

PHYSICAL REVIEW C

NUCLEAR PHYSICS

THIRD SERIES, VOLUME 29, NUMBER 3

MARCH 1984

Rank-one inverse scattering problem: Reformulation and analytic solutions

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(Received 11 March 1983)

Using the K -matrix formalism, we give a simplified reformulation of the S -wave rank-one inverse scattering problem. The resulting Cauchy integral equation, obtained differently by Gourdin and Martin in their first paper, is tailored to rational representations of $F(k)=k \cot(\delta_0)$. Use of such $F(k)$ permits a simple but general solution without integration, giving analytic form factors having a pole structure like the S matrix that are reducible to rational expressions using Padé approximants. Finally, we show a bound state pole condition is necessary, and makes the form factor unique.

Finite rank NN potentials are used primarily because they simplify calculations of structure and reactions.¹ Such potentials are being developed to reproduce NN phase shifts with increasing accuracy. In particular, rank-two and rank-three interactions possessing form factors that are rational functions of momentum have recently been introduced.² It would be desirable for there to be a convergent procedure for constructing analytic low-rank potentials that (a) are interchangeable with meson-theoretic potentials in an appropriate energy range and (b) have a simple and direct connection with two-body data. Such a procedure would simplify realistic nuclear calculations, in some cases decisively,³ while helping to justify the use of finite rank representations, known not to be fully equivalent, at higher energies, to local potentials.¹ In an approach which emphasizes the first goal, recent work based upon new Padé approximant methods^{4,5} for bound states can generate rational form factors for a variety of local potentials such as sums of Yukawas. Here we focus upon the second goal. We present a new formulation of rank-one inverse scattering theory that yields simple analytic form factors, and is designed to take advantage of the discovery that rational representations of the scattering function $F(k)=k \cot(\delta_0)$ can be accurate.⁶ Relating to the first goal, we employ our formulation to construct a rank-one potential phase-shift equivalent to a Yukawa potential.

Despite an interesting attempt with a rank-one interaction,⁷ it is generally accepted that at least a rank of two is required to reproduce one salient feature of the NN interactions: a long range attraction together with a strong short range repulsion.¹ A rank-two inverse scattering formalism, that of Fiedeldey,⁸ already exists. It allows for the initial introduction of a somewhat arbitrary long range

attraction such as could be constructed using the formulation we present here. We will address the rank-two inverse scattering problem in another paper.

We present a K -matrix⁹ formulation of rank-one inverse scattering theory that accomplishes several things: simpler physical and mathematical analysis than previous formulations;¹⁰ direct use of the known solutions of the inhomogeneous Riemann boundary value problem;¹¹ and presence of data in the form of $k \cot(\delta_0)$, which provides a context for introducing Padé approximants, facilitates analytical continuation, and enables a pole analysis of form factors not previously carried out. When the scattering function $F(k)$ is rational, we obtain the form factor *without integrations*. For a rational S matrix $S(k)$ we find that the poles of $S(k)$ and of the squared form factor $h(k)$ coincide and are of the same order in the upper k plane, except at the bound state pole, ik_B , and we obtain a general bound state pole condition: The order of the pole in $h(ik_B)$ must be one less than the order in $S(ik_B)$. This condition makes $h(k)$ unique, and as seen in our example, is easily enforced.

Our work is aligned most closely with the initial approach of Gourdin and Martin (GMI).¹² Using the Schrödinger equation, GMI arrived at our Eq. (5), which was analyzed extensively and partially solved for rational $F(k)$. The authors did not make use of Riemann boundary value theory, and their lengthy discussion of rational $F(k)$ did not utilize a bound state pole condition. Subsequently, Gourdin and Martin (GMII) (Ref. 12) transformed Eq. (5) into a form which enabled them to find a complete solution, with $h(k)$ proportional to

$$\sin\delta(k)\exp\left[-\frac{2}{\pi}\mathbf{P}\int_0^\infty dp p \frac{\delta(p)}{p^2-k^2}\right].$$

Although this expression can presumably be evaluated when $F(k)$ is given in rational form, it would appear immensely more practical to understand the simpler approach we present and to avoid integrations altogether. Subsequent work has led to the same expression for $h(k)$. Bolsterli and MacKenzie obtained this result starting with the T matrix,¹³ while Chadan and Sabatier explicitly referred to the Riemann problem while supplying extensions and more rigor to GMII.¹⁰ Our approach uses the same analyticity properties of scattering states but differs significantly in detail by working with a Cauchy integral equation. Coupled with the high accuracy of rational scattering functions, our formulation provides a powerful tool for inverse scattering problems.

