Study of Equivalent Local Potentials Obtained from Separable Two-Nucleon Interactions

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A method of equivalent local potentials is applied to separable interactions which fit the $^1S_0$ two-nucleon phase shifts. The method of generating equivalent local potentials used in this paper is independent of the boundary conditions imposed on the solutions of the nonlocal equation; consequently, all solutions of the nonlocal equation lead to the same equivalent local potential. The uniqueness of the equivalent local potential obtained by the present method is considered to be useful for the purpose of understanding nonlocal interactions. The Yamaguchi, the one- and two-term Tabakin, and the case-IV Mongan potentials are studied. The equivalent local Yamaguchi potential is similar to low-energy local two-nucleon interactions which do not have short-range repulsion. The two-term Tabakin potential results in an equivalent local potential with short-range repulsion. The occurrence of spurious states in the one-term Tabakin potential is related to a class of zeros of this separable interaction in coordinate space. The case-IV Mongan potential results in an equivalent local potential with strong short-range attraction. Further study of this interaction revealed a spurious state at 19.6 BeV which causes the wave function to have an additional node at experimentally relevant energies. The paper is concluded with an examination of the problem of constructing separable interactions which have short-range repulsion. It is shown that such interactions have a pronounced tendency to produce spurious states.

I. INTRODUCTION

The use of separable interactions to fit the two-nucleon phase shifts is well established,[1-3] and attention is now being directed toward the development of methods[4] which may provide a greater understanding of these interactions. In general these methods involve a comparison between the characteristics of separable or nonlocal potentials and the more familiar phenomenology of local potentials. However, the usual practice with regard to the nucleon-nucleon interaction has been to restrict the comparison to such features as the phase shifts or T matrix elements. Until recently, no attempt has been made to relate the separable form of the interaction to an "equivalent" local form in coordinate space. While an equivalent-local-potential analysis of the type considered here is quite common in optical-model studies[5] where only the average features of the interaction are of interest, a number of ambiguities in the definition of these potentials have rendered a detailed interpretation questionable. Recently, Coz and the present authors[6,7] have shown that a more precise definition of an equivalent local potential exists, and that a unique potential and solutions to an equivalent local equation can be obtained once two independent solutions of the nonlocal equation are known. The form of this equivalent local potential was first derived by Fiedeldey,[8] who used two particular solutions of the nonlocal equation. In this method of equivalent local potentials, the solutions to the nonlocal equation are related to the solutions of the equivalent local equation by

$$\psi_N(r) = A(r)\psi_L(r) .$$

The function $A(r)$, which is called the damping function, represents the main difference between the nonlocal and equivalent local equations in coordinate space.

Previously, we have applied the method of equivalent local potentials to several nonlocal Hartree-Fock nucleon-nucleus potentials.[6] The calculations yield equivalent local potentials which are qualitatively in agreement with conventional local potentials. Their energy dependence varies slightly over the energy range for which the original nonlocal potentials are applicable. The damping functions calculated in this application are smaller than one in the interior of the nucleus, and the nonlocal wave function is said to be damped relative to the equivalent local wave function. This is consistent with previous results obtained by other methods.[9] It has since been learned that the origin of this damping is probably due to the nonlocal two-nucleon interactions used in the original Hartree-Fock calculations rather than to the nonlocality which is the result of the exchange term in the Hartree-Fock equations. Subsequent calculations[10] with a local two-nucleon interaction indicate that the exchange term results in a small amount of antidamping $[A(r) > 1]$, much smaller than the damping $[A(r) \leq 1]$ obtained when a nonlocal two-nucleon interaction is used. The nonlocality of the two-nucleon interaction appears to be more important than the nonlocality due to the exclusion prin-

ciple. For this reason, we have made a study of several current nonlocal two-nucleon interactions.

In this paper, we apply the method of equivalent local potentials to separable interactions which fit the $^1S_0$ phase shifts[11] from 0 to 350 MeV. Our purpose is to determine if the equivalent local potentials obtained from these interactions are similar to phenomenological local two-nucleon interactions[12] throughout this energy range, and, in particular, to determine if short-range repulsion is reproduced. We would also like to see if the damping functions obtained by this method are similar to the damping functions obtained previously.[6]

II. SUMMARY OF THE METHOD

The method used to construct an equivalent local potential is described in previous work,[6,7] and in this summary we will state the results and mention some specific properties. For the purposes of this paper, a real and symmetric nonlocal potential is considered, and the results which will be stated are valid only for nonlocal potentials that satisfy these conditions. In the coordinate representation, the s-wave radial equation for a nonlocal potential is

$$\phi_N''(k,r)+k^2\phi_N(k,r)=\int_0^\infty V(r,s)\phi_N(k,s)ds\,. \quad (1)$$

The solutions of this equation are related to the solutions of an equivalent local equation

$$\phi_L''(k,r)+k^2\phi_L(k,r)=V(k,r)\phi_L(k,r) \quad (2)$$

by the expression

$$\phi_N(k,r)=A(k,r)\phi_L(k,r)\,. \quad (3)$$

$V(k,r)$ and $A(k,r)$ are called the equivalent local potential and damping function, respectively. Equation (3) is common to all methods of constructing an equivalent local potential which are based on a comparison of Eqs. (1) and (2) in the coordinate representation. The difference between this method and previous methods[5] is that in this method Eq. (3) is used to relate two linearly independent solutions of Eqs. (1) and (2). One of the features of this method is that any convenient pair of linearly independent solutions may be used. This is because the equivalent local potential and the functional form of the damping function are independent of the boundary conditions imposed on the solutions used to calculate them.[7]

