 2=0, 1, 2$ F^{-1}, and $r_0=0.95$ F. $\langle p|T|p\rangle$ is attractive and, in the range $pr_0=0$ to 2, is sensitive to the values of the boundary parameters b and f. For large pr_0, $\langle p|T|p\rangle$ continues in a manner similar to that obtained for a hard-shell potential, off the energy shell. For $b>1$, on the other hand, both the depth and the width of the attractive region, starting at small p, increase rapidly with b.

The corresponding situation for the off-shell wave function is that for $b<1$ the interior wave function ($r<r_0$) is appreciable only near the boundary r_0 whereas for $b>1$ the eigenvalues αr_0 are such that the amplitude of the interior wave function grows rapidly with increasing b. The growing attractiveness of T for $b>1$ is due simply to a strong coupling with the interior region, into which an increasing amount of wave function is fed as b increases. It is apparently not sufficient, in this problem, to represent the effect of the potential in $r<r_0$ by the boundary parameter b when $b \gg 1$. This description is different from what is encountered in a nuclear-matter calculation in which the density is held fixed as b is increased.[16] For fixed density the two-body Bethe-Goldstone wave function would go to zero at $r_0-\epsilon$ as b becomes infinite producing a repulsive T matrix so that the binding energy per nucleon at fixed density does not continue to rise with increasing b.

The experimental triton binding energy of 8.5 MeV corresponds to $\alpha r_0=0.43$ for $r_0=0.95$ F. We study the eigenfunctions of the three-nucleon system at this energy ($b=0.537$) and at the two neighboring solutions $b=0.50$, $\alpha r_0=0.29$ (3.86 MeV) and $b=0.60$, $\alpha r_0=0.65$ (19.4 MeV). This is just the region over which αr_0 varies most rapidly

with b. The spectator functions $H_1(q)$ and $H_2(q)$, the solutions of (3.9), are shown in Fig. 3 for these three values of b. The functions are normalized to $H_2(0)=1$. It is seen that H_2 is always appreciably larger in magnitude than H_1 for small q, a result which is traceable to the fact that for the boundary parameters considered the matrix $D(\gamma)$ is approximately $\begin{pmatrix} -1 & \frac{1}{4} \\ 1 & -\frac{1}{4} \end{pmatrix}$ at $q=0$.

For $q>5$ F^{-1} the spectator functions H_1 and H_2 execute oscillations about zero damped as approximately q^{-2} and with a period of about $4\pi/r_0$, characteristic of the kernel $K(q, q')$ (3.7) for q' fixed and $q\to\infty$. The behavior of the spectator functions for large q is essentially independent of b and f and arises from a two-body hard-shell interaction. This is seen from the two-body phase shift (2.10) which goes into the hard-core phase shift plus $\pi/2$ as p becomes infinite.

The b dependence of the spectator functions for $q\lesssim 3$ F^{-1} is quite apparent. Clearly H_2 drops off more slowly for $b=0.60$ than for $b=0.50$ meaning, as might be expected, that the amount of moderately large-q component in the wave function increases relative to the small-q component with increasing b and α. The details of the q dependence of H_1 and H_2 are important in the determination of the triton form factors and average kinetic energy. The values of the kinetic energy for the three values of b, calculated according to (3.16), are given in Table I. At the experimental binding energy, $\alpha r_0=0.43$, the average kinetic energy of 69 MeV is more than twice the value obtained by Brayshaw and Buck using local square-well potentials,[17] and indicates that there is too much high-q component in the spectator functions. This problem is discussed further in Sec. IV B in relation

