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Treatment of Two-Particle Continuum States in the Shell-Model Approach to Nuclear Reactions*

R. C. Fuller†

School of Physics, University of Minnesota, Minneapolis, Minnesota 55455

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To consider deuteron-induced reactions within the shell model, it is necessary to include single-particle states with two particles in continuum single-particle states. Expressions are derived for operators whose matrix elements give the transition amplitudes for the processes encountered in deuteron-induced reactions. Each of these operators can be written in terms of a single operator which satisfies an integral equation whose kernel involves the Green's function for two noninteracting particles in the presence of a common potential. This operator is discussed at length. Using single-particle Gamow states, a method is formulated to approximate the connected part of this operator by an operator of finite rank. This approximation is then tested numerically and is found to be effective. Finally, there is a discussion of the implications of the approximation in the scattering amplitudes of interest.

1. INTRODUCTION

The shell-model approach to nuclear reactions has generally been restricted to considering single-particle states with at most one particle in a scattering eigenstate of the single-particle Hamiltonian.¹ Therefore, only single-nucleon reaction channels may be considered within the model. Furthermore, one is restricted to considering only those virtual excitations in which all but one particle are in bound eigenstates of the single-particle Hamiltonian. However, since the deuteron is weakly bound, a shell-model treatment of deuteron-induced reactions must generally include single-particle states with two particles in continuum single-particle states. Therefore, in considering deuteron-induced reactions within the shell model, the three-particle aspects of the problem must be confronted. The complications which are inherent in existing formulations of the three-particle problem have meant that it is necessary in calculations to assume the simplest possible structure for the nucleus. This has restricted calculations to deuterons incident on an inert nucleus. Shanley² has considered the deuteron- α -particle system using an approach developed by Amado.³

This approach assumes that the dynamics of each two-particle subsystem is dominated by a few two-particle correlated states. In terms of the propagators for these states and their form factors, coupled equations which have the form of two-particle, multichannel Lippmann-Schwinger equations are written for the various amplitudes of interest. These equations represent the summing, to all orders, of diagrams for two-particle correlated states propagating along with the third particle which is free, and the subsequent breakup of the correlated state, giving three free particles. One member of the original pair then forms a correlated pair with the third particle, while the other originally paired particle propagates freely. This is all to say that Amado's approach assumes that the three-particle wave function is a superposition of two-particle correlated states with the third particle free, and then gives a prescription for calculating the amplitudes of these states which involves the solution of a set of coupled Lippmann-Schwinger-like integral equations. Since the number of coupled equations is proportional to the number of correlated pair amplitudes, it is generally necessary to assume that only a few two-particle correlated states in each two-particle system

are important in the three-particle wave function.

This paper describes an approach to deuteron-induced reactions which is suggested by the shell-model approach to nuclear reactions. It is assumed that a major portion of the three-particle wave function consists of a product of two single-particle shell-model states coupled by an effective interaction. As mentioned previously, two-particle continuum states must be considered and a large part of the paper is devoted to a discussion of their properties. In Sec. 2 expressions in terms of a single operator are derived for the amplitudes encountered for a deuteron incident on an infinitely massive, structureless target. In terms of single-particle states, both bound and in the continuum, it is shown that singular terms in the kernel of the integral equation for this operator can be isolated in a kernel of finite rank. For this development it is first necessary to discuss the Green's function for two noninteracting particles in a common potential. This is done in Sec. 3. In Sec. 4 we introduce a finite-rank approximation for part of this Green's function, and then we show the results of numerical calculations designed to test the approximation. In Sec. 5 we discuss within this approximation the operator introduced in Sec. 2.

We neglect the Coulomb part of the proton interaction with the core throughout the paper.

2. ELASTIC, BREAK-UP, AND STRIPPING AMPLITUDES

Primarily for the sake of completeness, a discussion of break-up and stripping as well as elastic scattering amplitudes for a deuteron incident on an inert core follows. The results apply to any three-particle system where two of the two-particle subsystems may be bound. However, things will be discussed in a way which is convenient for a shell-model approach to deuteron-induced reactions. Our approach is close to Baz's.⁴

We have a three-particle problem and we assume the only interactions are the pairwise interactions: V_n , the neutron-nucleus interaction; V_p , the proton-nucleus interaction; and V_{np} , the neutron-proton interaction. Then the Hamiltonian for the system is written as

$$H = H_0 + V_n + V_p + V_{np}, \quad (2.1)$$

where H_0 is the kinetic energy operator for the system. Let us denote the relative deuteron-nucleus momentum by \vec{K} . Then for a deuteron incident on the nucleus the wave function for the system satisfies the inhomogeneous equation

$$\psi_{D,\vec{K}}^{(+)}(E) = \Phi_{D,\vec{K}} + g_{np}^{(+)}(E)(V_n + V_p)\psi_{D,\vec{K}}^{(+)}(E), \quad (2.2)$$

where $\Phi_{D,\vec{K}}$ is the wave function of a free deuteron

incident on the nucleus and

$$g_{np}^{(+)}(E) = (E + i\epsilon - H_0 - V_{np})^{-1}. \quad (2.3)$$

$\psi_{D,\vec{K}}^{(+)}(E)$ also satisfies the homogeneous equation

$$\psi_{D,\vec{K}}^{(+)}(E) = G_{n+p}^{(+)}(E)V_{np}\psi_{D,\vec{K}}^{(+)}(E), \quad (2.4)$$

where

$$G_{n+p}^{(+)}(E) \equiv (E + i\epsilon - H_0 - V_n - V_p)^{-1}.$$

The energy E is related to the deuteron binding energy B_D and the momentum \vec{K} by

$$E = -B_0 + |\vec{K}|^2/2\mu, \quad (2.5)$$

where μ is the reduced deuteron-target mass. It is convenient to define the vectors $T_1(E, \vec{K})$ and $T_2(E, \vec{K})$

$$T_1(E, \vec{K}) = (V_n + V_p)\psi_{D,\vec{K}}^{(+)}(E), \quad (2.6a)$$

$$T_2(E, \vec{K}) = V_{np}\psi_{D,\vec{K}}^{(+)}(E), \quad (2.6b)$$

which satisfy the coupled equations

$$T_1(E, \vec{K}) = (V_n + V_p)G_{n+p}^{(+)}(E)T_2(E, \vec{K}), \quad (2.7a)$$

$$T_2(E, \vec{K}) = V_{np}\Phi_{D,\vec{K}} + V_{np}g_{np}^{(+)}(E)T_1(E, \vec{K}). \quad (2.7b)$$

Further note

$$\Phi_{D,\vec{K}}^\dagger V_{np} G_0^{(+)}(E) = \Phi_{D,\vec{K}}^\dagger, \quad (2.8)$$

which is just a way of writing the adjoint of the Schrödinger equation. In terms of the two-particle transition operator in the three-particle Hilbert space which satisfies the operator equation

$$t_{np}(W) = V_{np} + V_{np}G_0(W)t_{np}(W), \quad (2.9)$$

where

$$G_0(W) \equiv (W - H_0)^{-1}$$

is the three-particle free Green's function, one has the operator identity

$$V_{np}g_{np}(W) = t_{np}(W)G_0(W). \quad (2.10)$$

The elastic scattering amplitude $A_{\vec{K}',\vec{K}}(E)$ is given by

$$A_{\vec{K}',\vec{K}}(E) = (\Phi_{D,\vec{K}'}, (V_n + V_p)\psi_{D,\vec{K}}^{(+)}(E)). \quad (2.11)$$

Using Eqs. (2.6b) and (2.8) this amplitude may be written

$$A_{\vec{K}',\vec{K}}(E) = (\Phi_{D,\vec{K}'}, V_{np}G_0^{(+)}(E)T_1(E, \vec{K})). \quad (2.12)$$

Now define the operator $X^{(+)}(E)$ by

$$G_0^{(+)}(E)T_1(E, \vec{K}) \equiv X^{(+)}(E)V_{np}\Phi_{D,\vec{K}}. \quad (2.13)$$

Substituting Eq. (2.7a) into Eq. (2.7b) and using Eq. (2.10) gives

$$X^{(+)}(E)V_{np}\Phi_{D,\vec{k}} = G_0^{(+)}(E)[V_n + V_p]G_{n+p}^{(+)}(E) \\ \times [1 + t_{np}^{(+)}(E)X^{(+)}(E)]V_{np}\Phi_{D,\vec{k}}. \quad (2.14)$$

Then in terms of the operator $X(W)$ which satisfies the equation

$$X(W) = G_0(W)[V_n + V_p]G_{n+p}(W)[1 + t_{np}(W)X(W)], \quad (2.15a)$$

the elastic scattering amplitude is given by

$$A_{\vec{k}',\vec{k}}(E) = \lim_{W \rightarrow E + i\epsilon} (\Phi_{D,\vec{k}}, V_{np}X(W)V_{np}\Phi_{D,\vec{k}}). \quad (2.16)$$

For the break-up amplitude it is obviously appropriate to consider the vector $(V_n + V_p + V_{np})\psi_{D,\vec{k}}^{(+)}(E)$, where

$$(V_n + V_p + V_{np})\psi_{D,\vec{k}}(E) = T_1(E, \vec{K}) + T_2(E, \vec{K}). \quad (2.17)$$

A little operator algebra gives

$$T_1(E, \vec{K}) + T_2(E, \vec{K}) = [1 + (V_n + V_p)G_{n+p}^{(+)}(E)] \\ \times [1 + t_{np}^{(+)}(E)X^{(+)}(E)]V_{np}\Phi_{D,\vec{k}}. \quad (2.18)$$

Denoting the three-free-particle state by $\phi_{\vec{p}\vec{q}}$, where \vec{p} and \vec{q} are any two linearly independent momenta in the center-of-mass coordinate system, then the breakup amplitude is given by

$$B_{\vec{p}\vec{q}}(E) = (\phi_{\vec{p}\vec{q}}, T_1(E, \vec{K}) + T_2(E, \vec{K})). \quad (2.19)$$

The operator $1 + (V_n + V_p)G_{n+p}^{(+)}(E)$ is the Hermitian conjugate of the incoming wave operator for the system in the absence of the n - p interaction.