The linearly independent pair of solutions used in this paper are the "irregular solutions" $f_\pm(k,r)$ which satisfy the boundary conditions

$$\lim_{r\to\infty} e^{\mp ikr}f_\pm(k,r)=1\,. \quad (4)$$

These solutions have the property

$$f_-(k,r)=f_+^*(k^*,r)\,. \quad (5)$$

The quantity

$$J(k,r)=(2ik)^{-1}[f_-(k,r)f_+'(k,r)-f_+(k,r)f_-'(k,r)] \quad (6)$$

is the Wronskian of the pair of irregular solutions normalized so that

$$\lim_{r\to\infty} J(k,r)=1\,. \quad (7)$$

Equation (6) applies to the irregular solutions for both Eqs. (1) and (2); but for a local potential,

$$J_L(k,r)=1 \quad (8)$$

for all k and r, since the Wronskian of two linearly independent solutions is a constant. One of the important differences between nonlocal and local potentials is the fact that the Wronskian of two linearly independent solutions of the radial equation for a nonlocal potential is not a constant. The quantity $J_N(k,r)$ represents an intrinsic difference between nonlocal and local potentials, and deviations of $J_N(k,r)$ from 1 are a measure of the nonlocality of a potential as a function of k and r. An integral expression for $J(k,r)$,

$$J(k,r)=1-\int_r^\infty\int_0^\infty V(s,t)Q(k,s,t)dtds\,, \quad (9)$$

where

$$Q(k,r,s)=(2ik)^{-1}[f_-(k,r)f_+(k,s)-f_+(k,r)f_-(k,s)]\,, \quad (10)$$

shows the distinction between nonlocal and local potentials more clearly. In the limit as $V(s,t)\to U(s)\delta(s-t)$, $J(k,r)$ is 1. If the nonlocal potential is symmetric, $J(k,r)$ satisfies

$$J(k,0)=J(k,\infty)\,. \quad (11)$$

We mention these properties of the Wronskian for several reasons. First, the properties of the Wronskian are independent of any method of constructing an equivalent local potential. Moreover, since the construction of an equivalent local potential implies a transformation from Eq. (1), where the Wronskian is not a constant, to Eq. (2), where the Wronskian is a constant, the differing properties of Wronskians for nonlocal and local potentials should be taken into account in some way. Second, Wronskians are intimately related to linear independence, the one feature that distinguishes the present method of constructing an equivalent local potential from previous methods. Third, and most important, the Wronskian of the pair of irregular solutions is identical, apart from a constant factor, to the radial probability current den-

sity. The radial probability current density $S(k,r)$ is related to $J(k,r)$ by

$$S(k,r) = (\hbar k/m)J(k,r) .$$

The constancy of the Wronskian for a local potential, Eq. (8), is equivalent to the statement that real local potentials conserve radial probability current *density*, or conserve flux locally. Nonlocal potentials do not conserve flux locally, but symmetric nonlocal potentials conserve flux globally, or conserve probability current, by virtue of Eq. (11). Linear independence and Wronskians are directly related to the probability current density. Accordingly, we call $J(k,r)$ the radial probability current density, or radial current when no confusion can arise.

The equivalent local potential and damping function obtained by the present method are

$$V(k,r) = -\frac{1}{2}\frac{J''}{J} + \frac{3}{4}\left(\frac{J'}{J}\right)^2 - \frac{1}{J}\int_0^\infty V(r,s)\frac{d}{dr}Q(k,r,s)ds \tag{12}$$

and

$$A(k,r) = J(k,r)^{1/2} . \tag{13}$$

It may be shown[7] that the equivalent local potential is real for real values of k^2. Provided that the $J(k,r)$ is positive, the damping function is also real. Both $V(k,r)$ and $J(k,r)$ depend parametrically on k^2. This energy dependence is a reflection of the dispersion inherent in wave equations with nonlocal potentials relative to equations with energy-independent local potentials. It may be shown that the nonlocal and equivalent local potentials have the same bound-state energies and scattering phase shifts. Once the solutions to the nonlocal equation have been generated, the equivalent local potential and damping function are quite easy to evaluate.

III. SOLUTIONS OF THE NONLOCAL EQUATION

If the nonlocal potential has the separable form

$$V(r,s) = \sum_i \lambda_i n_i(r) n_i(s) , \tag{14}$$

then the solutions of Eq. (1) can be obtained by algebraic methods. All phenomenological separable two-nucleon interactions have this form. The irregular solutions satisfy the integral equation

$$f_\pm(k,r) = e^{\pm ikr} - \int_r^\infty G(k,r,r')\int_0^\infty V(r',s)f_\pm(k,s)dsdr' \tag{15}$$

and are given by

$$f_\pm(k,r) = e^{\pm ikr} - \int_r^\infty \frac{G(k,r,r')\rho_\pm(k,r')dr'}{D(k)} , \tag{16}$$

where

$$G(k,r,r') = k^{-1}\sin k(r-r') \tag{17}$$

and

$$\rho_\pm(k,r) = \sum_i \lambda_i n_i(r) E_i^\pm(k) .$$

$D(k)$ is the determinant of the matrix with elements

$$D_{ji}(k) = \delta_{ji} - \int_0^\infty\int_0^r G(k,r,s)n_j(r)\lambda_i n_i(s)dsdr , \tag{18}$$

and $E_l^\pm(k)$ is the determinant of the matrix obtained by replacing the lth column of the matrix with elements $D_{ji}(k)$ by $e_j^\pm(k,0)$, where

$$e_j^\pm(k,r) = \int_r^\infty e^{\pm iks}n_j(s)ds . \tag{19}$$

By substituting the irregular solutions into Eq. (9), the radial current for a separable nonlocal potential becomes

$$J(k,r) = 1 - \frac{1}{2ik}\int_r^\infty \frac{[f_-(k,r)\rho_+(k,s) - f_+(k,r)\rho_-(k,s)]ds}{D(k)} . \tag{20}$$