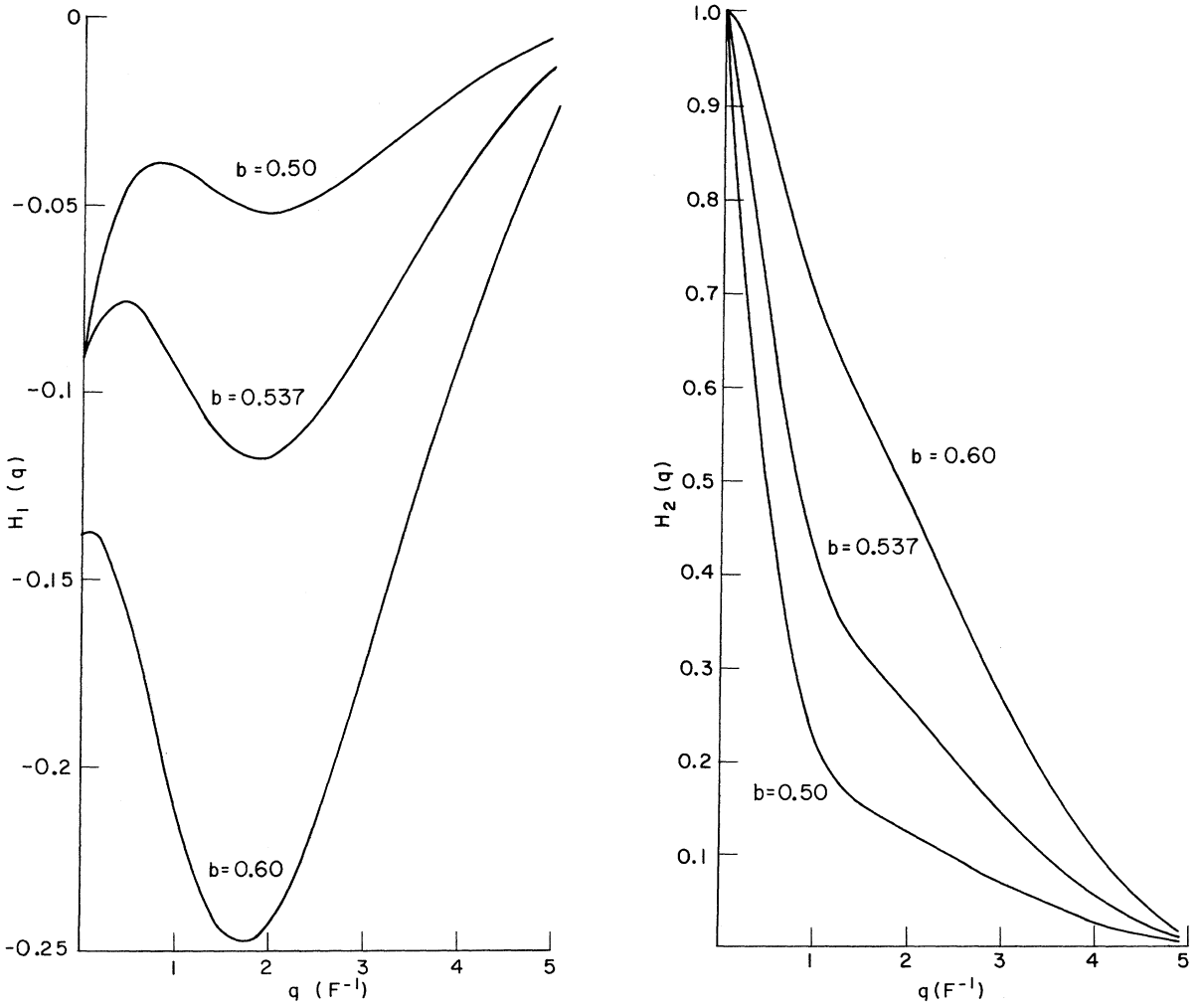

FIG. 3. The spectator functions $H_1(q)$ and $H_2(q)$ for several values of b. The functions for each value of b are normalized to $H_2(0)=1$. The triton binding energies are, respectively, 3.86, 8.5, and 19.4 MeV for b equal to 0.50, 0.537, and 0.6, taking $r_0=0.95$ F.

TABLE I. The triton eigenvalue αr_0 for $f=0.11$ and several values of the boundary-condition parameter b. The binding energy and the average kinetic energy are evaluated for the boundary radius $r_0=0.95$ F.

b	αr_0	Binding energy ($r_0=0.95$ F) (MeV)	Average kinetic energy ($r_0=0.95$ F) (MeV)
0.50	0.29	3.86	39.1
0.537	0.43	8.50	69.0
0.60	0.65	19.4	99.3

to the triton form factors.