For the neutron stripping amplitude, denote the wave function for a free proton incident on a bound neutron-nucleus state by $\beta_{n\vec{k}}$ where \vec{k} is the relative proton bound-state momentum. Then the stripping amplitude to the n th bound state is given by

$$S_{n,\vec{k},\vec{k}}(E) = (\beta_{n,\vec{k}}, [V_n + V_p]\psi_{D,\vec{k}}^{(+)}(E)). \quad (2.20)$$

Just a little operator algebra yields

$$S_{n,\vec{k},\vec{k}}(E) = \lim_{W \rightarrow E_{n,\vec{k}} + i\epsilon} (\beta_{n,\vec{k}}, [1 + V_p G_{n+p}(W)] \\ \times [1 + t_{np}(W)X(W)]V_{np}\Phi_{D,\vec{k}}), \quad (2.21)$$

where

$$E_{n,\vec{k}} = -B_n + |\vec{k}|^2/2\mu_p, \quad (2.22)$$

where μ_p is the proton-bound-state reduced mass.

In each amplitude the full three-particle dynamics is contained in the operator $X(W)$ which satisfies the operator equation

$$X(W) = \mathcal{G}_{n+p}(W) + \mathcal{G}_{n+p}(W)t_{np}(W)X(W), \quad (2.15b)$$

where

$$\mathcal{G}_{n+p}(W) = G_{n+p}(W) - G_0(W). \quad (2.23)$$

The kernel $\mathcal{G}_{n+p}(W)t_{np}(W)$ is connected since it involves only terms in which all three particles are interacting. Loosely speaking, the neutron and proton have been allowed to interact to all orders giving $t_{np}(W)$ and then the next interaction must involve the neutron or the proton and the core. Connectivity is a necessary condition for compactness^{5,6} which is a necessary property of the kernel if the usual methods for solving the equation are to be used.

It is convenient to decompose $\mathcal{G}_{n+p}(W)$ into partially and completely connected parts. From the operator identity

$$G_{n+p}(W) = G_0(W) + G_0(W)[V_n + V_p]G_{n+p}(W), \quad (2.24)$$

one obtains

$$\mathcal{G}_{n+p}(W) = G_0(W)[V_n + V_p]G_0(W) \\ + G_0(W)[V_n + V_p]\mathcal{G}_{n+p}(W). \quad (2.25)$$

Following Faddeev⁵ define the two operators $\mathcal{G}_{n+p}^n(W)$ and $\mathcal{G}_{n+p}^p(W)$ by

$$\mathcal{G}_{n+p}^n(W) = G_0(W)V_{n(p)}G_0(W) + G_0(W)V_{n(p)}\mathcal{G}_{n+p}(W). \quad (2.26)$$

Then

$$\mathcal{G}_{n+p}(W) = \mathcal{G}_{n+p}^n(W) + \mathcal{G}_{n+p}^p(W), \quad (2.27)$$

where

$$[1 - G_0(W)V_n]\mathcal{G}_{n+p}^n(W) \\ = G_0(W)V_n G_0(W) + G_0(W)V_n \mathcal{G}_{n+p}^p(W). \quad (2.28)$$

The two-particle transition matrix satisfies the equation

$$t_n(W) = V_n + t_n(W)G_0(W)V_n, \quad (2.29a)$$

therefore

$$1 + G_0(W)t_n(W) = [1 - G_0(W)V_n]^{-1}. \quad (2.29b)$$

Substituting Eq. (2.30) into Eq. (2.28) and using Eq. (2.29a) gives

$$\mathcal{G}_{n+p}^n(W) = G_0(W)t_n(W)G_0(W) + G_0(W)t_n(W)\mathcal{G}_{n+p}^p(W). \quad (2.30)$$

Interchanging n and p in this equation gives the equation satisfied by $\mathcal{G}_{n+p}^c(W)$. Therefore,

$$\mathcal{G}_{n+p}(W) = G_0(W)[t_n(W) + t_p(W)]G_0(W) + \mathcal{G}_{n+p}^c(W), \quad (2.31)$$

where $\mathcal{G}_{n+p}^c(W)$ is connected, i.e., any and all terms in the interaction of Eq. (2.30) beyond the inhomogeneous term involve the interaction of both the neutron and proton with the nucleus. The first two terms of Eq. (2.31) are partially connected. The first term includes the neutron-core interaction to all orders and the second, the proton-core interaction to all orders.

Complete numerical solution of the integral equation corresponding to Eq. (2.15b), as it stands, is not generally within the capability of existing computers. Therefore the problem is: How does one obtain reliable approximations for the required matrix elements of $X(W)$? An iterative solution is the first method which comes to mind. Roughly speaking, the validity of such an approximation depends on the "size" of the kernel $\mathcal{G}_{n+p}(W)t_{np}(W)$. This notion was formulated in a precise manner by Weinberg⁶ for the discussion of the convergence of iterative solutions to integral equations which arise in two-particle scattering theory. The kernel $\mathcal{G}_{n+p}(W)t_{np}(W)$ is much more complicated than the kernel considered by Weinberg, and therefore we follow a more intuitive approach which is close, in spirit, to the approach suggested by Weidenmüller and co-workers⁷ for including single-particle continuum states in the shell-model approach to nuclear reactions.

From this point on, unless explicitly stated, we assume the target or core is infinitely massive. $G_{n+p}(W)$ is then the Green's function for two non-interacting particles in a potential and is written in terms of the single-particle Hamiltonians H_n and H_p as

$$G_{n+p}(W) = (W - H_n - H_p)^{-1}. \quad (2.32)$$

Eigenstates of $H_n + H_p$ are the products of single-particle eigenstates of H_n and H_p , while eigenvalues of $H_n + H_p$ are the sums of single-particle eigenvalues. Therefore, matrix elements of $G_{n+p}(W)$ is singular at energies corresponding to the sum of single-particle Gamow state energies, i.e., $G_{n+p}(W)$ has a pole term at $E_{\gamma_n} + E_{\gamma_p}$:

$$G_{n+p}^{\circ}(W) = \frac{\Gamma_n \Gamma_p \tilde{\Gamma}_n^{\dagger} \tilde{\Gamma}_p^{\dagger}}{W - E_{\gamma_n} - E_{\gamma_p}}, \quad (2.33)$$

where $\Gamma_{n(p)}$ is the neutron (proton) single-particle Gamow state and $\tilde{\Gamma}_{n(p)}$ is the corresponding time-reversed state.⁸ Near such energies the pole contributions to $\mathcal{G}_{n+p}(W)t_{np}(W)$ cannot be treated by perturbation theory. At this point we cannot be explicit about the nearness. All that is required in the present discussion is the observation that there is a neighborhood where an iterative solution is not possible. When one proceeds along the lines suggested by Weidenmüller and co-workers⁷

or by Romo⁹ to treat single-particle Gamow states, several problems are encountered. Weidenmüller approximates the single-particle continuum contribution to the shell-model Green's function by a finite rank approximation using Weinberg states. Such an approach is generally not possible for deuteron-induced reactions where two-particle continuum states cannot be neglected. If one writes for $\mathcal{G}_{n+p}(W)$, following an approach analogous to Romo's,

$$\begin{aligned} \mathcal{G}_{n+p}(W) &= [\mathcal{G}_{n+p}(W) - \mathcal{G}_{n+p}^{\circ}(W)] + \mathcal{G}_{n+p}^{\circ}(W) \\ &= \tilde{\mathcal{G}}_{n+p}(W) + \mathcal{G}_{n+p}^{\circ}(W), \end{aligned} \quad (2.34)$$

where $\mathcal{G}_{n+p}^{\circ}(W)$ is generally a sum of terms like the single term in Eq. (2.33), and attempts to treat $\tilde{\mathcal{G}}_{n+p}(W)$ perturbatively, two problems arise. First the interaction $t_{np}(W)$ is not restricted to a region around the nucleus as is the residual interaction when only one single-particle continuum state is included. Therefore one would generally encounter difficulties in evaluating matrix elements of $t_{np}(W)$ in terms of the Gamow states in the residue of $\mathcal{G}_{n+p}^{\circ}(W)$, since it is generally necessary to include single-particle resonance Gamow states in $\mathcal{G}_{n+p}^{\circ}(W)$. The properties of such states are discussed in the Appendix, and we only note in passing that the radial part of a resonance Gamow state increases exponentially at large single-particle radial separation. Secondly $\tilde{\mathcal{G}}_{n+p}(W)$ must include all the branch points in $G_{n+p}(W)$. In this case there is not only a branch point at the two-particle continuum threshold which in this case is a logarithmic branch point; there are also square-root branch points at each single-particle Gamow state energy corresponding to one of the valence particles being in that single-particle Gamow state.⁸ In the next section the operator $\mathcal{G}_{n+p}(W)$ is discussed in detail and the preceding discussion is expanded.

3. DISCUSSION OF THE GREEN'S FUNCTION $G_{n+p}(W)$

The first two terms in Eq. (2.31), $G_0(W)[t_n(W) + t_p(W)]G_0(W)$, which we shall call the impulse terms, do not contain pole singularities since only one of the valence particles is interacting in each term. Therefore the pole singularities in $\mathcal{G}_{n+p}(W)$ are contained in the connected term $\mathcal{G}_{n+p}^c(W)$. Since the connected term involves both valence particle-core interactions, it presumably is well described by a few low-lying single-particle angular momentum states. This will shortly be made more precise.

In order to investigate the analytic properties of matrix elements of $\mathcal{G}_{n+p}(W)$ it is convenient to express $G_{n+p}(W)$ in terms of the single-particle

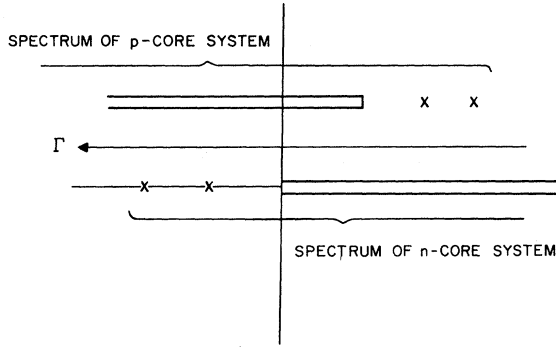


FIG. 1. The cut z plane for the integrand of the convolution in Eq. (3.3).

Green's functions

$$g_n(W) = (W - H_n)^{-1} \quad (3.1a)$$

and

$$g_p(W) = (W - H_p)^{-1} \quad (3.1b)$$

by the convolution¹⁰

$$G_{n+p}(W) = g_n(W) * g_p(W). \quad (3.2)$$

The convolution is defined by the integral

$$g_n(W) * g_p(W) = \frac{1}{2\pi i} \int_{\Gamma_z} dz g_n(z) g_p(W - z), \quad (3.3)$$

where the contour Γ_z encircles in counterclockwise direction singularities of $g_n(z)$ which lie on the real z axis. The integrand is defined in the cut z plane shown in Fig. 1, where the path Γ_z has been folded back from the lower lip of the p -core continuum cut and pulled around the p -core bound-state poles. This is allowed since the only singularities in the portion of the z plane through which the contour has been distorted are the bound-state poles. Anticipating the limit $W \rightarrow E + i\epsilon$, W has been given a positive imaginary part.