Another solution of Eq. (1) that will be used is the regular solution

$$\phi(k,r) = k^{-1}\sin kr + \int_0^r \frac{G(k,r,r')\rho(k,r')dr'}{D(k)} , \tag{21}$$

where

$$\rho(k,r) = \sum_i \lambda_i n_i(r) N_i(k) \tag{22}$$

and

$$N_i(k) = (2ik)^{-1}[E_i^+(k) - E_i^-(k)] . \tag{23}$$

A few comments about the determinant $D(k)$ are in order. $D(k)$ is the Fredholm determinant of the integral equations for the regular and irregular solutions to Eq. (1). The Fredholm determinant $D^\pm(k)$ of the integral equation for the physical wave function is often referred to in studies of both local and nonlocal potentials. The zeroes of $D^\pm(k)$ correspond to the spectrum of the integral equation for the physical wave function. The determinant $D(k)$ is rarely discussed in either nonlocal- or local-potential analyses. For local potentials this is understandable, since the integral equations for the regular and irregular solutions are Volterra equations, and the Fredholm determinant for a Volterra integral equation is identically equal to 1. For nonlocal potentials the integral equations for the regular and irregular solutions are Fredholm integral equations. Consequently, the integral equations for these solutions have a spectrum that corresponds to the zeros of $D(k)$.

The spectrum of $D(k)$ is important in the study of nonlocal potentials, in some cases as important as the spectrum of $D^+(k)$. The zeros of $D(k)$ correspond to effects that cannot occur for short-range local potentials, and are called spurious states for this reason. For short-range local potentials, $D^+(k)$ cannot be zero for real values of k. For nonlocal potentials, $D^+(k)$ can be zero for real values of k, and these zeros are called continuum bound states. Since $D^+(k)$ and $D(k)$ are related by

$$D^+(k) = D(k) + \sum_i \lambda_i N_i(k)\, e_i^+(k, 0) \tag{24}$$

and since $D(k)$ can be zero for real values of k, it is apparent that there may be some connection between spurious states and continuum bound states.

None of the currently acceptable separable two-nucleon interactions, which are used to approximate local potentials, have been studied for the purpose of determining whether these interactions admit spurious states. With regard to the present study, two properties of spurious states are important. First, for energies near spurious-state energies, the correspondence between nonlocal and local potentials is tenuous, and the equivalent local potentials obtained by the present method exhibit behavior which would be termed "pathological" if it occurred for a local potential. This can be seen by noting that $D(k)$ occurs in the denominator of $J(k,r)$ in Eq. (20). When $D(k)$ is near zero, it is possible for $J(k,r)$ to have zeros; and when $D(k)$ is zero, $J(k,r)$ is infinite. Since $J(k,r)$ occurs in the denominator of $V(k,r)$ in Eq. (12), it is clear that the zeros of $J(k,r)$ lead to poles of $V(k,r)$. Second, the occurrence of spurious states at one energy may cause effects at energies which are far removed from the spurious-state energy. These effects may be apparent in the equivalent local potential.

IV. YAMAGUCHI POTENTIAL

Some time ago, Yamaguchi[13] demonstrated that a single separable potential provides an adequate account of the binding energy of the deuteron and the low-energy two-nucleon scattering parameters. Since many of the more recent two-term separable potentials which fit the two-nucleon phase shifts use the Yamaguchi potential for the attractive term, it is convenient to study this potential first.

The Yamaguchi potential is[13]

$$V(r,s) = \lambda\, e^{-\beta(r+s)},$$

and a linearly independent pair of solutions follow directly from Eq. (16). The bound- or antibound-state energy is obtained from

$$\lambda = -2\beta(\beta+\alpha)^2,$$

where the wave number α is positive or negative according to whether the state is bound or antibound. The wave function for this state is

$$f^+(i\alpha, r) = e^{-\alpha r} - e^{-\beta r}.$$

The bound-state wave function for the Yamaguchi potential is identical to the wave function for the first bound state of the Hulthén potential with range $(\beta-\alpha)^{-1}$ and depth $-(\beta^2-\alpha^2)$. A Hulthén potential with these parameters is used for comparison with the equivalent local Yamaguchi potential. The parameters for the Yamaguchi potential which fit the deuteron binding energy (2.2255 MeV) and the $^3S_1$ scattering length (5.411 F) are $\lambda = -7.533\ \mathrm{F}^{-3}$ and $\beta = 1.4054\ \mathrm{F}^{-1}$. By taking the same range for the $^1S_0$ potential, the $^1S_0$ scattering length (−23.671 F) is fit with $\lambda = -5.237\ \mathrm{F}^{-3}$.

