PHYSICAL REVIEW C VOLUME 29, NUMBER 2 FEBRUARY 1984

Constant-volume constraint and many-body forces in the Nilsson model

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(Received 18 August 1983)

It is shown that the Nilsson prescription for calculating equilibrium deformations of nuclei under the constraint of constant volume may be interpreted in terms of a Hartree mean-field approximation applied to nucleons interacting via many-body forces. Such an interaction is obtained in closed form, having a Taylor expansion beginning with the familiar quadrupole-quadrupole interaction, and then successively adding three-body, four-body, etc., terms. Since the interaction is not uniquely determined, two interesting forms are discussed. One, having both monopole and quadrupole parts, has a Hartree field identical to the Nilsson potential. The other, which is purely quadrupole, permits a generalization to all orders of a result obtained by Moszkowski for small deformations. The scalar many-body interaction naturally explains the nonscalar effective two-body interactions used in random-phase approximation calculations of deformed nuclei.

I. INTRODUCTION

A very important aspect of the Nilsson model of the atomic nucleus is the determination of equilibrium shapes. In the most pristine version of the model, the equilibrium deformation parameters were obtained by minimizing the sum of the energies of individual nucleons moving independently in a phenomenological deformed potential, subject to the constraint that the volumes enclosed by equipotential surfaces remain constant as the deformation varies.[1,2] This volume-conservation (VC) constraint is intended to simulate the incompressibility of nuclear matter and provides a restoring force. The more modern version of the model, usually called the Nilsson-Strutinskii model,[3] incorporates pairing and Coulomb effects and Strutinskii averaging,[4] but such refinements are peripheral to the discussion at hand. The success of this intuitive prescription over the years has been "disconcertingly spectacular,"[5] especially since it appears at first sight to be rather different from conventional mean-field approximations, such as the Hartree-Fock approximation. In spite of efforts by Moszkowski,[6] and later Bassachis,[7] to relate the Nilsson model to the Hartree approximation for small deformations, this model in its full generality poses something of a mystery. The aim of this paper is to at least partially dispel some of the mystery.

It is shown that the energy in the Nilsson model corresponding to an equilibrium deformation may be interpreted as the expectation value in the Hartree sense of a Hamiltonian involving a many-body effective interaction. Although the interaction is not unique, two interesting closed forms are discussed, both of which have Taylor expansions beginning with the familiar two-body quadrupole-quadrupole interaction, with successive terms bringing in three-body, four-body, etc., components. One interaction has the property that its Hartree field precisely replicates the Nilsson Hamiltonian from which one started. This interaction has the minor esthetic drawback of including somewhat unorthodox monopole-quadrupole coupling terms. Kishimoto has independently derived the first two terms in the Taylor series of a similar interaction.[8] The other interaction contains pure quadrupole terms, but its Hartree field is not exactly identical to the Nilsson Hamiltonian. However, it is amenable to a generalization of Moszkowski's interpretation[6] of the relation of the Nilsson model for small deformations to the Hartree field of the usual two-body quadrupole-quadrupole interaction.

It shall also be shown that these rotationally invariant many-body interactions generate in a natural way the nonrotationally invariant effective two-body interactions which have been used in random-phase approximation (RPA) calculations of deformed nuclei. In addition, the closed form allows one to generate the many-body corrections required to maintain the Goldstone mode at zero energy when doing calculations beyond the RPA.

II. EQUILIBRIUM CONDITIONS IN THE NILSSON MODEL

The Nilsson Hamiltonian H_N, excluding for simplicity the hexadecapole term[9] and Coulomb effects, may be written in the form

$$H_N = H_0 + U_D , \tag{1}$$

where H_0 is the spherically symmetric Hamiltonian for A nucleons,

$$H_0 = \sum_{i=1}^{A} \left[\frac{p_i^2}{2m} + \frac{m\mathring{\omega}_0^2}{2} r_i^2 + C\,\vec{l}_i\cdot\vec{s}_i + D(l_i^2 - \langle l^2\rangle_N) \right] + H_{\text{pair}} . \tag{2}$$