Upon defining the single-particle free Green's functions by

$$g_{0n(p)}(W) = (W - H_{0n(p)})^{-1}, \quad (3.4)$$

where $H_{0n(p)}$ is the neutron (proton) kinetic energy operator, the single-particle Green's functions may be written

$$\begin{aligned} g_{n(p)}(W) &= g_{0n(p)}(W) + g_{0n(p)}(W) V_{n(p)} g_{n(p)}(W) \\ &= g_{0n(p)}(W) + \tilde{g}_{n(p)}(W). \end{aligned} \quad (3.5)$$

This is taken as the definition of $\tilde{g}_{n(p)}(W)$, which thus includes to any and all orders the $n(p)$ -core interaction. The convolution (3.2) may then be written as a sum of four terms

$$\begin{aligned} G_{n+p}(W) &= g_{0n}(W) * g_{0p}(W) + g_{0n}(W) * \tilde{g}_p(W) \\ &\quad + \tilde{g}_n(W) * g_{0p}(W) + \tilde{g}_n(W) * \tilde{g}_p(W), \end{aligned} \quad (3.6)$$

where

$$G_0(W) = g_{0n}(W) * g_{0p}(W). \quad (3.7)$$

The impulse terms $G_0(W)[t_n(W) + t_p(W)]G_0(W)$ are given by

$$I_{n+p}(W) = g_{0n}(W) * \tilde{g}_p(W) + \tilde{g}_n(W) * g_{0n}(W), \quad (3.8)$$

and the completely connected part, $\mathcal{G}_{n+p}^c(W)$, by the convolution of the single-particle operators which contain the valence particle-core interactions

$$\mathcal{G}_{n+p}^c(W) = \tilde{g}_n(W) * \tilde{g}_p(W). \quad (3.9)$$

For the sake of simplicity, we neglect spin and assume the valence particle-core interactions are central, then the single-particle operators are diagonal in the orbital angular momentum. Using Eq. (3.3) it follows that in configuration space

$$\begin{aligned} G_{n+p}(W, \vec{r}_n, \vec{r}_p | \vec{r}'_n, \vec{r}'_p) &= \sum_{\substack{i_n m_n \\ i_p m_p}} Y_{i_n m_n}(\hat{r}_n) Y_{i_p m_p}(\hat{r}_p) Y_{i_n m_n}^*(\hat{r}'_n) Y_{i_p m_p}^*(\hat{r}'_p) \\ &\quad \times \frac{1}{r_n r_p r'_n r'_p} g_{i_n}(W, r_n, r'_n) * g_{i_p}(W, r_p, r'_p), \end{aligned} \quad (3.10)$$

where the convolution is written in terms of the single-particle partial-wave Green's functions which are discussed in the Appendix. Using the partial-wave decomposition of Eq. (3.5), the convolution appearing in Eq. (3.10) may be decomposed into disconnected, partially connected, and connected contributions. For the potentials generally used in the shell-model approach to nuclear reactions \tilde{g} is well represented by a few low-lying partial waves. And thus $\mathcal{G}_{n+p}^c(W)$ can be expected to be well approximated by a sum over a limited number of single-particle partial waves.

We now turn to the evaluation of the connected part of the convolution in Eq. (3.10),

$$\mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p) = \tilde{g}_{i_n}(W, r_n, r'_n) * \tilde{g}_{i_p}(W, r_p, r'_p), \quad (3.11)$$

where, according to the definition in Eq. (3.3),

$$\begin{aligned} \mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p) &= \frac{1}{2\pi i} \int_{\Gamma_z} dz \tilde{g}_{i_n}(z, r_n, r'_n) \tilde{g}_{i_p}(W - z, r_p, r'_p). \end{aligned} \quad (3.12)$$

Since the single-particle Green's function has a square-root branch point at threshold in the energy plane, it is convenient to introduce the single-particle momentum variables in the usual manner,

$$\begin{aligned} k_n &= (2m_n z)^{1/2}, \\ k_p &= [2m_p(W-z)]^{1/2}, \end{aligned} \quad (3.13)$$

and rewrite Eq. (3.12) as

$$\begin{aligned} \mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p) \\ = \frac{1}{2\pi i} \int_{\Gamma_k} dk_n k_n m_n^{-1} \bar{g}_{i_n}(k_n, r_n, r'_n) \bar{g}_{i_p}(k_p, r_p, r'_p). \end{aligned} \quad (3.14)$$

Let us assume the single-particle potentials are less singular than r^{-2} at the origin and vanish identically beyond some finite distance. Then the single-particle Green's functions are meromorphic in the entire k plane.¹¹ This assumption is made to simplify the discussion. The results are certainly applicable for a finite-range single-particle potential like the Woods-Saxon potential if some care is exercised. Now the integrand of Eq. (3.14) is a meromorphic function of k_n and k_p . As a function of k_n , it is apparent from Eq. (3.13) that the integrand has branch points at $\pm\Delta$, where

$$\Delta = (2m_p W)^{1/2}. \quad (3.15)$$

It is convenient to introduce a variable in terms of

which the integrand is single-valued. In doing this we are guided by work done for the two-channel S matrix by Cox¹² and Kato¹³ as discussed by Newton.¹⁴ The transformation

$$u = (k_n - \Delta)(k_p + \Delta)^{-1} \quad (3.16)$$

moves one of the branch points to the origin and the other to infinity. At this point we let $m_n = m_p = m$. Then

$$\begin{aligned} k_n &= \Delta(1+u)(1-u)^{-1}, \\ k_p &= 2i\Delta u^{1/2}(1-u)^{-1}. \end{aligned} \quad (3.17a)$$

This new Riemann surface is opened up by setting $u = t^2$, so that

$$\begin{aligned} k_n &= \Delta(1+t^2)(1-t^2)^{-1}, \\ k_p &= 2i\Delta t(1-t^2)^{-1}. \end{aligned} \quad (3.17b)$$

It is then convenient to introduce the variable ω by

$$\omega = (1-t)(1+t)^{-1}, \quad (3.18)$$

so that

$$\begin{aligned} k_n(\omega) &= \Delta(\omega + \omega^{-1})/2, \\ k_p(\omega) &= \Delta(\omega - \omega^{-1})/(2i). \end{aligned} \quad (3.19)$$

The convolution (3.14) may now be written as

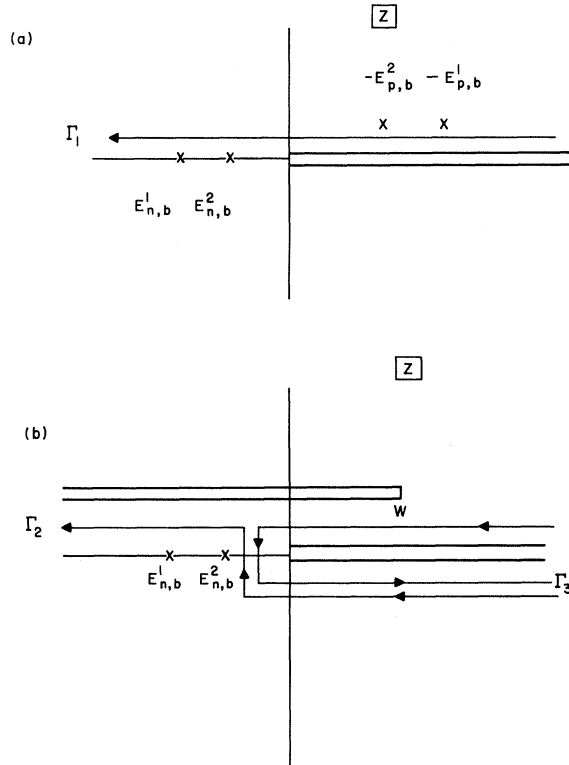


FIG. 2. The cut z planes and the contours of integration for integrals in Eq. (3.25).

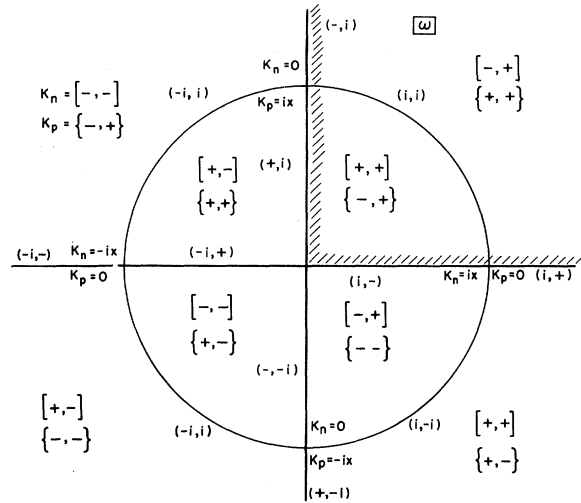


FIG. 3. The ω plane for negative energy. The signs of the real and imaginary parts of k_n within a given region indicated in the brackets [], while the signs for k_p are indicated within the braces { }. The parentheses along the axis indicate the sign and whether k_n and k_p are pure real or imaginary, respectively. The physical region which corresponds to both k_n and k_p having positive imaginary parts is indicated by the shaded region.