The construction of the equivalent local Yamaguchi potential is not difficult. The radial current for the Yamaguchi potential is obtained from Eq. (20); it is

$$J(k,r) = 1 + \frac{\lambda\, e^{-\beta r} \sin(kr)}{k(\beta^2 + k^2 - \lambda/2\beta)}.$$

The equivalent local potential is obtained from Eq. (12) and is plotted in Fig. 1. The $^3S_1$ parameters were used for this plot. Results for the $^1S_0$ param-

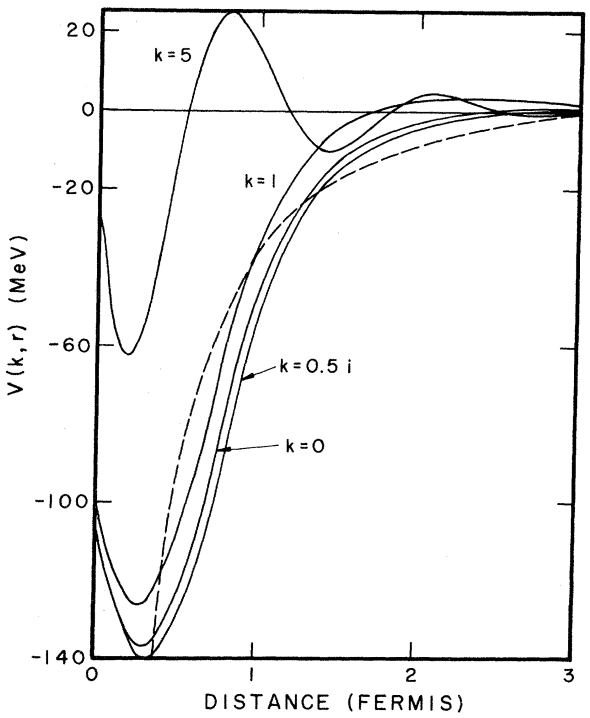

FIG. 1. Equivalent local Yamaguchi potential. The dashed line is the Hulthén potential which has the same bound-state energy and wave function as the Yamaguchi potential.

eters are not substantially different, since the nonlocal potentials for both partial waves have the same range. The smaller strength of the $^1S_0$ potential leads to a corresponding smaller strength of its local equivalent and an almost imperceptible shift in the extremal points of the radial current and equivalent local potential.

Both the radial current and equivalent local potential become oscillatory for $k > 1$. For $k < 1$, the equivalent local potential is reasonable and tracks with the Hulthén potential for distances greater than 0.5 F. The deviations from the Hulthén potential curve for these distances can be attributed to the differences in the ranges of the Yamaguchi and Hulthén potentials. The equivalent local potential and the Hulthén potential differ for distances less than 0.5 F. This is due to the fact that a continuous nonlocal potential cannot lead to poles in the equivalent local potential unless the radial current has zeros. Since the two-nucleon interaction is not well known for distances less than 0.5 F, the structure of the equivalent local potential for small distances does not prejudice its physical content. It is somewhat interesting that the deviations begin to occur at about the same distance that the two-nucleon interaction begins to be poorly determined.

V. TWO-TERM TABAKIN POTENTIAL

Tabakin[1] has parametrized two-term separable interactions which fit the nucleon-nucleon scattering data reasonably well. Our analysis is restricted to the $^1S_0$ partial wave where the first term is an attractive interaction of the Yamaguchi type. Tabakin's purpose in introducing the second term was to construct a repulsive part of the interaction which would dominate at high energies in much the same way as a repulsive core in a local interaction. Since the equivalent local Yamaguchi potential does not exhibit short-range repulsion, its occurrence in the equivalent local Tabakin potential would have to be due to the second term in Tabakin's interaction.

The nonlocal Tabakin interaction to be used in Eq. (1) is given in coordinate space by

$$V(r,s) = -g_0(r)g_0(s) + h_0(r)h_0(s)\,,$$

where

$$g_0(r) = \gamma e^{-ar}$$

and

$$h_0(r) = \beta e^{-br}\left[\frac{d^2 - b^2}{2db}\sin dr + \cos dr\right];$$

all parameters are defined in Tabakin's paper.[1] The resulting $^1S_0$ equivalent local Tabakin interaction is shown in Fig. 2 as a function of r for incident laboratory energies of 10, 60, 160, 260, and 310 MeV. The variation with energy is smooth and the repulsive core is evident, becoming stronger with increasing energy. The form of the separable interaction seems to be such that attractive and repulsive effects are spread out in the equivalent local potential and tend to weaken one another. Perhaps the most striking difference between the equivalent local potential and the usual local interactions that we have come to accept[12] is the lack of both height in the repulsive core and depth in the attractive region of the potential. However, the $^1S_0$ phase shifts have recently been fit with local potentials that have "supersoft" cores.[14] Apart from its energy dependence, the equivalent local Tabakin potential is similar to these supersoft nucleon-nucleon potentials.

The damping function $A(k, r)$ is shown in Fig. 3 for the same energies. It is remarkably similar to the damping function for the Yamaguchi poten-

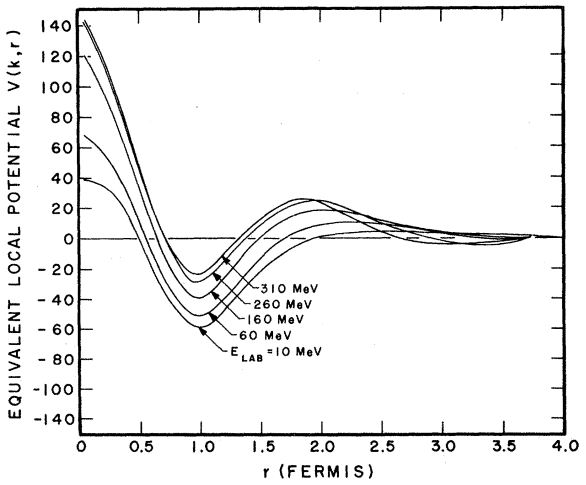

FIG. 2. Equivalent local potential for the two-term Tabakin potential.