We specialize to S -wave scattering states and an attractive interaction. The repulsive case is easily traced through upon changing the sign in Eq. (1). Given a scattering function $F(k)=k \cot(\delta_0)$, we find the rank-one interaction producing $F(k)$, if it exists, by (a) solving the S -wave K -matrix equation for a rank-one interaction, and (b) treating the formal solution as an integral equation in the squared form factor, $h(k)=g^2(k)$. The S -wave interaction is

$$V_0(k',k)=-\frac{4\pi}{M}g(k')g(k), \quad (1)$$

where $\hbar=c=1$ and $M/2$ is the reduced mass. We solve

$$K_0(k',k,E)=V_0(k',k) + \int_0^\infty dy y^2 \frac{V_0(k',y)K_0(y,k,E)}{E-y^2/M}, \quad (2)$$

where Cauchy principal values are understood, obtaining

$$K_0(k',k,E)=-\frac{4\pi}{M} \frac{g(k')g(k)}{1+2\pi J(ME)}, \quad (3)$$

where we have defined $h(k)$ to be an even function and

$$J(\omega)=\int_{-\infty}^\infty dy \frac{y^2 h(y)}{\omega-y^2}. \quad (4)$$

The fully on-shell K matrix is related to the scattering function by

$$K_0(k,k,k^2/M)=-2/[\pi M F(k)],$$

giving

$$1+2\pi \int_{-\infty}^\infty dy \frac{h(y)y^2}{k^2-y^2}=2\pi^2 F(k)h(k). \quad (5)$$

Equation (5) can be put in the standard form of the dominant integral equation of the Cauchy type for which a general solution is known.¹¹ We write

$$a(k)h(k)+\frac{b(k)}{\pi i} \int_{-\infty}^\infty dy \frac{h(y)}{y-k}=f(k), \quad (6)$$

with $a(k)=k \cot(\delta_0)$, $b(k)=ik$, and $f(k)$ becoming the constant

$$f \equiv f(k)=\frac{1}{2\pi^2}-\frac{1}{\pi} \int_{-\infty}^\infty dy h(y). \quad (7)$$

In turn, Eq. (6) is reducible to a special case of the inhomogeneous Riemann boundary value problem, which is to

find functions $\phi^+(z)$ and $\phi^-(z)$ that are analytic, respectively, in the upper (D^+) and lower (D^-) half complex plane with limiting values on the real k axis, $\phi^\pm(k)$, that satisfy

$$\phi^+(k)=G(k)\phi^-(k)+t(k). \quad (8)$$

The coefficient of the Riemann problem is

$$G(k)=\frac{a(k)-b(k)}{a(k)+b(k)}, \quad (9)$$

which is simply related to the S matrix by $G(k)=S_0^*(k)$. The free term $t(k)$ of the Riemann problem is proportional to the scattering amplitude:

$$t(k)=\frac{f(k)}{a(k)+b(k)}. \quad (10)$$

Finally, the form factor is given upon application of the well-known Plemelj formulas,¹¹ also used to establish the connection between the Riemann problem and our integral equation:

$$h(k)=\phi^+(k)-\phi^-(k), \quad (11)$$

$$\frac{1}{\pi i} \int_{-\infty}^\infty dy \frac{h(y)}{y-k}=\phi^+(k)+\phi^-(k).$$

As with local interactions the S matrix links functions analytic in D^+ and D^- .⁹