$$\begin{aligned} \mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p) \\ = \frac{1}{2\pi i} \int_{\Gamma_\omega} d\omega f(\omega) \bar{g}_{i_n}(k_n, r_n, r'_n) \bar{g}_{i_p}(k_p, r_p, r'_p), \end{aligned} \quad (3.20)$$

where

$$f(\omega) = ik_n(\omega)k_p(\omega)/m\omega. \quad (3.21)$$

The partial-wave single-particle Green's function has a pole singularity at a positive imaginary momentum related to the single-particle binding energy $-E_b$ by

$$k_b = i(-2mE_b)^{1/2}. \quad (3.22)$$

For a bound state in a given partial wave of the n -core system, the pole term may be written in terms of the normalized radial bound-state wave

$$\mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p) = \frac{1}{2\pi i} \int_{\Gamma_1} d\omega f(\omega) \bar{g}_{i_n}(k_n, r_n, r'_n) \mathcal{B}_{i_p}(k_p, r_p, r'_p) + \frac{1}{2\pi i} \int_{\Gamma_2 + \Gamma_3} d\omega f(\omega) \bar{g}_{i_n}(k_n, r_n, r'_n) \bar{g}'_{i_p}(k_p, r_p, r'_p), \quad (3.25)$$

where the integrand of the first term is defined in the cut z plane shown in Fig. 2(a) and the second in Fig. 2(b). The integration contours in each term are indicated. Again W has been given a positive imaginary part which is taken to zero after the integral is evaluated. The figures shown are appropriate for positive energy. The corresponding figures for negative energy are obvious and are not shown. The ω plane for negative energies is shown in Figs. 3 and 4. The shadowed region is a map of the physical region. For positive energies the ω plane is shown in Figs. 5 and 6 with the physical region again shadowed. For both positive and negative energies the contours Γ_1 and Γ_2 may be closed. Γ_1 may be closed by a large semicircle in the upper half of the z plane and Γ_2 by a large semicircle in the lower. The contribution from the semicircle vanishes in each case, since the contours are in the physical region of both single-particle Green's functions. The only singularities within the closed contours are the single-particle bound-state poles, and therefore the integrals corresponding to Γ_1 and Γ_2 may be immediately evaluated by the residue theorem. Although it really is not necessary, we shall evaluate these integrals in the ω plane. The discussion will be useful later when contributions from other Gamow-state poles are evaluated. The contours Γ_1 and Γ_2 in the ω plane are shown in Fig. 4 for negative energies and Fig. 6 for positive energies. To evaluate the residue of the Γ_1 integral we first consider the denominator, $k^2 - k_{p,b}^2$, of one of the pole terms in Eq. (3.4) which near the pole may be written as

function $\beta_{i_n}(r_n)$ as

$$\mathcal{P}_{i_n, b}(k_n, r_n, r'_n) = \frac{2m\beta_{i_n}(r_n)\beta_{i_n}(r'_n)}{k_n^2 - k_{n,b}^2}. \quad (3.23)$$

Of course the same expression holds for a p -core bound-state pole. The proton bound-state pole terms in a given partial wave may be subtracted from $\bar{g}_{i_p}(k_p, r_p, r'_p)$ to give

$$\begin{aligned} \bar{g}'_{i_p}(k_p, r_p, r'_p) &= \bar{g}_{i_p}(k_p, r_p, r'_p) - \sum_{b_p} \mathcal{P}_{i_p}(k_p, r_p, r'_p), \\ &\equiv \bar{g}_{i_p}(k_p, r_p, r'_p) - \mathcal{B}_{i_p}(k_p, r_p, r'_p). \end{aligned} \quad (3.24)$$

Since the pole terms depend on k_p^2 , they do not have a branch point at threshold. Therefore we decompose the integral as

$$\begin{aligned} 2k_{p,b}(k_p - k_{p,b}) &= 2k_{p,b}[\Delta(\omega - \omega^{-1})(2i)^{-1} - k_{p,b}], \\ &= k_{p,b}\Delta\{\omega^2 - 1 - 2i\omega k_{p,b}\Delta^{-1}\}(i\omega)^{-1}. \end{aligned} \quad (3.26)$$

The factor in braces has zeros given by

$$\begin{aligned} \omega_{p,b}^\pm &= ik_{p,b}\Delta^{-1} \pm (\Delta^2 - k_{p,b}^2)^{1/2}\Delta^{-1} \\ &= ik_{p,b}\Delta^{-1} \pm k_n(\Delta, p_b)\Delta^{-1}. \end{aligned} \quad (3.27)$$

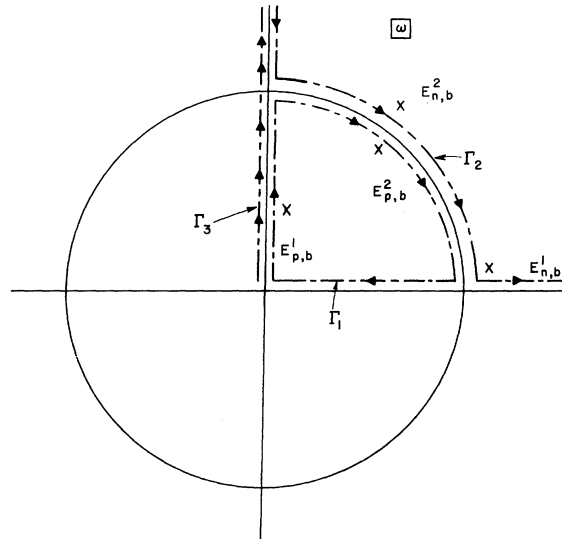


FIG. 4. The ω plane for negative energies showing the paths of integration in Eq. (3.25) and the positions of the bound states.

From Eq. (3.19) we have

$$k_n + ik_p = \Delta \omega. \quad (3.28)$$

Therefore, the pole position in the ω plane is given by $\omega_{p,b}^+$. Rewriting Eq. (3.26) as

$$2k_{p,b}(k_p - k_{p,b}) = k_{p,b}\Delta(\omega - \omega_{p,b}^-)(\omega - \omega_{p,b}^+)(i\omega)^{-1}, \quad (3.29)$$

which near the pole appears as

$$2k_{p,b}k_n(\Delta, p_b)(\omega - \omega_{p,b}^+)(i\omega)^{-1}.$$

Therefore, the integral Γ_1 is just

$$\sum_{p_b} \left[-\frac{1}{2\pi i} \int_{\Gamma_1} d\omega \frac{k_n k_p}{k_n(\Delta, p_b) k_{p,b}} \frac{\beta_{i_p}(r_p) \beta_{i_p}(r'_p)}{\omega - \omega_{p,b}^+} \bar{g}_{i_n}(k_n, r_n, r'_n) \right] = \sum_{p_b} \beta_{i_p}(r_p) \beta_{i_p}(r'_p) \bar{g}_{i_n}^{(+)}[k_n(\Delta, p_b), r_n, r'_n]. \quad (3.30a)$$

The minus sign is canceled in the evaluation since the contour Γ_1 encircles the pole in a counterclockwise direction. From Fig. 2(a) it is apparent that $\bar{g}_{i_n}(k_n, r_n, r'_n)$ is evaluated on the upper lip of its continuum cut and therefore one uses $\bar{g}_{i_n}^{(+)}(k_n, r_n, r'_n)$ in Eq. (3.30a). The argument may be repeated for Γ_2 to obtain

$$\sum_{p_n} \frac{1}{2\pi i} \int d\omega \frac{k_n k_p}{k_{n,b} k_p(\Delta, n_b)} \frac{\beta_{i_n}(r_n) \beta_{i_n}(r'_n)}{\omega - \omega_{n,b}^+} \bar{g}_{i_p}^{(+)}(k_p, r_p, r'_p) = \sum_{n_b} \beta_{i_n}(r_n) \beta_{i_n}(r'_n) \bar{g}_{i_p}^{(+)}[k_p(\Delta, n_b), r_p, r'_p]. \quad (3.30b)$$

In this way all the pole terms of $\mathcal{G}_{n+p}^c(W)$ corresponding to one of the valence particles being in a bound state may be isolated. Note that poles corresponding to both valence particles in bound states are contained only in Eq. (3.30a), since in Eq. (3.30b) the p -core bound-state poles have been subtracted from the p -core Green's function. Each term in Eq. (3.30a) and Eq. (3.30b) contains a cut starting from the energy of the single-particle bound state in that term. The momentum is evaluated on the upper lip of the cut when appropriate and is either positive real or positive imaginary.

Neither Eq. (3.30a) nor Eq. (3.30b) contains the poles of $\mathcal{G}_{n+p}^c(W)$ corresponding to both valence particles in resonance Gamow states. They are

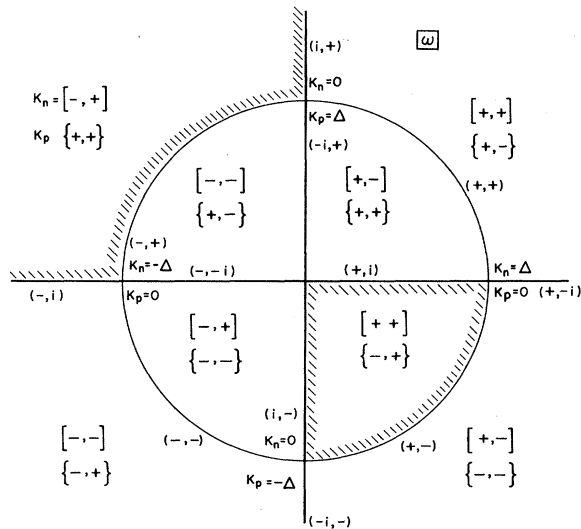


FIG. 5. The ω plane for positive energy with the same conventions enumerated in Fig. 3.

therefore contained in the contribution to Eq. (3.25) from the Γ_3 contour. We may isolate some of these contributions by distorting the path Γ_3 to pick up terms corresponding to certain nonbound n -core Gamow states. First consider the integral Γ_3 for positive energies. The zero of energy has been set at the two-particle threshold in $G_{n+p}(W)$. Referring to Fig. 3 one sees that for ω in the second quadrant, k_p takes on values in its upper-half plane; and k_n takes on values in its lower-half plane. Therefore the only singularities in the inte-

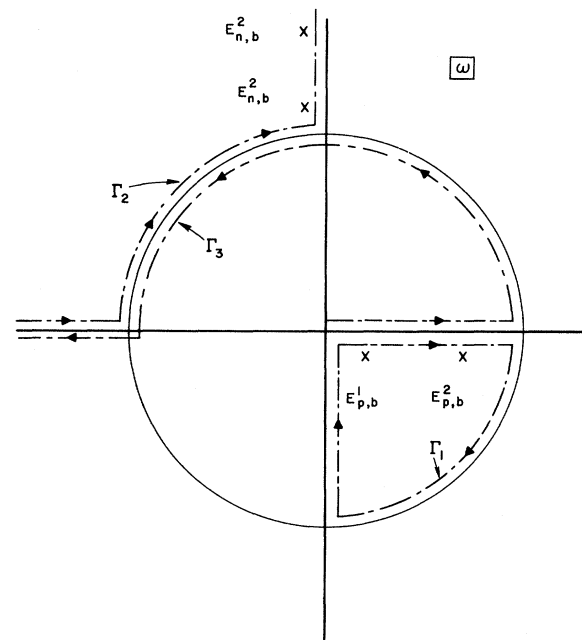


FIG. 6. The ω plane for positive energy with the same conventions enumerated in Fig. 4.

grand of the Γ_3 integral arise from poles in \bar{g}'_{i_n} since \bar{g}'_{i_p} does not contain any singularities in the upper-half plane of its momentum variable. Again from Fig. 3, for ω in the second quadrant, one sees that n -core resonance-momenta lie inside the line $|\omega|=1$; virtual state momenta, on the line $|\omega|=1$ and time-reversed-resonance-momenta lie outside this line. Proceeding just as when the bound-state pole contributions were evaluated, the non-bound neutron-core Gamow-state pole contributions may be evaluated to give for the Γ_3 contour

$$\sum_{n\Gamma} \Gamma_{i_n}(\mathbf{r}_n) \Gamma_{i_n}(\mathbf{r}'_n) \bar{g}'_{i_p}[k_p(\Delta, n_\Gamma), r_p, r'_p] + \frac{1}{2\pi i} \int_{\Gamma'_3} d\omega f(\omega) \bar{g}'_{i_n}(k_n, r_n, r'_n) \bar{g}'_{i_p}(k_p, r_p, r'_p).$$

The contour Γ'_3 is the contour Γ_3 after Γ_3 has been moved through the poles included in the sum. In this manner the pole terms in $\mathcal{G}_{n+p}^{\pm}(W)$ corresponding to a finite set of n -core Gamow states can be written in terms of the p -core Green's functions. These Green's functions have cuts in the total energy plane emanating from the n -core Gamow-state energies. Further note that for ω in the second quadrant and $\text{Im}\omega > -\text{Re}\omega$, $\text{Im}k_p > -\text{Im}k_n$. Using Eq. (3.19), it is easy to show all neutron-core resonances whose momenta satisfy $|\text{Im}k_n| < |\text{Re}k_n|$ are included in this segment for all negative energies. These are the so-called proper resonances.