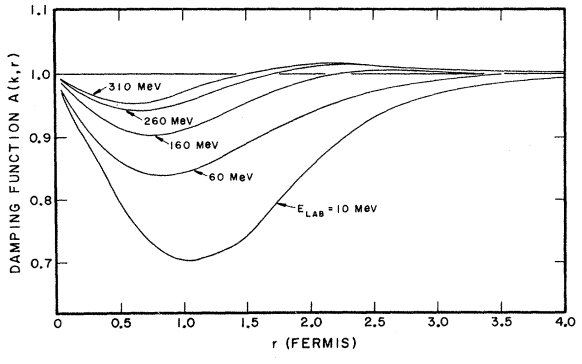

FIG. 3. Damping function for the two-term Tabakin potential.

tial. The shape, depth at the minimum point, and energy dependence are almost identical, and in both cases the damping of the nonlocal wave function with respect to the equivalent local wave function tends to disappear at high energies. The damping function is also similar to those obtained from the nonlocal nucleon-nucleus potentials[6] derived from the Hartree-Fock calculations with nonlocal two-nucleon interactions. Since these calculations used the two-term Tabakin potential for the two-nucleon interaction, this similarity indicates that the damping is due to the nonlocality of the Tabakin potential and is not a property of the Hartree-Fock exchange term. This is consistent with the observations in Ref. 10.

VI. ONE-TERM TABAKIN POTENTIAL

Tabakin[3] has also fit the $^1S_0$ phase shifts with a one-term separable potential. While a one-term separable potential is computationally simpler than the two-term potentials[1,2] used to simulate the repulsive core, the one-term Tabakin potential differs from interactions usually considered in that it has a spurious or continuum bound state. The physical significance of these states, which do not occur for short-range local potentials, is uncertain. Bolsterli[15] has emphasized that a continuum bound state is the same as a resonance with zero width, and that an additional interaction will spread the width and lead to an observable resonance in the cross section. Brady *et al.*[16] have argued that the wave function for the one-term Tabakin potential, which has an additional node, is more like the wave function for an attractive local potential with one bound state than a potential with no bound states and a repulsive core. The objections to the one-term Tabakin potential are cogent, and the efficacy of using separable potentials with spurious states is questionable. Our purpose in studying the one-term Tabakin potential is to determine if the method of equivalent local potentials can be used to provide information about separable potentials with spurious states.

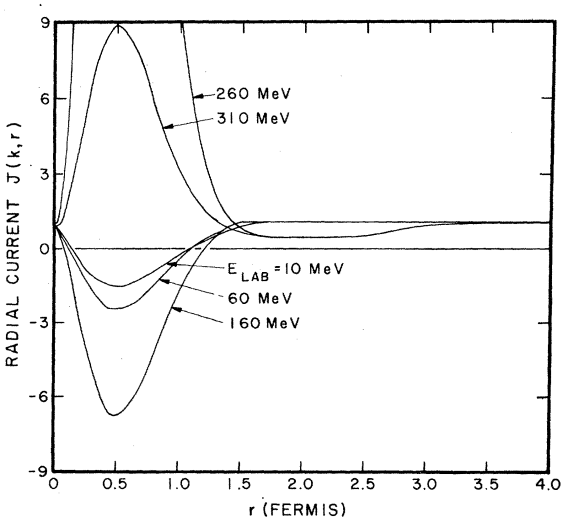

FIG. 4. Radial current for the one-term potential.

Figure 4 shows $J(k, r)$ for the one-term Tabakin potential as a function of r for the energies used previously. Since $J(k, r)$ can be negative, the damping function (13) and the relation between the nonlocal and equivalent local wave functions (3) are undefined. The behavior of $V(k, r)$ as a function of r is dominated by poles which are the result of $J(k, r)$ passing through zero. It is clear from Fig. 4 that the equivalent local potential can bear no resemblance to a typical local nucleon-nucleon interaction at any energy. The radial current is drastically different from those obtain previously; however, both $J(k,r)$ and $V(k,r)$ are well defined by Eqs. (9) and (12) and provide useful information for studying spurious states.

Referring to Fig. 4, we note that the extremal point of $J(k, r)$ at 0.5 F is invariant to changes in the energy. This extremal point coincides with a zero of $g_0(r)$, and it has been shown[7] that extremal points of this type can lead to zeros of $J(k, r)$ as the strength of the nonlocal potential is increased from zero. [$J(k, r)$ is identically equal to 1 if the strength of the potential is zero.] The two-term Tabakin potential also has zeros, but they are not of the type discussed here. The zeros of $g_0(r)$ correspond to zeros of a nonlocal potential $V(r, r')$ which are independent of one of the coordinates, whereas the zeros of the two-term Tabakin potential depend on both of the coordinates. The other two zeros of $g_0(r)$ near 2.0 F also lead to zeros of $J(k, r)$ which are not shown in Fig. 4, as the effect occurs between 240 and 260 MeV. We have not been able to show that zeros of potentials of the type that occur in the one-term Tabakin potential necessarily lead to zeros of $J(k,r)$; we can only state that these zeros are related to spurious states and are a possible source of trouble.