The pairing Hamiltonian H_{pair}, the $\vec{l}\cdot\vec{s}$, and the l^2 terms do not play a role in the ensuing arguments, but are included for generality to emphasize the validity of the results even in the presence of these terms.[10] The term U_D

is that part of the oscillator potential $U_{\rm osc}$ involving deformation effects:

$$U_D = U_{\rm osc} - \frac{m\mathring{\omega}_0^2}{2} R^2 = \frac{m}{2}\sum_{i=1}^{A}(\omega_x^2 x_i^2 + \omega_y^2 y_i^2 + \omega_z^2 z_i^2 - \mathring{\omega}_0^2 r_i^2) , \tag{3}$$

with the notation

$$R^2 \equiv \sum_{i=1}^{A} r_i^2 . \tag{4}$$

Let $\langle\ \rangle$ denote the expectation value with respect to either an exact ground state of (1) or else a variational approximation, as, for example, a Bardeen-Cooper-Schrieffer state to take care of $H_{\rm pair}$.[3] In either case, the Hellmann-Feynman theorem[11] is valid, so that the minimization of $\langle H_N\rangle$ with respect to the oscillator frequencies ω_x, ω_y, and ω_z, subject to the constant-volume constraint

$$\omega_x\omega_y\omega_z = \mathring{\omega}_0^3 , \tag{5}$$

may be written in the form

$$\left\langle \frac{\partial H_N}{\partial \omega_k}\right\rangle - \mu\frac{\partial}{\partial\omega_k}(\omega_x\omega_y\omega_z) = 0,\quad k = x,y,z , \tag{6}$$

where μ is a Lagrange multiplier whose subsequent elimination yields the following equations for an equilibrium deformation:

$$\omega_x^2\left\langle \sum_{i=1}^{A} x_i^2\right\rangle = \omega_y^2\left\langle \sum_{i=1}^{A} y_i^2\right\rangle = \omega_z^2\left\langle \sum_{i=1}^{A} z_i^2\right\rangle . \tag{7}$$

This is just the condition that the shape of the oscillator potential and that of the density distribution coincide, the justification for which is the short range of nuclear forces.[12] Equation (7) is equivalent to the following:

$$\begin{aligned}
\left\langle \sum_{i=1}^{A} x_i^2\right\rangle &= \frac{\mathring{\omega}_0^2}{\omega_x^2}\frac{\langle R^2\rangle_0}{3} ,\\
\left\langle \sum_{i=1}^{A} y_i^2\right\rangle &= \frac{\mathring{\omega}_0^2}{\omega_y^2}\frac{\langle R^2\rangle_0}{3} ,\\
\left\langle \sum_{i=1}^{A} z_i^2\right\rangle &= \frac{\mathring{\omega}_0^2}{\omega_z^2}\frac{\langle R^2\rangle_0}{3} ,\\
\langle R^2\rangle_0 &= 3\left[\left\langle \sum_{i=1}^{A} x_i^2\right\rangle\left\langle \sum_{i=1}^{A} y_i^2\right\rangle\left\langle \sum_{i=1}^{A} z_i^2\right\rangle\right]^{1/3} .
\end{aligned} \tag{8}$$

From Eq. (8), the total energy corresponding to an equilibrium deformation may be written as

$$E = \langle H_N\rangle = \langle H_0\rangle + \langle U_D\rangle , \tag{9}$$

where

$$\langle U_D\rangle = \tfrac{3}{2}m\mathring{\omega}_0^2\left[\left\langle \sum_{i=1}^{A} x_i^2\right\rangle\left\langle \sum_{i=1}^{A} y_i^2\right\rangle\left\langle \sum_{i=1}^{A} z_i^2\right\rangle\right]^{1/3} - \tfrac{1}{2}m\mathring{\omega}_0^2\langle R^2\rangle . \tag{10}$$

III. DERIVATION OF EFFECTIVE INTERACTIONS

It shall now be shown that (10) can be written as the Hartree expectation value of a rotationally invariant many-body interaction. The first step is to express (10) in terms of the expectation values of the monopole operator (4) and the mass quadrupole operators

$$Q_{2\mu} = \sum_{i=1}^{A} r_i^2 Y_{2\mu}(\Omega_i) ,\quad \mu = 0, \pm 1, \pm 2 . \tag{11}$$