For consideration of positive energies, it is necessary to discuss the analytic properties of $G_{n+p}(W)$ in more detail. It has been shown that matrix elements of $G_{n+p}(W)$ contain cuts in the W plane starting from the single-particle binding energies. These cuts may be taken along the real W axis in the direction of increasing energy. In addition, there is a logarithmic branch point at the two-particle threshold.⁸ Returning to Fig. 1 which shows the cut z plane for the integrand in the convolution (3.12), one sees that continuation in W corresponds to moving the proton threshold branch point about in the z plane. Branch points in $G_{n+p}(W)$ arise from pinching the contour of integration between the single-particle bound-state poles of one particle and the threshold branch point of the other as the integrand is continued in W . The two-particle threshold branch point of $G_{n+p}(W)$ arises from the pinch which occurs when the two single-particle threshold branch points are brought together. Therefore one goes from the upper lip to the lower lip of the two-particle continuum cut in $G_{n+p}(W)$ as one continues the integrand in W so as to bring the proton threshold branch point from the upper lip of the neutron continuum cut around the neutron threshold to the low-

er lip of this cut.

Poles in $G_{n+p}(W)$ arise from pinching the contour between two single-particle poles as the integrand is continued in W . Such a pinch occurs at physical energies when the contour is pinched between two single-particle bound-state poles. Poles at unphysical energies arise from the pinch between two single-particle poles, at least one of which corresponds to a nonbound single-particle Gamow state. Above the two-particle threshold in $G_{n+p}(W)$ the nearest pinches occurring at unphysical energies are reached by continuing the integrand in W as shown in Fig. 7(a). The first pinches in such a continuation occur when some proton Gamow-states pole pinches the contour against a neutron Gamow state encountered after going through the neutron continuum cut from above. In terms of the neutron energy Riemann surface, these are neutron states whose Gamow energies lie in the lower half of the unphysical neutron energy sheet. The corresponding single-particle momentum has a positive real part and a negative imaginary part. These are the neutron resonance states. On the other hand pinches with time-reversed neutron Gamow states would be encountered by continuing the integrand in W so as to pull the proton branch point around the neutron threshold and then up through the neutron continu-

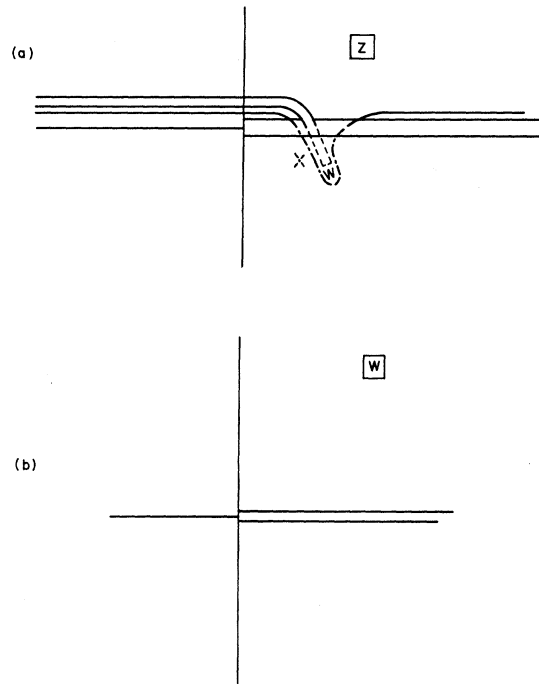


FIG. 7. The distortion of the n -core continuum cut and the path of integration as the convolution integrand is analytically continued in W is shown in (a). The two-particle continuum cut in $G_{n+p}(W)$ is shown in (b).

um cut from below. In the cut energy plane for $G_{n+p}(W)$ shown in Fig. 7(b), where the bound-state poles and cuts have been omitted for simplicity, the first analytic continuation proceeds from the upper lip of the continuum cut while the second requires continuation of $G_{n+p}(W)$ around the two-particle threshold. Therefore above this threshold, poles in $G_{n+p}(W)$ closest to the physical re-

gion, which is the upper lip of the cut, generally correspond to pinches which include a neutron resonance pole. These poles lie within the circular segment $|\omega|=1$ in the first quadrant of the ω plane. Contributions from these poles are picked up as one distorts the contour Γ_3 within this region. Therefore we have

$$\begin{aligned} \mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p) = & \sum_{p_b} \beta_{i_p}(r_p) \beta_{i_p}(r'_p) \bar{g}_{i_n}^{(+)}[k_n(\Delta, p_b), r_n, r'_n] + \sum_{n_b} \beta_{i_n}(r_n) \beta_{i_n}(r'_n) \bar{g}_{i_p}^{(+)}[k_p(\Delta, n_b), r_p, r'_p] \\ & + \sum'_{n_\gamma} \Gamma_{i_n}(r_n) \bar{\Gamma}_{i_n}(r'_n) * \bar{g}_{i_p}^{(+)}[k_p(\Delta, n_\gamma), r_p, r'_p] + \frac{1}{2\pi i} \int_{\Gamma_3} d\omega f(\omega) \bar{g}_{i_n}^{(+)}(k_n, r_n, r'_n) \bar{g}_{i_p}^{(+)}(k_p, r_p, r'_p), \end{aligned} \quad (3.31)$$

where the prime on the sum indicates the integration path Γ_3 depends on neutron Gamow states included. By including appropriate neutron states in the sum, one can isolate the poles in $\mathcal{G}_{i_n i_p}^c(W)$ within a given energy region in terms involving the connected part of the single-particle Green's functions. In addition these terms include some of the cuts which are contained in $\mathcal{G}_{i_n i_p}^c(W)$. Therefore the term involving the integral over the contour Γ_3 can be expected to vary slowly with energy.

4. FINITE RANK APPROXIMATION FOR $\mathcal{G}_{i_n i_p}^c(W)$

As discussed in Sec. 2, the poles in $\mathcal{G}_{i_n i_p}^c(W)$ give rise to large contributions to the kernel $\mathcal{G}_{n+p}(W) t_{np}(W)$. This is to say that near poles of $\mathcal{G}_{i_n i_p}^c(W)$ the perturbation series for Eq. (2.15a) does not converge. Therefore these contributions must be treated nonperturbatively which is straightforward if they can be included in a kernel of finite rank. From Eq. (3.31), it is apparent that the poles close to the physical region can be included in a kernel of finite rank if the single-particle Green's functions can be approximated by operators of finite rank which include the single-particle Gamow states close to the physical single-particle energy region.

The residue of $\mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p)$ at poles corresponding to a neutron Gamow state with energy e_n and proton Gamow state with energy e_p follows from Eq. (3.31) as

$$\begin{aligned} \lim_{W \rightarrow e_n + e_p} (W - e_n - e_p) \mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p) \\ = \Gamma_{i_n}(r_n) \Gamma_{i_p}(r_p) \bar{\Gamma}_{i_n}^*(r'_n) \bar{\Gamma}_{i_p}^*(r'_p). \end{aligned} \quad (4.1)$$

$\Gamma(r)$ is a normalized Gamow-state wave function, either bound or unbound, as discussed in the Appendix. $\bar{\Gamma}(r)$ is the corresponding time reversed

Gamow state. If either or both e_n and e_p were binding energies, the corresponding normalized bound-state wave functions would appear.

A finite rank contribution to the kernel results from extracting pole contributions from the connected part of the single-particle Green's functions as

$$\bar{g}(k, r, r') = m \sum_i \frac{\gamma_i(r) \gamma_i(r')}{2k_{\gamma_i}(k - k_{\gamma_i})} + \bar{g}^R(k, r, r'), \quad (4.2)$$

where the partial-wave and particle subscripts have been omitted. $\gamma_i(r)$ is the normalized Gamow-state wave function^{9, 15, 16} for the momentum k_i ; and $\bar{\gamma}_i(r)$, the wave function for the time-reversed momentum $-k_i^*$. The sum generally includes bound, resonance, and virtual states. For deuteron-induced reactions, we have argued that two-particle continuum states cannot be excluded. Therefore, if the pole terms were extracted in this manner, it would generally be necessary to evaluate matrix elements in terms of product states of two unphysical single-particle Gamow states. Here unphysical simply means Gamow states whose momenta lie in the lower-half k plane. Both wave functions in such a product grow exponentially and therefore, it would be necessary to resort to the special methods which have been introduced^{9, 15} for evaluating the integrals encountered in evaluating the matrix elements.