The spurious state of the one-term Tabakin potential occurs at a laboratory energy of 240 MeV. As shown in Fig. 5, this state corresponds to a zero of the Fredholm determinant $D(k)$. Bolsterli[15] has pointed out that the Fredholm determinant $D^{\pm}(k)$ is also zero at 240 MeV, but it is clear from Eq. (24) that $D^{\pm}(k)$ cannot be zero unless both $D(k)$ and $N_1(k)$ are zero at the same energy. The confluence of zeros of $D(k)$ and $N_1(k)$ is a very special case. It occurs in the one-term Tabakin potential because the parameters of the potential were intentionally adjusted so that the spurious-state energy [$D(k) = 0$] coincides with the energy at which the relative phase shift passes through zero [$N_1(k)$

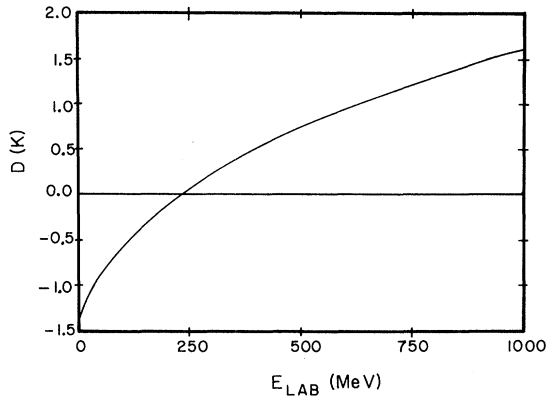

FIG. 5. Fredholm determinant $D(k)$ for the one-term Tabakin potential plotted as a function of energy.

$=0]$. Since $D(k)$ can be zero independent of $D^{\pm}(k)$, it is the zeros of $D(k)$ that cause spurious states. We will return to the discussion of spurious states later; at this point, it is sufficient to note that the undesireable properties[15,16] of the one-term Tabakin potential can be related to the zeros of $D(k)$.

VII. CASE-IV MONGAN POTENTIAL

Mongan[2] has recently constructed a number of separable interactions which fit the two-nucleon phase shifts. The case-II and case-IV $^1S_0$ potentials are identical and are referred to as the case-IV potential in this paper. This potential is interesting because both the attractive and repulsive terms of the interaction are of the Yamaguchi type. Exact solutions of the nonlocal equation can be obtained easily once the solutions for the Yamaguchi potential are known. The case-IV Mongan potential is

$$V(r,r') = \lambda_R e^{-\beta_R(r+r')} - \lambda_A e^{-\beta_A(r+r')}$$

in the coordinate representation. The values of the parameters used to fit the $^1S_0$ phase shifts are $\lambda_R = 3454.8\ \mathrm{F}^{-3}$, $\beta_R = 6.157\ \mathrm{F}^{-1}$, $\lambda_A = 28.293\ \mathrm{F}^{-3}$, $\beta_A = 1.786\ \mathrm{F}^{-1}$. The inverse ranges are as given in Table IV of Mongan's second paper.[2] The strengths are modified in the transformation to the coordinate representation. The strength parameters used in this paper are related to the parameters in Mongan's paper by

$$\lambda = \tfrac{1}{2}\pi C^2/41.468 .$$

Exact expressions for the solutions of the nonlocal equation were used in this study of the case-IV Mongan potential.

The equivalent local potential for the case-IV Mongan potential is shown in Figs. 6 and 7. Two plots were necessary to show the very strong short-range *attraction*. The change in the equivalent local potential with energy is continuous and no significant repulsion was observed in this energy interval. While the equivalent local potential is a continuous function of r for each energy in the energy range for which the case-IV Mongan potential fits the phase shifts, the very strong attraction is surprising, since the original separable potential is supposed to exhibit short-range repulsion. In fact, the case-IV Mongan equivalent local potential seems strong enough to have a bound state. For this reason, the potential was examined in greater detail.

A test of the case-IV Mongan potential was made to determine if this potential has spurious states. The test used is very simple and may be convenient to apply to other potentials. The test is based on the observation[7] that the Fredholm determinant $D(k)$ satisfies

$$\lim_{k\to\infty} D(k) = 1 .$$

Then, if

$$\lim_{k\to 0} D(k) < 0 ,$$

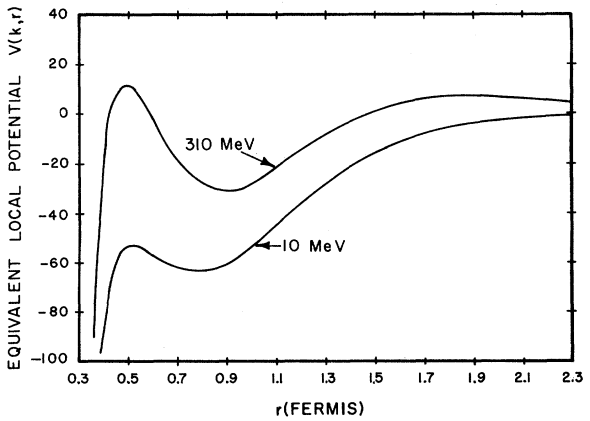

FIG. 6. Long-range part of the equivalent local potential for the case-IV Mongan potential.

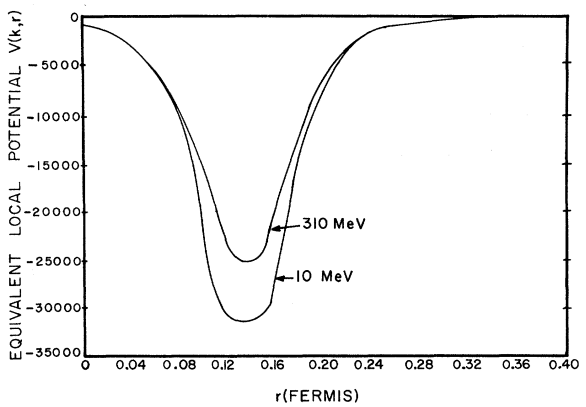

FIG. 7. Short-range part of the equivalent local potential for the case-IV Mongan potential.