Since $U_{\rm osc}$ is aligned along its principal axes, the conditions $\langle Q_{2\pm1}\rangle = 0$ and $\langle Q_{22}\rangle = \langle Q_{2-2}\rangle$ are fulfilled. Then, $\langle Q_{20}\rangle$, $\langle Q_{22}\rangle$, and $\langle R^2\rangle$ may be expressed as linear combinations of $\langle \sum_i x_i^2\rangle$, $\langle \sum_i y_i^2\rangle$, and $\langle \sum_i z_i^2\rangle$. These relations may be inverted as follows:

$$\begin{aligned}
\left\langle \sum_{i=1}^{A} x_i^2\right\rangle &= \tfrac{1}{3}\left[\langle R^2\rangle - \left[\frac{4\pi}{5}\right]^{1/2}\langle Q_{20}\rangle + \left[\frac{24\pi}{5}\right]^{1/2}\langle Q_{22}\rangle\right] ,\\
\left\langle \sum_{i=1}^{A} y_i^2\right\rangle &= \tfrac{1}{3}\left[\langle R^2\rangle - \left[\frac{4\pi}{5}\right]^{1/2}\langle Q_{20}\rangle - \left[\frac{24\pi}{5}\right]^{1/2}\langle Q_{22}\rangle\right] ,\\
\left\langle \sum_{i=1}^{A} z_i^2\right\rangle &= \tfrac{1}{3}\left[\langle R^2\rangle + 2\left[\frac{4\pi}{5}\right]^{1/2}\langle Q_{20}\rangle\right] .
\end{aligned} \tag{12}$$

Upon substituting Eqs. (12) into (10) and expanding, one finds

$$\langle U_D\rangle = \tfrac{1}{2}m\mathring{\omega}_0^2\left[\langle R^2\rangle^3 - \frac{12\pi}{5}\langle R^2\rangle(\langle Q_{20}\rangle^2 + 2\langle Q_{22}\rangle^2) + 2\left[\frac{4\pi}{5}\right]^{3/2}(\langle Q_{20}\rangle^3 - 6\langle Q_{20}\rangle\langle Q_{22}\rangle^2)\right]^{1/3} - \tfrac{1}{2}m\mathring{\omega}_0^2\langle R^2\rangle . \tag{13}$$

The aim is to write (13) as the *Hartree expectation value* of a rotational scalar, involving direct factorization. This means that the expectation value of a product of one-body operators is approximated by the corresponding product of expectation values of these operators. Hence, in any function of one-body operators, these operators are replaced by c numbers. Now, there are only two scalar invariants that can be formed from the quadrupole operators, namely,

$Q_2 \cdot Q_2 = \sqrt{5}(Q_2 Q_2)_0$ and $(Q_2 Q_2 Q_2)_0$, where $(\ )_0$ denotes angular-momentum coupling to spin zero. It is a straightforward exercise to show that the Hartree expectation values of these invariants are

$$\langle Q_2 \cdot Q_2 \rangle = \langle Q_{20} \rangle^2 + 2\langle Q_{22} \rangle^2 , \quad \langle (Q_2 Q_2 Q_2)_0 \rangle = -(\tfrac{2}{35})^{1/2}(\langle Q_{20} \rangle^3 - 6\langle Q_{20} \rangle \langle Q_{22} \rangle^2) . \tag{14}$$

Equation (13) can now be written as the Hartree expectation value of a rotational scalar interaction V,

$$\langle U_D \rangle = \langle V \rangle , \tag{15}$$

but V is not unique. In fact, there are an infinite number of possibilities depending on the treatment of the $\langle R^2 \rangle$ terms. Here, two limiting cases are considered, which are also the most interesting. In the first case, $\langle R^2 \rangle$ in (13) is directly replaced by R^2 to give

$$\begin{aligned} V &= \frac{m\mathring{\omega}_0^2}{2} \left[(R^2)^3 - \frac{12\pi}{5} R^2 (Q_2 \cdot Q_2) - \frac{8\pi}{5} (14\pi)^{1/2} (Q_2 Q_2 Q_2)_0 \right]^{1/3} - \frac{m\mathring{\omega}_0^2}{2} R^2 \\ &= \frac{m\mathring{\omega}_0^2}{2} \langle R^2 \rangle \left[1 + \frac{(R^2)^3 - \langle R^2 \rangle^3}{\langle R^2 \rangle^3} - \frac{12\pi}{5} R^2 \frac{(Q_2 \cdot Q_2)}{\langle R^2 \rangle^3} - \frac{8\pi}{5} (14\pi)^{1/2} \frac{(Q_2 Q_2 Q_2)_0}{\langle R^2 \rangle^3} \right]^{1/3} - \frac{m\mathring{\omega}_0^2}{2} R^2 . \end{aligned} \tag{16}$$