B. Charge Form Factor and Radius

The body form factor $F(q^2)$ was calculated according to (3.12), and the triton charge form factor $F_{ch}^{H^3}(q^2)$ according to (3.11) using $F_{ch}^n=0$ and the proton form factor F_{ch}^p from the three-pole fit of Janssens *et al.*[18] The integrations were performed to an accuracy of about 5%. The resultant $F_{ch}^{H^3}(q^2)$ is plotted in Fig. 4 and Table II contains values computed for $q^2 \geq 1$ F^{-2} and $b=0.5$, 0.537, and 0.6,

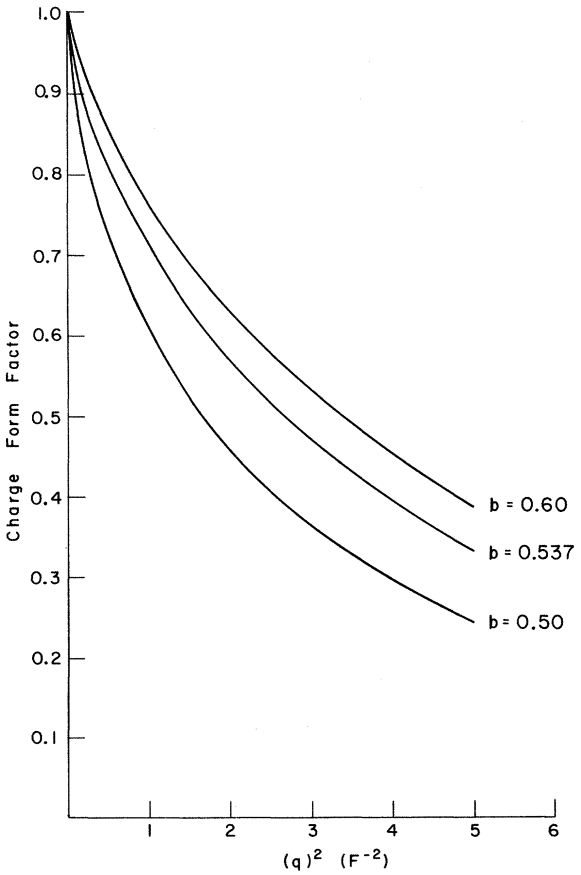

FIG. 4. Triton charge form factor $F_{ch}^{H^3}(q^2)$ for several values of b, and $r_0=0.95$ F.

as well as the experimental values of the H^3 and He^3 charge form factors.[19] If one excludes Coulomb and meson exchange effects, then the several percent of S' state needed to explain the difference between the H^3 and the He^3 charge form factors must come from the spin dependence of the two-nucleon interaction. No distinction is made here between the singlet and triplet forces, and so some sort of average of the two experimental charge form factors should be used for comparison. In either case the disagreement is large for $q^2>1$ F^{-2}. At $q^2=1$ F^{-2} and b between 0.5 and 0.537 there is approximate agreement with experiment. But none of the calculated form factors drop off as fast as the experimental form factor for $q^2>1$ F^{-2}. This slow decrease of the form factors clearly indicates that the spectator functions H_1 and H_2 contain relatively too much high-q component.

The body radius a was determined by fitting polynomial curves to $F(q^2)$ for small q^2 and taking their slope as $q^2 \to 0$. From (3.11) the triton charge radius is given by $a_{ch}^2(H^3)=a^2+2a_{ch}^2(n)+a_{ch}^2(p)$, where $a_{ch}(n)$ and $a_{ch}(p)$ are the neutron and proton charge radii. Table III contains the values of a and $a_{ch}(H^3)$ calculated using $a_{ch}(p)=0.8$ F and $a_{ch}^2(n)=-0.126$ F^2. The experimental values are 1.70 F for $a_{ch}(H^3)$[19] and 1.88 F for $a_{ch}(He^3)$.[20] The charge radii calculated for He^3 differ from the triton radii only in the amount of $a_{ch}(n)$. As should be expected the radius decreases with b. The charge radius at the experimental triton binding energy for $r_0=0.95$ is larger than the experimental radius, but the experimental value does lie between the values of $a_{ch}(H^3)$ calculated for $b=0.537$ and $b=0.6$.

The fact that agreement with experiment occurs at larger b for the charge radius than for the form factor at $q^2=1$ F^{-2} is indicative of the large curvature that has been obtained in $F_{ch}^{H^3}$ as seen in Fig. 4. This curvature implies that the charge distribution has a long tail, not unlike the experimental

TABLE II. The triton charge form factor $F_{ch}^{H^3}(q^2)$ for $f=0.11$ and $r_0=0.95$ F, calculated for several values of b. The first three columns are computed values of the form factor, and the last two columns are the experimental charge form factors for the triton and helium-3 (Ref. 19).