$D(k)$ must have at least one zero in the interval $0 \leq k \leq \infty$. In other words, there must be at least one spurious state. For the case-IV Mongan potential, we find

$$\lim_{k \to 0} D(k) = -9.48,$$

and conclude that the case-IV Mongan potential has a spurious state. If

$$\lim_{k \to 0} D(k) > 0,$$

the test would have been inconclusive, since a positive value of $D(k)$ at one energy may not guarantee that it is positive at all energies. Nevertheless, it is quite easy to calculate $D(k)$ at zero energy, and the test provides a quick way of checking potential before more complicated calculations are attempted. A plot of $D(k)$ for the case-IV Mongan potential is shown in Fig. 8. The spurious state is at 19.6 BeV.

In view of the very high energy of the spurious state, its relevance for a two-nucleon interaction below 0.5 BeV can be questioned. Figure 9 shows the zero-energy wave function for the case-IV Mongan potential as calculated from Eq. (21). The fact that this wave function has a node demonstrates that the spurious state at 19.6 BeV does have an effect at experimentally relevant energies. Most likely, the extra node is due to the negative sign of $D(k)$. This is consistent with the results for the one-term Tabakin potential which also has negative $D(k)$ and an additional node. The case-IV Mongan potential is not like local potentials which

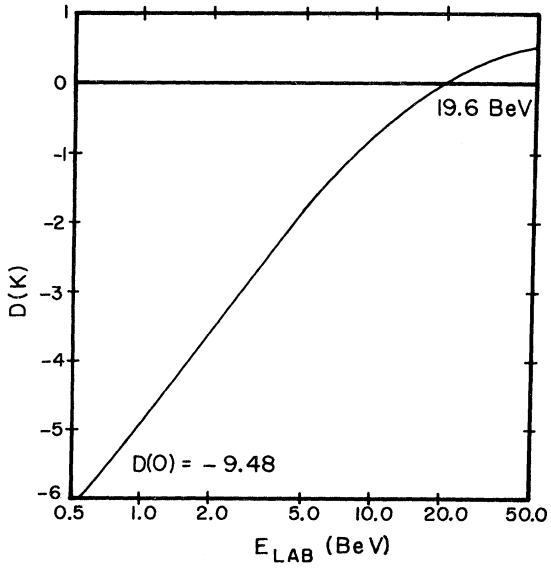

FIG. 8. Fredholm determinant $D(k)$ for the case-IV Mongan potential plotted as a function of energy on a semilog scale.

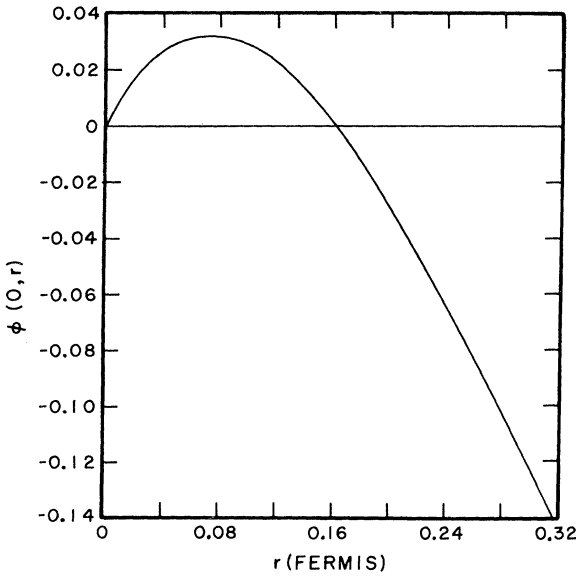

FIG. 9. Zero-energy wave function for the case-IV Mongan potential.

have short-range repulsion, long-range attraction, and no bound states. The correspondence between the node in the zero-energy wave function and the strong short-range attraction in the equivalent local potential is consistent with our suspicions. Under ordinary circumstances, two-nucleon calculations are not checked to insure that the high-energy behavior of a given interaction is consistent with the nonrelativistic potential model. The main reason we checked was because the case-IV Mongan equivalent local potential did not look like a local potential that would fit the $^1S_0$ phase shifts.

Sprung[17] has used the Marchenko[18] method to construct an equivalent local potential from the S matrix element of the case-IV Mongan potential. The potential obtained (see Fig. 2 of Ref. 17) is very similar to the Reid soft-core potential[12] and quite different from the potential obtained by the method used in this paper. The "Marchenko equivalent" local potential of the case-IV Mongan potential would not yield a zero-energy wave function with a node. The reason for this difference between the properties of the two types of equivalent local potentials can be understood by examining the Marchenko method. The starting point for the Marchenko method is the integral

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} [S(k) - 1] e^{ikx} dk,$$

where $S(k)$ is the S matrix element. This integral is evaluated by transforming it to a contour integral in the upper half of the k plane and determining the residues of the poles of the integrand, a procedure that is meaningful only if the number of poles of $S(k)$ inside the contour is stable. If the

symmetry of the case-IV Mongan potential is destroyed by applying a small perturbation, for example, by writing one of the terms as

$$e^{-\beta r}e^{-(\beta+\epsilon)r'},$$

where ϵ may be arbitrarily small but finite, then $S(k)$ for the perturbed potential will have a pole on the real axis at the zero of $D(k)$. This pole can alter the resulting Marchenko equivalent local potential significantly. It should be emphasized that the instability just mentioned is a property of the case-IV Mongan potential, and is not a defect in the Marchenko method. Sprung[17] has also constructed the Marchenko equivalent local potential for the two-term Tabakin potential. In this case, the comparison between the local potentials obtained by the two methods is more favorable.