The interaction V may be expanded in powers of $\langle R^2 \rangle^{-1}$ as follows:

$$\begin{aligned} V = -\tfrac{1}{2}\chi \Bigg[& Q_2 \cdot Q_2 - \left[\frac{R^2 - \langle R^2 \rangle}{\langle R^2 \rangle} \right] (Q_2 \cdot Q_2) + \tfrac{2}{3}(14\pi)^{1/2} \frac{(Q_2 Q_2 Q_2)_0}{\langle R^2 \rangle} + \left[\frac{R^2 - \langle R^2 \rangle}{\langle R^2 \rangle} \right]^2 (Q_2 \cdot Q_2) \\ & - \tfrac{4}{3}(14\pi)^{1/2} \left[\frac{R^2 - \langle R^2 \rangle}{\langle R^2 \rangle} \right] \frac{(Q_2 Q_2 Q_2)_0}{\langle R^2 \rangle} + \frac{4\pi}{5} \frac{(Q_2 \cdot Q_2)^2}{\langle R^2 \rangle^2} + 0 \left[\frac{1}{\langle R^2 \rangle^3} \right] \Bigg] , \end{aligned} \tag{17}$$

where

$$\chi = \frac{4\pi}{5} \frac{m\mathring{\omega}_0^2}{\langle R^2 \rangle} . \tag{18}$$

The leading term in expansion (17) is just the familiar quadrupole-quadrupole interaction with the strength χ appropriate to taking all nucleons into account.[13] The expansion parameter for spherical nuclei is reasonably small, being of the order of the quadrupole zero-point amplitude β_2.

Another possible choice of V satisfying Eq. (15) may be obtained from (13) by keeping $\langle R^2 \rangle$ as a parameter but replacing the $\langle Q_{2\mu} \rangle$ by operators in accordance with Eq. (14). This choice of V, denoted by V', is

$$V' = \frac{m\mathring{\omega}_0^2}{2} \langle R^2 \rangle \left[1 - \frac{12\pi}{5} \frac{Q_2 \cdot Q_2}{\langle R^2 \rangle^2} - \frac{8\pi}{5} (14\pi)^{1/2} \frac{(Q_2 Q_2 Q_2)_0}{\langle R^2 \rangle^3} \right]^{1/3} - \frac{m\mathring{\omega}_0^2}{2} \langle R^2 \rangle . \tag{19}$$

Its expansion may be obtained from (17) by dropping all terms involving the radial fluctuations $(R^2 - \langle R^2 \rangle)/\langle R^2 \rangle$. Hence (19) is a special case of (16). However, (19) is free of the somewhat unusual monopole-quadrupole cross terms contained in (16). On the other hand, these pose no mathematical difficulties and give rise to important properties discussed below. Both interactions have the feature that the successive terms in the expansion bring in many-body forces of increasingly higher order. Thus, the leading-order quadrupole-quadrupole interaction contains two-body and one-body terms; the next order brings in three-body as well as two- and one-body interactions, etc.

IV. THE HARTREE APPROXIMATION

Turning first to the interaction V given by Eq. (16), it shall be shown that it gives rise to a self-consistent Hartree potential which is identical to the Nilsson oscillator potential. The first step is to express $U_{\rm osc}$ at equilibrium in terms of appropriate deformation parameters. From the inverse of Eq. (12) together with (8), one obtains

$$\begin{aligned} \langle Q_{20} \rangle &= \left[\frac{5}{16\pi} \right]^{1/2} \left[\frac{2}{\omega_z^2} - \frac{1}{\omega_x^2} - \frac{1}{\omega_y^2} \right] \mathring{\omega}_0^2 \frac{\langle R^2 \rangle_0}{3} , \\ \langle Q_{22} \rangle = \langle Q_{2-2} \rangle &= \left[\frac{15}{32\pi} \right]^{1/2} \left[\frac{1}{\omega_x^2} - \frac{1}{\omega_y^2} \right] \mathring{\omega}_0^2 \frac{\langle R^2 \rangle_0}{3} , \\ \langle R^2 \rangle &= \left[\frac{1}{\omega_x^2} + \frac{1}{\omega_y^2} + \frac{1}{\omega_z^2} \right] \mathring{\omega}_0^2 \frac{\langle R^2 \rangle_0}{3} . \end{aligned} \tag{20}$$