Another approach is suggested by the observation that a finite rank approximation to the connected part of the single-particle Green's function follows from a finite rank approximation to the valence particle-core potential. Weinberg states⁶ could presumably be used to construct such an approximation. However a more straightforward approach involves the use of single-particle Gamow states to construct a finite rank approximating potential. This is discussed fully in Ref. 15, and we only present a review here. Denote the row

vector whose elements are a finite set of Gamow states for a given partial wave by $\gamma(r)$:

$$\gamma(r) = (\gamma_1(r), \dots, \gamma_n(r)). \quad (4.3)$$

$\bar{\gamma}(r)$ is the row vector of the corresponding time-reversed states. Introduce a finite rank potential by

$$V^s(r, r') = V(r)\gamma(r)A^{-1}\bar{\gamma}^\dagger(r')V(r'), \quad (4.4)$$

where $V(r)$ is the valence particle-core interaction and A is the matrix

$$A = \int_0^\infty dr \bar{\gamma}^\dagger(r)V(r)\gamma(r). \quad (4.5)$$

The partial-wave Green's function for this potential is given by

$$g^{s(+)}(k, r, r') = g_0^{(+)}(k, r, r') + s(k, r)\mathfrak{D}^{-1}(k)\bar{s}^\dagger(r'). \quad (4.6)$$

$\mathfrak{D}(k)$ is the matrix

$\mathfrak{D}(k)$

$$= A - \int_0^\infty dr \int_0^\infty dr' \bar{\gamma}^\dagger(r)V(r)g_0^{(+)}(k, r, r')V(r')\gamma(r'). \quad (4.7)$$

The partial-wave free Green's function is given by

$$g_{0i}^{(+)}(k, r, r') = 2mk^{-1}(-)^{l+1}u_l(kr_<)w_l^{(+)}(kr_>), \quad (4.8)$$

where $r_<$ ($r_>$) is the lesser (greater) of r and r' , and $u_l(kr)$ and $w_l^{(+)}(kr)$ are the Bessel-Riccati and Hankel-Riccati functions, respectively.¹⁷ For a cut-off potential, the matrix elements in $\mathfrak{D}(k)$ are entire functions of the momentum k .¹⁵ $s(k)$ is the row vector

$$s(k, r) = \int_0^\infty dr' g_0^{(+)}(k, r, r')V(r')\gamma(r'); \quad (4.9)$$

and $\bar{s}^\dagger(k, r)$, the column vector

$$\bar{s}^\dagger(k, r) = \int_0^\infty dr' g_0^{(+)}(k, r, r')V(r')\bar{\gamma}^\dagger(r'). \quad (4.10)$$

The Green's function $g^{s(+)}(k, r, r')$ has poles corresponding to the Gamow states included in the potential $V^s(r, r')$ at the positions and with the residues of the partial-wave Green's function for the potential $V(r)$. This result is independent of the normalization of the Gamow states.

For large r , an element of either $s(k, r)$ or $\bar{s}^\dagger(k, r)$ is proportional to e^{ikr} . Returning to Eq. (3.31), we see that for negative energies in $\mathcal{G}_{n+p}^c(W)$, the single-particle Green's functions which appear with bound-state wave functions are evaluated at either positive real or positive imaginary momenta. The proton Green's function associated with the unphysical neutron Gamow states

is evaluated at momenta corresponding to ω in the second quadrant of the ω plane shown in Fig. 3. These momenta are in the upper-half k plane. Using Eq. (3.19), as previously mentioned it is easily shown that for $\text{Im}\omega > -\text{Re}\omega$ in the second quadrant, $\text{Im}k_p > -\text{Im}k_n$. Further it is easily shown that this region of the ω plane includes, for all negative energies, all the so-called proper neutron resonances. Turning to positive energies for $\mathcal{G}_{n+p}^c(W)$ the single-particle Green's functions appearing with the bound-state wave functions are evaluated at positive real momenta. The proton Green's function associated with neutron Gamow states is evaluated at momenta in the upper-half momentum plane corresponding to $|\omega| < 1$ within the first quadrant of the ω plane shown in Fig. 5. Again it is easily shown using Eqs. (3.19) that for $\text{Im}\omega < \text{Re}\omega$ in this region, $\text{Im}k_p > -\text{Im}k_n$. This discussion will be useful later when we discuss matrix elements in terms of the states which form the finite rank approximation to $\mathcal{G}_{n+p}^c(W)$.

One would like to approximate $\mathcal{G}_{n+p}^c(W)$ by a finite rank operator. The previous discussion shows how a finite rank operator can be constructed which includes the poles and some of the cuts in $\mathcal{G}_{n+p}^c(W)$ close to physical energies. It is difficult to determine with any rigor how well and under what conditions the finite rank operator can approximate $\mathcal{G}_{n+p}^c(W)$. Therefore certain matrix elements of a finite rank operator constructed using the methods described were compared with these same matrix elements of $\mathcal{G}_{n+p}^c(W)$ which were evaluated numerically.

The matrix element $\mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p)$ may be evaluated numerically using Eqs. (3.20), (3.25), and (3.30). The discontinuity of $\bar{g}_{i_n}^s(k_n, r_n, r'_n)$ across the neutron continuum cut in Fig. 1 is given by

$$\begin{aligned} & -4imk_n^{-1}\Delta\bar{g}_{i_n}^{s(+)}(k_n, r_n, r'_n) \\ & = \bar{g}_{i_n}^{s(+)}(k_n, r_n, r'_n) - \bar{g}_{i_n}^{s(+)}(-k_n, r_n, r'_n) \\ & = -4imk_n^{-1}[\psi_{i_n}^{s(+)}(k_n, r_n)\psi_{i_n}^{s(+)}(k_n, r'_n) - u_{i_n}(k_n r_n) \\ & \quad \times u_{i_n}(k_n r'_n)], \end{aligned} \quad (4.11)$$

where $\psi_{i_n}^{s(+)}(k, r)$ is the scattering wave function discussed in the Appendix. The contribution from the contour Γ_3 in Eq. (3.25) as shown in Fig. 2(b) may then be written as

$$\begin{aligned} C_3(E, r_n, r_p | r'_n, r'_p) \\ = \frac{2}{\pi} \int_0^\infty dk_n \Delta\bar{g}_{i_n}^{s(+)}(k_n, r_n, r'_n)\bar{g}_{i_p}^{s(+)}(k_p, r_p, r'_p), \end{aligned} \quad (4.12)$$

TABLE I. Valence-particle-core potential parameters. V_0 is the depth of the square-well potential of radius a . m is taken as the proton mass.

$S=2mV_0a^2=(6.2)^2$	$S=2mV_0a^2=(3.5)^2$
$a=3\text{ F}$	$a=3\text{ F}$
$V_0=88.3\text{ MeV}$	$V_0=24.2\text{ MeV}$

where

$$\begin{aligned} k_p &= (2mE - k_n^2)^{1/2} ; k_n^2 < 2mE, \\ &= i(k_n^2 - 2mE)^{1/2} ; k_n^2 > 2mE. \end{aligned} \quad (4.13)$$

Then $\mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p)$ is the sum of the bound-state contributions given by Eqs. (3.30a) and (3.30b), and $C_3(E, r_n, r_p | r'_n, r'_p)$ which generally must be evaluated numerically.

From the equation for $X(W)$, Eq. (2.15b) and the matrix elements involving $X(W)$ for the various deuteron-induced reaction amplitudes given by Eqs. (2.16), (2.18), and (2.21), one sees that the neutron-proton interaction V_{np} always appears to the right and left of $\mathcal{G}_{n+p}(W)$. Since this is a short-range interaction relative to the range of the valence particle-core reactions, results are most sensitive to the matrix elements of $\mathcal{G}_{i_n i_p}(W, r_n, r_p | r'_n, r'_p)$ for which $r_n \approx r_p$ and $r'_n \approx r'_p$. Therefore we only compare the finite rank pole approximation for $r_n = r_p, r'_n = r'_p$ with $\mathcal{G}_{i_n i_p}^c(W, r, r | r', r')$.

Square-well valence particle-core potentials were used for the numerical comparison, since it is easy to solve the single-particle problem for this potential. All calculations were performed for p -wave valence particle states. The radius of both valence particle-core interactions was chosen to be 3 F. The first set of calculations assume the same potential depth, which is just insufficient to give two bound states, for both particles. The parameters for this potential are given in the first column of Table I. The momenta of the four lowest-lying Gamow states for this potential are given in the first column of Table II. These four momenta correspond to the bound, virtual, resonance, and time-reversed-resonance states for this potential. Since this potential supports a

TABLE II. Valence-particle-core Gamow-state parameters used in calculations. S is defined in Table I. l is the orbital angular momentum. k is the momentum, and a is the radius of the square-well potential.

$S=(6.2)^2$	$S=(3.5)^2$
$l=1$	$l=1$
$ka = \pm 0.5986 - 0.12i$	$ka = \pm 4.7912 - 1.3992i$
$ka = 0 + 4.8734i$	$ka = 0 + 1.1318i$
$ka = 0 - 2.7751i$	$ka = 0 - 0.6438i$

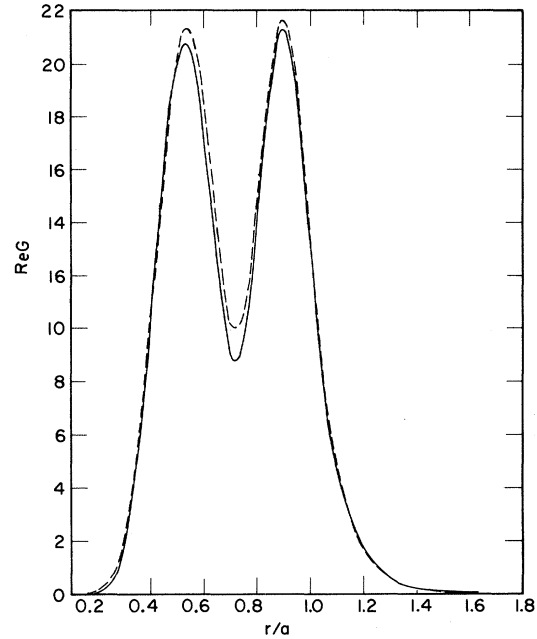


FIG. 8. Plot of the real part of the connected Green's function $G_{11}^c(E, r, r | r, r)$ at $E = -51.3$ MeV vs r/a . The exact result is indicated by the solid line; the approximation described in the text, by the dashed line. The numbers on the ordinate are appropriate for $\hbar=c=1$.