VIII. APPEARANCE OF SPURIOUS STATES IN SEPARABLE INTERACTIONS WITH STRONG REPULSIVE TERMS

The spurious state obtained for the case-IV Mongan potential is not of the same type as the spurious state for the one-term Tabakin potential. In the latter potential, the spurious state can be related to a class of zeros of the nonlocal potential in the coordinate representation. The spurious state for the case-IV Mongan potential is due entirely to the magnitude of the "repulsive" term in the interaction. To show how separable potentials with a "repulsive" sign can lead to spurious states, we return to the Yamaguchi potential.

For the Yamaguchi potential, the Fredholm determinant of the matrix given by Eq. (18) is

$$D(k) = 1 - \frac{\lambda}{2\beta}\frac{1}{k^2+\beta^2}\,.$$

If $\lambda \leq 0$, then $|D(k)| \geq 1$ for all k in the strip $-\beta < \mathrm{Im}(k) < \beta$. There is no possibility of $D(k)$ being zero in the domain of definition of the solutions to the nonlocal equation.[7] If $\lambda > 0$, then $|D(k)| \leq 1$ for all k in the strip, and it is possible for $D(k)$ to be zero or negative. It is apparent in this case that the zeros of $D(k)$ occur for k^2 real; they are on either the real or imaginary axis in complex k space. As λ is increased from zero, a pair of zeros will move from $\mathrm{Im}k = \pm\beta$, and will meet at $k = 0$ when $\lambda = 2\beta^3$. As λ is increased further, the zeros of $D(k)$ will move out onto the real axis of the complex k plane. The energy of this spurious state moves to higher energies as λ is increased. For all energies below the spurious state, $D(k)$ will be negative. It is clear that there is a limit to the amount of repulsion that can be allowed without introducing a spurious state.

While we have used a very simple potential as an example, it is apparent that similar effects can occur for any separable interaction whose strength has a repulsive sign. Indeed, they have occurred for the case-IV Mongan potential. It should be emphasized that the occurrence of spurious states is very dependent on the numerical values of the parameters used in a given nonlocal potential. Consequently, any attempt to fit phase shifts with potentials that are not local should be accompanied by an investigation of the number of nodes of the wave function.

IX. CONCLUSION

In this paper, we have applied a method of equivalent local potentials to separable two-nucleon interactions which fit the $^1S_0$ phase shifts. For those separable interactions which do not admit spurious states, we have found a reasonable correspondence between the equivalent local potentials obtained from these interactions and phenomenological local potentials which fit the same phase shifts. This result is consistent with previous observations on other nonlocal potentials and their local equivalents.[6,10] Most of the equivalent local potentials obtained thus far are reasonable in terms of a correspondence with phenomenological local potentials. For those which are not reasonable, the method has been used to detect spurious states which occur in the original nonlocal equation. We conclude that the method is a useful tool for studying the properties of nonlocal potentials.

In these applications, we have encountered problems with separable potentials that are supposed to exhibit the short-range repulsion that is characteristic of local two-nucleon interactions. While we have not made a thorough study, our results indicate that there are definite limits to the amount of repulsion that can be safely considered with separable nonlocal interactions.

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Regge Poles and Strong Absorption in Heavy-Ion and α-Nucleus Scattering*

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Regge poles describing shape resonances ("quasimolecular states") are found to play a prominent role in many optical-model scattering amplitudes and appear to be directly related to the Gruhn-Wall "dip" often observed in the reflection coefficient $\eta(l)$. The Blair smooth-cutoff model is generalized to include such a Regge pole and used to fit angular distributions for elastic $^{16}O+^{16}O$ and $\alpha+^{16}O$ scattering, in the 20–30-MeV (c.m.) energy range. It also appears that the smooth-cutoff (strong-absorption) model itself can be interpreted as the result of many overlapping inelastic Regge resonances.

I. INTRODUCTION

A long-standing problem in elastic α-nucleus scattering at energies well above the Coulomb barrier is the strong rise of the angular distributions at backward angles. Although both the optical model with strong absorption and the smooth-cutoff model[1] reproduce the diffractionlike angular distribution in the forward hemisphere, neither has so far been very successful in simultaneously describing the strong, oscillatory backscattering. Bryant and Jarmie[2] have noted that the shape of this backward-angle scattering is strongly reminiscent of the glory effect, which is a grazing-ray phenomenon,[3] and indeed Gruhn and Wall[4] found that a narrow dip superimposed on the smooth-cutoff model at an l value near $l=kR$ aided materially in raising the backward-angle scattering. More significantly, Cowley and Heymann[5] have recently shown that adding a Regge pole at $l\approx kR$ to the smooth-cutoff model permits an impressive fit to their elastic $\alpha+^{16}O$ cross sections over the full angular range.

A direct-channel Regge pole is simply a convenient means of describing resonances in several adjacent l values simultaneously; the fact that the data seem to call for such resonances near $l=kR$ implies considerable transparency of the interaction at the nuclear surface, in spite of its strong absorption near the center. This is reminiscent of the interaction responsible for heavy-ion scattering, for a growing body of opinion[6-8] suggests that it, too, is strongly absorptive at small impact parameters but highly transparent at large