It is convenient to choose as deformation parameters the ratios σ_0 and σ_2 defined by

$$\begin{aligned} \sigma_0 &= \left[\frac{4\pi}{5} \right]^{1/2} \langle Q_{20} \rangle / \langle R^2 \rangle , \\ \sigma_2 &= \left[\frac{4\pi}{5} \right]^{1/2} \langle Q_{22} \rangle / \langle R^2 \rangle . \end{aligned} \tag{21}$$

In the case of axial symmetry, when $\sigma_2 = 0$, the usual Nilsson parameter δ is related to σ_0 by

$$\delta=\frac{3\sigma_0}{2(1+\sigma_0)} . \tag{22}$$

Then, from (20), (21), and the VC condition (5), one may solve for the oscillator frequencies as functions of σ_0 and σ_2 as follows:

$$\begin{aligned}\omega_x^2&=(1+2\sigma_0)(1-\sigma_0-\sqrt{6}\sigma_2)\omega_0^2(\sigma_0,\sigma_2) ,\\ \omega_y^2&=(1+2\sigma_0)(1-\sigma_0+\sqrt{6}\sigma_2)\omega_0^2(\sigma_0,\sigma_2) ,\\ \omega_z^2&=(1-\sigma_0+\sqrt{6}\sigma_2)(1-\sigma_0-\sqrt{6}\sigma_2)\omega_0^2(\sigma_0,\sigma_2) ,\end{aligned} \tag{23}$$

where

$$\omega_0^2(\sigma_0,\sigma_2)=[(1+2\sigma_0)(1-\sigma_0+\sqrt{6}\sigma_2)(1-\sigma_0-\sqrt{6}\sigma_2)]^{-2/3}\mathring{\omega}_0^2 . \tag{24}$$

In terms of the parametrization (23), $U_{\rm osc}$ may be written in the form

$$\begin{aligned}U_{\rm osc}&=\tfrac{1}{2}m\mathring{\omega}_0^2R^2+U_D\\ &=\tfrac{1}{2}m\omega_0^2(\sigma_0,\sigma_2)\left[(1-\sigma_0^2-2\sigma_2^2)R^2-(\sigma_0-\sigma_0^2+2\sigma_2^2)\left[\frac{16\pi}{5}\right]^{1/2}Q_{20}-\sigma_2(1+2\sigma_0)\left[\frac{16\pi}{5}\right]^{1/2}(Q_{22}+Q_{2-2})\right] .\end{aligned} \tag{25}$$

If, on the other hand, one starts with the nuclear Hamiltonian H,

$$H=H_0+V , \tag{26}$$

the Hartree potential U_H arising from V may be defined by

$$U_H=\left\langle\frac{\partial V}{\partial R^2}\right\rangle R^2+\sum_\mu\left\langle\frac{\partial V}{\partial Q_{2\mu}}\right\rangle Q_{2\mu} , \tag{27}$$

where $\langle\ \rangle$ denotes the Hartree expectation value with respect to the self-consistent ground state. It is a straightforward exercise to show that

$$\begin{aligned}\left\langle\frac{\partial V}{\partial R^2}\right\rangle&=\tfrac{1}{2}m\omega_0^2(\sigma_0,\sigma_2)(1-\sigma_0^2-2\sigma_2^2)-\frac{m\mathring{\omega}_0^2}{2} ,\\ \left\langle\frac{\partial V}{\partial Q_{20}}\right\rangle&=-\tfrac{1}{2}m\omega_0^2(\sigma_0,\sigma_2)(\sigma_0-\sigma_0^2+2\sigma_2^2)\left[\frac{16\pi}{5}\right]^{1/2} ,\\ \left\langle\frac{\partial V}{\partial Q_{22}}\right\rangle&=\left\langle\frac{\partial V}{\partial Q_{2-2}}\right\rangle\\ &=-\tfrac{1}{2}m\omega_0^2(\sigma_0,\sigma_2)\sigma_2(1+2\sigma_0)\left[\frac{16\pi}{5}\right]^{1/2} ,\end{aligned} \tag{28}$$

with all other expectation values vanishing, and with Eq. (21) playing the role of the Hartree self-consistency conditions. From Eqs. (25), (27), and (28) it is seen that U_D and U_H are identical in form. It still remains to be proven that the Hartree self-consistency conditions (21) are equivalent to the VC conditions (7). This result easily follows by combining Eqs. (12) with (21) to yield