If G and t are Hölder continuous,¹¹ then the existence and uniqueness of solutions are determined in a straightforward way, and depend upon the index of G ,

$$\chi=\text{ind}G=\Delta[\arg(G)]/2\pi,$$

where the total change of the argument of $G(k)$ over the real line is denoted by $\Delta[\arg(G)]$. The number χ is easily calculated in terms of phase shifts, becoming

$$\chi=2[\delta(0)-\delta(\infty)]/\pi.$$

For nucleons, it is realistic to assume there is no zero energy resonance and $|\delta(0)-\delta(\infty)| \leq \pi$, and we shall make these assumptions here. Then χ is twice the number of bound states.^{9,10} If $\chi=0$, the requirement that $h(k) \rightarrow 0$ as $k \rightarrow \infty$ makes the solution unique. If $\chi=2$, our method will be shown to give the unique form factor consistent with the binding energy.

The general algorithm for solving the Riemann problem is somewhat complicated, especially in the case $\chi \neq 0$, requiring the introduction of ancillary functions.¹¹ When $F(k)$ is rational, the algorithms become simple. As the basic application of our formulation, we specialize in what follows to a rational $F(k)$. Then

$$G(k)=\frac{c(k)}{d(k)}=\frac{c_+(k)c_-(k)}{d_+(k)d_-(k)}, \quad (12)$$

where polynomials c_+, d_+ (c_-, d_-) have roots in the upper (lower) half plane. Then $\chi=m_+-n_+$, where m_+ (n_+) is the number of zeros of c_+ (d_+), and the number of bound states is half the difference between the number of poles and zeros of the S matrix in the upper half plane. When poles and zeros are counted here, it is their multiplicities that are added.

The decomposition of G in Eq. (12) permits the use of

the generalized Liouville theorem.¹¹ One writes the Riemann equation as

$$\frac{d_-(k)}{c_-(k)}\phi^+(k) - \frac{c_+(k)}{d_+(k)}\phi^-(k) = \frac{d_-(k)}{c_-(k)}g(k). \quad (13)$$

The terms on the left are boundary values of functions analytic in D^+ and D^- . It is convenient to specify the analytic structure of Eq. (13) by requiring $F(k)$ to have the form $P_L(k^2)/Q_M(k^2)$, just $[L/M]$ in Padé approximant notation,⁴ where $P_L(k^2)$ and $Q_M(k^2)$ are polynomials in k^2 of degrees L and M . We require $M < L$ to ensure the vanishing of $\delta(k)$ at infinity. Any such $F(k)$ leads uniquely to a rational S matrix with the structure discussed following Eq. (16), and conversely. Then the right side of Eq. (13) is the rational function

$$fQ_M(k^2)/d_+(k)c_-(k),$$

expressible as $\psi^+(k) - \psi^-(k)$, where

$$\psi^+(k) = x(k)/c_-(k),$$

$$\text{degree}[x(k)] \leq \text{degree}[c_-(k)],$$

and

$$\psi^-(k) = -[\psi^+(k)]^*.$$

Equation (13) expresses the equality on the real axis of functions analytic in D^+ and D^- , which must therefore together represent an entire function $R(z)$. On the real axis,

$$\begin{aligned} R(k) &= \frac{d_-(k)}{c_-(k)}\phi^+(k) - \psi^+(k) \\ &= \frac{c_+(k)}{d_+(k)}\phi^-(k) - \psi^-(k). \end{aligned} \quad (14)$$

We require $\phi^\pm(\infty) = 0$ to restrict to solutions $h(k)$ that also vanish at infinity. Consequently, by the generalized Liouville theorem, $R(z)$ can be an arbitrary polynomial of degree $\chi - 1$, written $R_{\chi-1}(z)$, but with $R_{-1}(z) = 0$. The $\phi^\pm(k)$ are found by taking the real limit of z , $z \rightarrow k^\pm$, of those functions analytic in D^+ and D^- .