q^2 (F^{-2})	$b=0.50$ $\alpha r_0=0.29$	$b=0.537$ $\alpha r_0=0.43$	$b=0.60$ $\alpha r_0=0.65$	$F_{ch}^{H^3}$(exp.)	$F_{ch}^{He^3}$(exp.)
1.0	0.601	0.707	0.763	0.622	0.567
2.0	0.458	0.569	0.631	0.387	0.329
3.0	0.360	0.467	0.529	0.267	0.209
4.0	0.296	0.390	0.449	0.175	0.1326
5.0	0.240	0.329	0.384	0.118	0.0813

TABLE III. The body radius a of the triton, and the charge radius $a_{ch}(H^3)$ for several values of b ($f = 0.11$, $r_0 = 0.95$ F.) For comparison the experimental charge radii of H^3 and He^3 are (Refs. 19 and 21) $a_{ch}(H^3) = 1.70$ F and $a_{ch}(He^3) = 1.88$ F.

b	a (F)	$a_{ch}(H^3)$ (F)
0.50	2.45	2.53
0.537	1.84	1.95
0.60	1.45	1.58

result recently obtained by McCarthy *et al.*[21] for the charge form factor of He^3.

The q dependence of the form factor can be most easily adjusted by increasing the BC boundary radius, but this cannot be done without altering the fit to the two-body data. Fortunately, when the tensor force is included in the two-body BC interaction, the boundary parameter has the value $r_0 = 1.2$–1.3 F[4]. In the present calculation with $r_0 = 0.95$ F the spectator function H_1 for $\alpha r_0 = 0.43$ (8.5 MeV) drops to half its peak ($q = 0$) value at $q \approx 0.8$ F^{-1} (Fig. 3). This may be compared with the substantially more rapid decrease in the spectator functions obtained by Gupta, Bhakar, and Mitra[14] using separable central and tensor potentials, where the drop to half peak ($q = 0$) value occurs at $q \approx 0.44$ F^{-1}.

V. DISCUSSION AND CONCLUSION

The results obtained here for the triton binding energy, wave function, charge form factor, and charge radius show a pronounced dependence on the off-energy-shell boundary parameter b, the logarithmic derivative of the two-nucleon wave function inside the boundary radius r_0. Using an average singlet and triplet two-body interaction with boundary radius $r_0 = 0.95$ F, the values of b of physical interest are found to be in the range from b equals 0.5 to 0.6. The predictions are not entirely consistent with experiment and, in particular, the charge form factor at the experimental binding energy falls off too slowly as a function of q^2. This property of the form factor can be traced to the presence of relatively too much large-q component in the spectator functions $H_1(q)$ and $H_2(q)$.

The inclusion of a tensor force in the BC interaction could significantly alter the results obtained here and their quantitative dependence on the interior boundary condition. Calculations for the three-nucleon problem[14] and for nuclear matter (Haftel and Tabakin[1]) indicate that a tensor force should decrease the equilibrium density and the magnitude of the binding energy for a system of more than two nucleons. Thus a BC interaction described in terms of the boundary conditions f_0 for singlet states and the 2×2 matrix f for triplet states plus a single interior boundary condition b should result in a triton with a larger radius and a smaller binding energy than obtained here for the same value of b, necessitating an increase in b to fit the data. Such an interaction would give some S' state in the triton wave function and would make reasonable the calculation of the doublet scattering length, which is particularly sensitive to off-shell effects,[22,23] as well as, hopefully, improving the fit to the form factors. Once the interior boundary condition is also altered to include the tensor force in terms of a parameter b_0 for the singlet state and a 2×2 matrix for the triplet states, the expectation is that the presence of three more adjustable parameters should provide substantial leeway for fitting the triton and helium-3 properties.

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[1] F. Coester, S. Cohen, B. Day, and C. M. Vincent, Phys. Rev. C 1, 769 (1970); M. Haftel and F. Tabakin, to be published.

[2] J. A. Tjon, B. F. Gibson, and J. S. O'Connell, Phys. Rev. Letters 25, 540 (1970).

[3] E. L. Lomon and H. Feshbach, Ann. Phys. (N.Y.) 48, 94 (1968); 29, 19 (1964).

[4] H. Feshbach and E. L. Lomon, Phys. Rev. 102, 891 (1956).

[5] M. M. Hoenig and E. L. Lomon, Ann. Phys. (N.Y.) 36, 363 (1966).

[6] L. D. Faddeev, Zh. Eksperm. i Theor. Fiz. 39, 1459 (1960) [transl.: Soviet Phys.—JETP 12, 1014 (1961)].