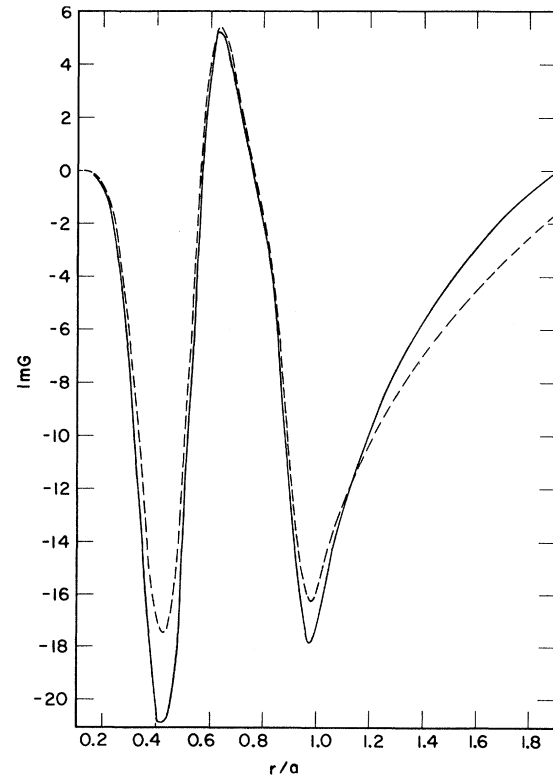


FIG. 9. Plot the imaginary part of the connected Green's function $G_{11}^c(E, r, r | r, r)$ at $E = 2.39$ MeV with the same conventions as in Fig. 8.

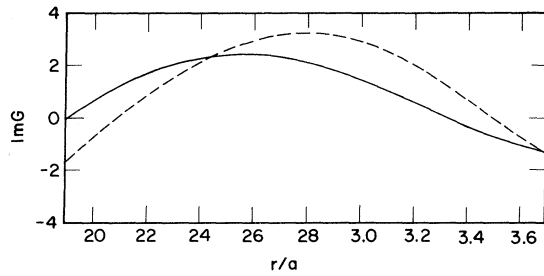


FIG. 10. The same plot as Fig. 9 for the values of r/a indicated.

bound state and a resonance, $\mathcal{G}_{11}^c(W, r_n, r_p | r'_n, r'_p)$ has a pole corresponding to both valence particles being bound, a second pole close to the physical region and below the two-particle continuum threshold corresponding to one particle bound and the other resonating, and a third pole close to the physical region above the two-particle threshold corresponding to both valence particles resonating with the core. The bound state occurs at -109.4 MeV, the bound-resonance pole at $W = -53.8 - 0.34i$ MeV, and the resonance-resonance pole at $W = 1.58 - 0.68i$ MeV. Figure 8 shows the real part of the completely diagonal matrix element $\mathcal{G}_{11}^c(E + i\epsilon, r, r | r, r)$ plotted against r/a for $E = -51.3$ MeV which is close to the bound-resonance pole energy. Also shown is the finite rank approximation obtained by replacing the single-particle Green's functions which appear in the contributions from the contours Γ_1 and Γ_2 in the first two terms of Eq. (3.31) by the Green's functions given by Eq. (4.6). The four Gamow states whose momenta are given in Table II were used to construct the finite rank potential in Eq. (4.4). This approximation for $\mathcal{G}_{11}^c(W, r, r | r, r)$ includes only poles corresponding to one of the valence particle bound and the other in one of the Gamow states

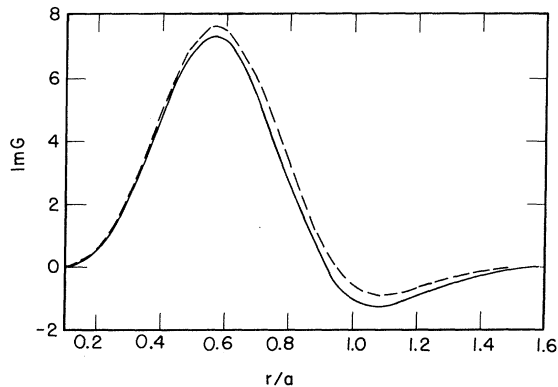


FIG. 11. Plot of the imaginary part of the off-diagonal connected Green's function $G_{11}^c(E, r, r | 2, 2)$ for $E = 2.39$ MeV with same conventions as followed in previous Green's function plots.

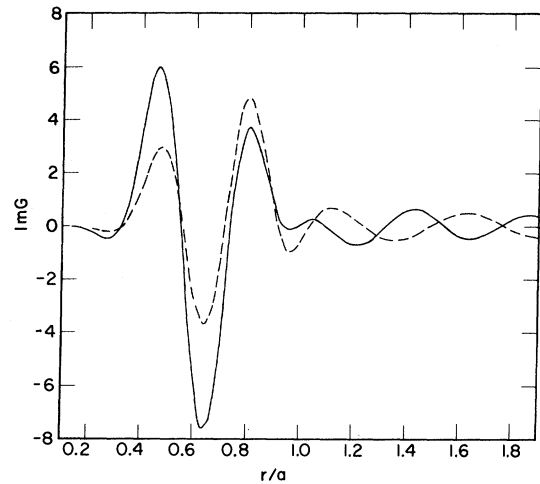


FIG. 12. Plot of imaginary part of the diagonal connected Green's function vs r/a for $E = 71.1$ MeV with same conventions as previous Green's function plots.

whose wave function is included in the separable potential. The approximation for the imaginary part which is better than the approximation for the real part is not shown. We may conclude that two-particle continuum states are relatively unimportant well below the two-particle threshold.

Figures 9 and 10 show the imaginary part of the completely diagonal matrix element $\mathcal{G}_{11}^c(E, r, r | r, r)$ plotted against r/a . The energy in this case is 2.39 MeV which is close to the resonance-resonance pole. In this case the finite rank approximation includes in addition to the first two terms in Eq. (3.31) the resonance, neutron Gamow state in the third term of Eq. (3.31). The single-particle Green's functions are approximated in the same way as those in Fig. 8. Figure 11 shows the imaginary part of the off-diagonal matrix element $\mathcal{G}_{11}^c(E, r, r | 2, 2)$ at the same energy plotted against r/a . The finite rank approximation is calculated just as in Figs. 9 and 10. This approximation includes the resonance-resonance pole. The agreement for the real part for both the diagonal and off-diagonal terms is at least as good as the imaginary part shown. Figure 12 again shows a comparison of diagonal matrix elements as a function of r/a . The finite rank approximation was calculated in exactly the same way as those shown in Figs. 9–11. The energy in this case is 71.1 MeV, far above the energy region of the poles included in the finite rank approximation. The result indicates the finite rank approximation which includes only the poles close to the physical region does not grossly misrepresent the connected part of the Green's function at energies far from these poles.

Figure 13 shows a comparison between the imaginary part of the finite rank approximation con-

structured in the same manner as in Fig. 12. In this case the n -core interaction is the square-well potential whose parameters appear in the second column of Table I while the p -core interaction is the same as previously used. The Gamow-state momenta for the n -core interaction are given in the second column of Table II. This potential is just sufficient to bind a single p -wave state. Therefore $\mathcal{G}_{11}^c(W, r_n, r_p | r'_n, r'_p)$ has a bound-state pole at -57.7 MeV and a second pole close to the physical region slightly below the two-particle continuum threshold at $W = -2.16 - 0.335i$ MeV corresponding to a bound neutron state and a resonating proton state. The Green's function in Fig. 11 is evaluated at -0.493 MeV. The finite rank approximation includes the first two terms and the resonance neutron state in the third term of Eq. (3.31). The proton Green's function is again calculated with the four Gamow-state separable potential. The approximation does not include the two-parti-

cle continuum cut and this calculation indicates the pole contributions are more important than contributions from this cut.

These calculations indicate that the connected part of the Green's function $\mathcal{G}_{i_n i_p}^c(W, r_n, r_p | r'_n, r'_p)$ is well approximated by this method for energies in the neighborhood of the poles of the Green's function.

5. CONCLUSIONS

We now turn to a discussion of the implications of the finite rank approximation for $\mathcal{G}_{n+p}^c(W)$. We will find that it is possible to write matrix elements of $X(W)$ as a sum of two types of terms – one associated with the impulse or partially connected contributions and the other resembling a sum of resonance contributions. Following Glöckle and Heiss¹⁸ we write the set of partial-wave equations for Eq. (2.15a) as

$$\begin{aligned} {}_{i_n i_p} X_{i'_n i'_p}(L, W, r_n, r_p | r'_n, r'_p) &= \mathcal{G}_{i_n i_p}(L, W, r_n, r_p | r'_n, r'_p) \delta_{i_n i'_n} \delta_{i_p i'_p} + \sum_{\tilde{i}_n \tilde{i}_p} \int d\tilde{r}'_n d\tilde{r}'_p \int dr_n dr_p \mathcal{G}_{i_n i_p}(L, W, r_n, r_p | \tilde{r}_n, \tilde{r}_p) \\ &\times {}_{i_n i_p} t_{\tilde{i}_n \tilde{i}_p}(L, W, \tilde{r}_n, \tilde{r}_p | \tilde{r}'_n, \tilde{r}'_p) \tilde{i}_n \tilde{i}_p X_{i'_n i'_p}(L, W, \tilde{r}'_n, \tilde{r}'_p | r'_n, r'_p), \end{aligned} \quad (5.1a)$$

where we have assumed the angular momentum of the core is zero. Then the total angular momentum L is just the sum of the individual, valence-particle angular momenta. To obtain ${}_{i_n i_p} t_{\tilde{i}_n \tilde{i}_p}(L, W)$ one is confronted with the usual problem of obtain-

ing the appropriate matrix element of the interaction of two particles in a potential.¹⁸ In this case the interaction is energy dependent and nonlocal. We rewrite Eq. (5.1a) in the usual schematic fashion

$$\begin{aligned} {}_i X_{i'}(L, W) &= \delta_{ii'} \mathcal{G}_i(L, W) + \sum_{\tilde{i}} \mathcal{G}_i(L, W) \\ &\times {}_i t_{\tilde{i}}(L, W) \tilde{i} X_{i'}(L, W). \end{aligned} \quad (5.1b)$$

In terms of the impulse or partially connected term and the connected term we have

$$\mathcal{G}_i(L, W) = I_i(L, W) + \mathcal{G}_i^c(L, W). \quad (5.2)$$

Referring to Eq. (3.1a) and the discussion in the previous section, we approximate $\mathcal{G}_i^c(L, W)$ by an operator of finite rank

$$\tilde{\mathcal{G}}_i^c(L, W) = |x_i(L, W)\rangle \Lambda_{i,L}(E)^{-1} \langle \tilde{x}_i(L, W)|, \quad (5.3)$$

where $|x_i(L, W)\rangle$ [$\langle \tilde{x}_i(L, W)|$] is generally a row [column] vector of ket [bra] vectors, and $\Lambda_{i,L}(E)$ is generally a matrix. Within this approximation we introduce an operator ${}_i \tilde{X}_{i'}(L, W)$ whose matrix elements are to approximate those of ${}_i X_{i'}(L, W)$ as the solution to the equation

$$\begin{aligned} {}_i \tilde{X}_{i'}(L, W) &= [I_i(L, W) + \tilde{\mathcal{G}}_i^c(L, W)] \delta_{ii'} \\ &+ \sum_{\tilde{i}} [I_i(L, W) + \tilde{\mathcal{G}}_i^c(L, W)] {}_i t_{\tilde{i}}(L, W) \tilde{i} \tilde{X}_{i'}(L, W). \end{aligned} \quad (5.4)$$