$$\frac{\left\langle\sum_{i=1}^A x_i^2\right\rangle}{1-\sigma_0+\sqrt{6}\sigma_2}=\frac{\left\langle\sum_{i=1}^A y_i^2\right\rangle}{1-\sigma_0-\sqrt{6}\sigma_2}=\frac{\left\langle\sum_{i=1}^A z_i^2\right\rangle}{1+2\sigma_0} , \tag{29}$$

which is readily converted to (7) with the aid of Eqs. (23). It may therefore be concluded that

$$U_H=U_D . \tag{30}$$

The point has often been made that in the Nilsson model, the total energy (apart from the pairing energy) is the expectation value of the independent-particle Hamiltonian, in contrast to the Hartree (-Fock) method in which the energy contains a correction term subtracting $\frac{1}{2}$ of the expectation value of the two-body interaction to prevent double counting.[6,7] Since it has just been shown that the Nilsson model is equivalent to a Hartree approximation, is there a contradiction? The answer is negative for the simple reason that the Hartree potential is arbitrary up to an additive constant, and the choice (27) already tacitly includes the constant which compensates for overcounting many-body interactions. It is worthwhile to understand just how this occurs. From Eq. (17), it is seen, first of all, that since V contains no pure monopole terms and that since the monopole-quadrupole cross terms depend on powers of $R^2-\langle R^2\rangle$, they do not contribute to $\langle V\rangle$ in the Hartree approximation, which therefore contains contributions only from pure quadrupole interactions. Second, the Hartree potential obtained by direct factorization of (16) or (17) differs from (27) by the replacement $R^2\to R^2-\langle R^2\rangle$, and therefore this term would give a vanishing expectation value. Hence, in the actual choice of U_H given by (27), the constant $\langle\partial V/\partial R^2\rangle\langle R^2\rangle$ has been added on already. Since, as follows from (15) and (30), $\langle U_H\rangle=\langle V\rangle$, this constant compensates for overcounting of the pure quadrupole interactions in the quadrupole field of U_H. One must therefore have the identity

$$\langle V\rangle=\langle\partial V/\partial R^2\rangle\langle R^2\rangle+\sum_\mu\langle\partial V/\partial Q_{2\mu}\rangle\langle Q_{2\mu}\rangle , \tag{31}$$

which is proven immediately by applying Euler's theorem on homogeneous functions to V, followed by Hartree factorization.

Keeping in mind the discussion of the preceding paragraph, consider next the alternate interaction V' [Eq. (19)]. If the Hartree approximation is applied to the Hamiltonian

$$H'=H_0+V' , \tag{32}$$

the Hartree potential U_H' arising from V' may be taken as

$$U_H' = \sum_\mu \left\langle \frac{\partial V'}{\partial Q_{2\mu}} \right\rangle' Q_{2\mu} = \sum_\mu \left\langle \frac{\partial V}{\partial Q_{2\mu}} \right\rangle' Q_{2\mu} , \qquad (33)$$

where $\langle \ \rangle'$ denotes the Hartree average with respect to the self-consistent Hartree ground state, which, as shown below, is not identical to the Nilsson-model ground state. Equation (33) has exactly the same form as the quadrupole field of the Nilsson potential as a function of parameters σ_0 and σ_2 given by equations of the form (21) (with $\langle \ \rangle$ replaced by $\langle \ \rangle'$), which play the role of self-consistency conditions. However, the total spatial one-body potential, including the "external" spherical oscillator, is given by

$$\begin{aligned} U'_{\text{osc}} &= \tfrac{1}{2} m \mathring{\omega}_0^2 R^2 + U_H' \\ &= \tfrac{1}{2} m \mathring{\omega}_0^2 R^2 - \tfrac{1}{2} m \omega_0^2(\sigma_0,\sigma_2) \left[\frac{16\pi}{5} \right]^{1/2} [(\sigma_0 - \sigma_0^2 + 2\sigma_2^2) Q_{20} + \sigma_2(1+2\sigma_0)(Q_{22}+Q_{2-2})] . \end{aligned} \qquad (34)$$

Obviously, $U'_{\text{osc}} \neq U_{\text{osc}}$, and U'_{osc} does not conserve the volume beyond first order, so that the Hartree ground-state wave function for V' is not identical to the Nilsson one, and, consequently, the equilibrium values of σ_0 and σ_2 would differ also, although, one might hope, by not too much.