[7] A. N. Mitra, Nucl. Phys. 32, 521 (1962).

[8] C. Lovelace, Phys. Rev. 135, B1225 (1964).

[9] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (U. S. Government Printing Office, Washington, D. C., 1964).

[10] M. G. Fuda, Phys. Rev. 166, 1064 (1968).

[11] V. S. Bhasin, Nucl. Phys. 58, 636 (1964).

[12] A. N. Mitra, G. L. Schrenk, and V. S. Bhasin, Ann. Phys. (N.Y.) 40, 357 (1966).

[13] L. I. Schiff, Phys. Rev. 133, B802 (1964).

[14] V. K. Gupta, B. S. Bhakar, and A. N. Mitra, Phys.

Rev. 153, 1114 (1967).
[15]E. L. Lomon and M. McMillan, Ann. Phys. (N.Y.) 23, 439 (1963).
[16]The properties of nuclear matter as a function of the interior BC have recently been studied with the complete BC-model interaction. E. L. Lomon, private communication.
[17]D. D. Brayshaw and B. Buck, Phys. Rev. Letters 24, 733 (1970).
[18]T. Janssens, R. Hofstadter, E. B. Hughes, and M. R. Yearian, Phys. Rev. 142, 922 (1966).
[19]H. Collard, R. Hofstadter, E. B. Hughes, A. Johansson, and M. R. Yearian, Phys. Rev. 138, B57 (1965).
[20]C. de Vries, R. Hofstadter, A. Johansson, and R. Herman, Phys. Rev. 134, B848 (1948).
[21]J. S. McCarthy, I. Sick, R. R. Whitney, and M. R. Yearian, Phys. Rev. Letters 25, 884 (1970).
[22]V. A. Alessandrini and C. A. Garcia Canal, Nucl. Phys. A133, 590 (1969).
[23]T. Brady, E. Harms, L. Laroze, and J. S. Levinger, Phys. Rev. C 2 59 (1970).

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Multiple Scattering of Pions by Deuterons*

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The general theory of multiple scattering of pions from nuclei is expressed in a way which does not require the use of a series expansion. In an on-shell approximation this theory may be reduced to the solution of a set of A coupled integral equations. As a demonstration of the method the equations are specialized to the case of the deuteron and solved to give comparison with the experimental data. The deuteron tensor T_{20} is shown to be sensitive to the percentage of D state for deuterons scattered at 0°.

I. INTRODUCTION

In the energy region of 0–500-MeV pion kinetic energy there exists no completely acceptable theory of pion-nucleus scattering. The high-energy eikonal approximation due to Glauber[1,2] has been successful in explaining high-energy scattering data at small angles. In spite of a number of attempts to increase its range of validity,[3] it is doubtful if this basic approach would be of value for large angles in this energy range.

The Watson multiple-scattering series has been used with some success for pion-deuteron scattering by evaluating the double-scattering term in various approximations.[4] However, it is difficult to go beyond the second term and, furthermore, for large nuclei and near a resonance it is not certain that the series even converges.

Optical-model calculations[5] have been the most successful in this region, but their application is limited to large-A nuclei. Their use near a resonance also has questionable validity.

For these reasons the present paper attempts to develop a theory with the following aims:
(i) There should be no small-angle or forward-scattering approximations, as these are not appropriate for this energy regime.
(ii) There should be no truncation of the multiple-scattering series to avoid questions of convergence near a resonance.
(iii) There should be only on-shell information required (at least in the first-order theory) to make the calculation as simple as possible.

In order to develop such a theory we shall use the form of multiple-scattering theory used some years ago by Foldy and Brueckner[6] and more recently by Seki.[7] This method expresses the multiple-scattering amplitude from one of the nucleons as the simple-scattering amplitude plus a term which looks very much like a double-scattering amplitude. The difference between this second term and an actual double-scattering term is that one of the simple amplitudes has been replaced by the multiple-scattering amplitude. Thus, one has implicit equations for the multiple-scattering amplitudes. With the aid of some approximations these equations can be brought into a solvable form.

These general equations are developed in Sec. II and it will be seen that they may be written as coupled integral equations over angular variables. In Sec. III the equations are specialized to the case of the deuteron and solved in double scattering at high energies to compare with Glauber theory. Here also in Sec. III the case of π-deuteron scattering just below the (3, 3) resonance is calculated and compared with experiment.

II. THEORY

Let us consider the scattering from A fixed nu-