For a detailed discussion of the solution both for $\chi = 0$ and $\chi = 2$, we make use of the function

$$h(k) = \{[x(k)c_+(k) + x(k)^*d_-(k)] + [c_-(k)c_+(k) - d_-(k)d_+(k)]R_{\chi-1}(k)\} / d_-(k)c_+(k). \quad (17)$$

When $\chi = 0$, then $R_{-1}(k) = 0$, and $h(k)$ is an even function. When $\chi = 2$, then $h(k)$ is real and even only if $R_1(k)$ is imaginary and odd. Hence, $R_1(k) = ikC_1$, where C_1 is real. The bound state condition, which we establish now, is used to determine C_1 . A negative energy bound state pole would occur at $z = ik_B$, with $k_B > 0$. It follows directly from the Schrödinger equation for the separable interaction that $A(ik_B) = 0$. Since D^+ represents the domain of continuation of the function $A_-(k)$, then ik_B is a pole of the analytically-continued S matrix. Clearly, ik_B is a simple zero of $A(z)$ because $dA(z)/dz \neq 0$ if $z \neq 0$. The numerator of $S(k)$ at ik_B becomes $-4\pi^2 h(ik_B)k_B$, and therefore if N is the order of the pole of $S(ik_B)$, then $N - 1$ is the order of the pole of $h(ik_B)$. From Eqs. (12)

$$A(z) = 1 + 2\pi \int_{-\infty}^{\infty} dy \frac{h(y)y^2}{z^2 - y^2}, \quad (15)$$

which is analytic^{7,11} in D^+ and D^- for the rational solutions $h(v)$ we obtain. $A(z)$ has the limits $A_-(k)$ [or $A_+(k)$] on the real axis approached from D^+ (or D^-), given by

$$A_\pm(k) = 1 + 2\pi \int_{-\infty}^{\infty} dy \frac{h(y)y^2}{k^2 - y^2} \pm 2\pi^2 ikh(k). \quad (16)$$

The S matrix

$$S(k) = \exp[2i\delta(k)]$$

is written from Eq. (5) as

$$S(k) = A_+(k)/A_-(k).$$

Since $A_+(k) = A_-(-k)$, it follows that $S(-k) = 1/S(k)$, while from unitarity, $S(k)^* = 1/S(k)$. It follows by substitutions from Eq. (12) that we may require

$$c_+(k) = d_-(k)^* = d_-(-k)$$

and

$$c_-(k) = d_+(k)^* = d_+(-k).$$

Furthermore, the zeros must be distributed in each of the functions $c_\pm(k)$ and $d_\pm(k)$ symmetrically about the imaginary z axis.

We are now able to examine the symmetry and analytic structure of the solution $h(k)$ as a consequence of properties of the S matrix, including well-defined behavior of $S(k)$ at the poles and real zeros of the analytically continued rational solution $h(z)$. If $h(z)$ has a zero for real $z = \pm k_c$, then $A_+(k_c) = A_-(k_c)$ and $\delta(k_c) = 0$. The existence of a zero of $\delta(k)$, analyzed in detail in Ref. 7, can be associated with a strong short-range repulsion, and will not be dealt with here. Then $h(k)$ has no zeros and no sign changes for real finite k , and if attractive, satisfies $h(k) > 0$. A solution with $h(k) < 0$ for finite k would correspond to a repulsive interaction, as seen by starting with a different sign in Eq. (1).

The explicit solution is given from Eqs. (14) and (11):

and (17) it is evident that $h(k)$ has the same poles as those of $S(k)$ in D^+ , and that these poles have the same order unless the remaining constant C_1 in $R_1(z)$ is assigned the value that creates a common factor $k^2 + k_B^2$ in the numerator. This bound state condition in practice is simple to apply.