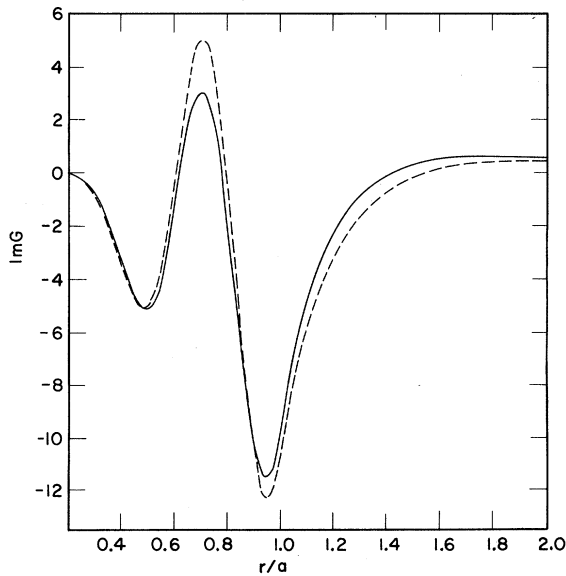


FIG. 13. Plot of imaginary part of the diagonal connected Green's function for the valence-particle potential parameters indicated in the text. The energy is $E = -0.493$ MeV. The same conventions apply as in previous Green's function plots.

Proceeding in the usual manner,¹⁹ the solution may be written in terms of the operator ${}_i\Omega_{i'}(L, W)$ which satisfies the equation

$${}_i\Omega_{i'}(L, W) = \delta_{ii'} + \sum_{\bar{i}} I_{\bar{i}}(L, W) \times {}_i t_{\bar{i}}(L, W) {}_{\bar{i}}\Omega_{i'}(L, W) \quad (5.5a)$$

and its transpose ${}_i\Omega_{i'}^T(L, W)$, which satisfies

$${}_i\Omega_{i'}^T(L, W) = \delta_{ii'} + \sum_{\bar{i}} {}_i\Omega_{\bar{i}}^T(L, W) {}_{\bar{i}}t_{i'}(L, W) I_{\bar{i}}(L, W) \quad (5.5b)$$

as

$${}_i\bar{X}_{i'}(L, W) = \sum_{\bar{i}} {}_i\Omega_{\bar{i}}(L, W) I_{\bar{i}}(L, W) + \sum_{\bar{i}} {}_i\Omega_{\bar{i}}(L, W) |x_{\bar{i}}(L, W)\rangle [\Delta^{-1}(L, W)]_{\bar{i}i'} \langle \bar{x}_{\bar{i}}(L, W) | {}_{\bar{i}}\Omega_{i'}^T(L, W), \quad (5.6)$$

where the matrix $\Delta(L, W)$ is given by

$$(\Delta(L, W))_{ii'} = \Lambda_{i,L}(E)\delta_{ii'} + \langle \bar{x}_i(L, W) | \sum_{\bar{i}} {}_i t_{\bar{i}}(L, W) {}_{\bar{i}}\Omega_{i'}(L, W) |x_{i'}(L, W)\rangle. \quad (5.7)$$

Matrix elements of the kernel $I(L, W)t(L, W)$ appearing in Eq. (5.5a) do not contain singularities in W close to the physical region. The poles close to the physical region in the full kernel $\mathcal{G}_{n+p}(W)t_{np}(W)$ have been isolated in the finite rank approximation to $\mathcal{G}_{n+p}^c(W)$. Even so further investigation is required to determine when, and even if, $\Omega(L, W)$ may be obtained by a perturbation expansion of Eq. (5.5a). In any case the first term in Eq. (5.6) is associated with the impulse term in the valence particle-core Green's function $G_{n+p}(W)$. The second term contains the contributions from the two-particle states associated with the poles of $\mathcal{G}_{n+p}^c(W)$ close to the physical region. These states are modified through the impulse terms by $\Omega^L(W)$ and mixed by the effective interaction

$${}_i T_{i'}(L, W) = \sum_{\bar{i}} {}_i t_{\bar{i}}(L, W) {}_{\bar{i}}\Omega_{i'}(L, W). \quad (5.8)$$

Further note that matrix elements of ${}_i\bar{X}_{i'}(L, W)$ are singular at those energies W_s where

$$\det[\Delta(L, W_s)] = 0. \quad (5.9)$$

We note in passing that the approximation ${}_i\Omega_{i'}(L, W) = 1$ gives for the deuteron elastic scattering amplitude, Eq. (2.16), a sum of impulse scattering amplitudes and resonance-like terms.

In evaluating matrix elements which arise in Eq. (5.6), one should refer to the discussion in Sec. 4 on the behavior in configuration space of the product states which appear in Eq. (5.3). The results of the discussion in Sec. 4 indicate that for a sufficiently short-range $n-p$ interaction the relevant matrix elements in Eq. (5.6) are certainly well defined for all negative energies if only proper neutron resonance and time-reversed resonance states are considered along with the valence particle bound states. For positive energies, when necessary, matrix elements may be obtained by analytic continuation from negative energies.

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APPENDIX

In this Appendix we shall present a short discussion of the Gamow-state wave function. Chapter 12 of Newton's book¹¹ is a good reference for the general statements made here concerning potential scattering. The regular solution of the radial Schrödinger equation for the l th partial wave satisfies the boundary condition

$$\lim_{r \rightarrow 0} r^{-1} \varphi_l(k, r) = 1. \quad (A1)$$

Two linearly independent solutions which generally are not regular at $r=0$ may be introduced by boundary conditions at infinity

$$\lim_{r \rightarrow \infty} e^{i\pi/2} f_l^{(\pm)}(k, r) = 1. \quad (A2)$$

These two solutions span the solution space and, therefore, $\varphi_l(k, r)$ may be written as the linear combination

$$\varphi_l(k, r) = (2ik)^{-1} [\mathcal{F}_l^{(-)}(k) f_l^{(+)}(k, r) - \mathcal{F}_l^{(+)}(k) f_l^{(-)}(k, r)]. \quad (A3a)$$

The partial-wave scattering solution corresponding to an outgoing scattered wave may be written as

$$\psi_l^{(+)}(k, r) = k i^l \varphi_l(k, r) [\mathcal{F}_l^{(+)}(k)]^{-1}, \quad (A4)$$

$$\psi_l^{(+)}(k, r) \xrightarrow{r \rightarrow \infty} \frac{1}{2} i^{l+1} \{ e^{-ikr} - \mathcal{F}_l^{(-)}(k) [\mathcal{F}_l^{(+)}(k)]^{-1} e^{ikr} \}. \quad (A5)$$

Therefore the S matrix is given by

$$S_l(k) = (-)^l \mathcal{F}_l^{(-)}(k) [\mathcal{F}_l^{(+)}(k)]^{-1}. \quad (A6)$$

For the cut-off potentials considered in this paper, $\mathfrak{F}_i^{(+)}(k)$ are entire functions of the momentum k . Therefore poles of $S_i(k)$ correspond to zeros of $\mathfrak{F}_i^{(+)}(k)$. The zeros of $\mathfrak{F}_i^{(+)}(k)$ in the upper half of the k plane lie along the positive imaginary axis, are finite in number, and correspond to bound states. Those in the lower half k plane are either pure imaginary and correspond to virtual states or they occur in pairs symmetric about the imaginary axis and correspond to resonances. From Eqs. (A1)–(A3), it follows that there exist solutions regular at the origin and purely outgoing at large r for those momenta k_α such that

$$\mathfrak{F}_i^{(+)}(k_\alpha) = 0, \quad (\text{A7})$$

since

$$\begin{aligned} \varphi_i(k_\alpha, r) &= (2ik_\alpha)^{-1} \mathfrak{F}_i^{(-)}(k_\alpha) f_i^{(+)}(k_\alpha, r) \\ &= c_\alpha f_i^{(+)}(k_\alpha, r). \end{aligned} \quad (\text{A3b})$$

The two-particle Green's function is given by

$$g_i^{(+)}(k, r, r') = -\varphi_i(k, r) f_i^{(+)}(k, r) [\mathfrak{F}_i^{(+)}(k)]^{-1}. \quad (\text{A8})$$

Since $\varphi_i(k, r)$ and $f_i^{(+)}(k, r)$ are proportional at k_α , the residue of $g_i^{(+)}(k, r, r')$ at k_α is a separable function of r and r' . For

$$\mathfrak{F}_i^{(+)'}(k_\alpha) \equiv \left. \frac{d}{dk} \mathfrak{F}_i^{(+)}(k) \right|_{k=k_\alpha}, \quad (\text{A9})$$

the residue follows as

$$\begin{aligned} R_{i,\alpha}(r, r') &\equiv \lim_{k \rightarrow k_\alpha} (k - k_\alpha) g_i^{(+)}(k, r, r'), \\ &= -\varphi_i(k_\alpha, r) \varphi_i(k_\alpha, r') [c_\alpha \mathfrak{F}_i^{(+)'}(k_\alpha)]^{-1}. \end{aligned} \quad (\text{A10a})$$

Using the extended definition of the inner product introduced in Ref. 15 (also see Romo⁹), it was shown in this reference that the inner product $I_{i,\alpha\alpha}\{\varphi\}$ of the Gamow-state wave functions $\varphi_i(k_\alpha, r)$ satisfies the equation

$$c_\alpha \mathfrak{F}_i^{(+)'}(k_\alpha) = -2k_\alpha I_{i,\alpha\alpha}\{\varphi\}. \quad (\text{A11})$$

Further

$$\varphi_i(-k_\alpha^*, r) = \varphi_i^*(k_\alpha, r).$$

Therefore Eq. (A10a) may be written as

$$R_{i,\alpha}(r, r') = \varphi_i(k_\alpha, r) \varphi_i^*(-k_\alpha^*, r') (2k_\alpha I_{i,\alpha\alpha}\{\varphi\})^{-1}, \quad (\text{A10b})$$

which is the result used in Eq. (4.2) since $\varphi_i(-k_\alpha^*, r)$ is the time-reversed solution corresponding to $\varphi_i(k_\alpha, r)$.

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