Our first example is for a shape-independent $F(k)$ consisting of the first two terms in the effective range expansion. The choice of scattering length $a > 0$ corresponds to a low energy approximation of the triplet np interaction with one bound state. Then

$$c_+ = (r_0/2)[k - i(\alpha + \beta)][k - i(\alpha - \beta)]$$

and $c_- = 1$, where $\alpha = 1/r_0$ and

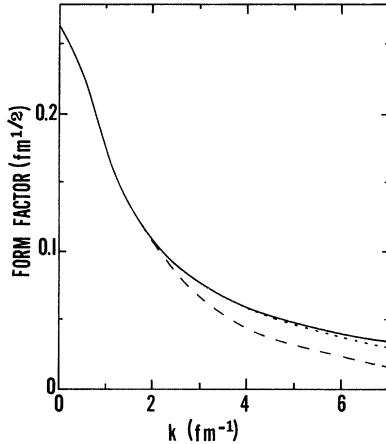


FIG. 1. The form factor $g(k)$ for the shape-independent approximation using 3S_1 np parameters. The exact curve (solid line) and Padé approximants shown: [1/2] (dashed line) and [4/5] (dotted line).

$$\beta = [1 - 2r_0/a]^{1/2}/r_0.$$

If we assume $k_B = \alpha - \beta$, then

$$h(k) = \frac{\alpha/\pi^2}{[k^2 + (\alpha + \beta)^2]}. \quad (18)$$

Although the form factor $g(k) = \sqrt{h(k)}$ is hardly rational, Padé approximants of $g(k)$ converge rapidly, as seen in Fig. 1, where the values¹⁴ $a = 5.414$ fm and $r_0 = 1.750$ fm are used. Here, Padé approximants express $g(k)$ as a combination of functions of the type $(k^2 + \gamma_j^2)^{-1}$.

The shape-independent approximation illustrates two other cases. When $a < 0$, as characteristic of the 1S_0 np state, we find $\beta > \alpha$, and the poles shift so that $\chi = 0$. The analysis, simpler because the bound state pole condition is not needed, results in the same expression for $h(k)$ given in Eq. (18). Finally, when $\beta = 0$, $S(k)$ has a double pole at ik_B . The bound state pole condition, easily applied, lowers the order of the pole of $h(k)$ at ik_B to unity. The solution is still given by Eq. (18) with $\beta = 0$. These simple results may be compared with those given in Ref. 10.

As a final example we consider a Yukawa potential that fits the 1S_0 np effective range parameters.¹⁴ Figure 2

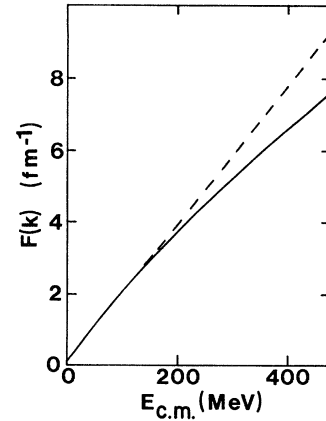


FIG. 2. The scattering function $F(k) = k \cot(\delta_0)$ for the attractive Yukawa with 1S_0 np parameters (solid line) and its [3/2] approximant (dashed line). The exact curve and [4/3] approximant coincide in the energy range shown.

shows our rational least squares fits of $F(k)$ that we computed for this potential. The [4/3] approximant curve is indistinguishable from the data in the energy range shown, while the [3/2] approximant gives a reasonable fit for $E_{c.m.} < 120$ MeV. The [3/2] approximant to $F(k)$ is exactly reproduced by

$$h(k) = \frac{\alpha_1}{k^2 + k_1^2} + \frac{\alpha_2 + \alpha_3 k}{(k + k_{21})^2 + k_{22}^2} + \frac{\alpha_2 - \alpha_3 k}{(k - k_{21})^2 + k_{22}^2}, \quad (19)$$

with $\alpha_1 = 0.0201873$ fm⁻¹, $\alpha_2 = 0.000811496$ fm⁻¹, $\alpha_3 = -0.0175540$, $k_1 = 0.0667608$ fm⁻¹, $k_{21} = 0.585295$ fm⁻¹, $k_{22} = 1.20083$ fm⁻¹. Here, rational approximants to $g(k)$ would be most easily obtained by least squares methods.⁶

We are indebted to Jill C. Bonner for useful comments, and we also wish to thank R. G. Newton for suggestions which have led to substantial improvements in the manuscript.

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