It is obvious from Eq. (33) that

$$\langle U_H' \rangle' = \sum_\mu \left\langle \frac{\partial V}{\partial Q_{2\mu}} \right\rangle' \langle Q_{2\mu} \rangle' \neq \langle V' \rangle' , \qquad (35)$$

but a constant can always be added to (33) to ensure the equality. To find this constant, one need only note that $\langle V' \rangle' = \langle V \rangle'$ in the sense of Hartree expectation values, and that Eq. (31) is valid with $\langle \ \rangle$ replaced by $\langle \ \rangle'$. Therefore, the constant is $\langle \partial V/\partial R^2 \rangle' \langle R^2 \rangle'$. Thus, the redefined potential

$$\begin{aligned} U_H'' &= \langle \partial V/\partial R^2 \rangle' \langle R^2 \rangle' + U_H' \\ &= \tfrac{1}{2} [m\omega_0^2(\sigma_0,\sigma_2)(1-\sigma_0^2-2\sigma_2^2) - m\mathring{\omega}_0^2] \langle R^2 \rangle' + U_H' , \end{aligned} \qquad (36)$$

satisfies $\langle U_H'' \rangle' = \langle V' \rangle' = \langle V \rangle'$.

If one were to choose V' instead of V, the VC minimization could be interpreted as an *approximation* to a Hartree calculation for V', with the term

$$\tfrac{1}{2} [m\omega_0^2(\sigma_0,\sigma_2)(1-\sigma_0^2-2\sigma_2^2) - m\mathring{\omega}_0^2] R^2 \qquad (37)$$

in the Nilsson potential serving the function of giving rise to the constant in Eq. (36) when the expectation value of the Hamiltonian is taken. Of course, the addition of the operator (37) rather than a constant has its side effects—changing the wave functions and the equilibrium deformations—but that is where the approximation lies. Essentially this kind of viewpoint was advanced many years ago by Moszkowski,[6] except that only the leading term of V', the quadrupole-quadrupole force, was used, and correspondingly, the Nilsson potential was expanded only to second order in the deformation. Thus, it is seen that with V', Moszkowski's interpretation can be generalized to all orders.

As mentioned previously, there is an infinite number of interactions intermediate between V and V', but these cannot be readily related to the Nilsson model via the Hartree approximation. Of the two, V seems more satisfying since it exactly regenerates the volume-conserving Nilsson potential which spawned it in the first place. Of course, whether V or V' is a better schematic interaction must ultimately be decided by comparing other theoretical properties with experiment.

V. EFFECTIVE INTERACTIONS IN DEFORMED NUCLEI

Although the conventional quadrupole-quadrupole interaction has often been employed in RPA calculations for deformed nuclei, the excitation energies are not very well reproduced unless the interaction is modified so that different components have unequal strengths. While this conclusion has been arrived at from purely empirical studies,[14] interactions with this property have also been derived by several techniques, starting with the vibrating potential model (VPM) (Ref. 15) and including adaptations of Landau's Fermi-liquid theory for finite systems[16,17] and sum-rule methods.[18] These interactions have played an important role in accounting for the splitting of the giant quadrupole resonance.[16,18] Since the interactions are not rotationally invariant (although they keep the rotational Goldstone mode at zero excitation energy in the RPA) the question arises as to how such interactions can arise from a rotationally invariant interaction. A related question is how the Goldstone mode can be kept at zero energy if one wishes to go beyond the RPA.

The rotationally invariant interaction V provides a natural and simple answer to the above questions. This interaction can be expanded in a Taylor series about the equilibrium values $\langle R^2 \rangle$ and $\langle Q_{2\mu} \rangle$; thus, through second order

$$V = V^{(0)} + V^{(1)} + V^{(2)} + \cdots , \qquad (38)$$

where

$$\begin{aligned} V^{(0)} &= \langle V \rangle , \\ V^{(1)} &= \left\langle \frac{\partial V}{\partial R^2} \right\rangle (R^2 - \langle R^2 \rangle) + \sum_\mu \left\langle \frac{\partial V}{\partial Q_{2\mu}} \right\rangle (Q_{2\mu} - \langle Q_{2\mu} \rangle) , \end{aligned} \qquad (39)$$

and

$$V^{(2)}=\frac{1}{2}\left\langle\frac{\partial^2 V}{\partial(R^2)^2}\right\rangle(R^2-\langle R^2\rangle)^2+\sum_\mu\left\langle\frac{\partial^2 V}{\partial R^2\partial Q_{2\mu}}\right\rangle(R^2-\langle R^2\rangle)(Q_{2\mu}-\langle Q_{2\mu}\rangle)$$
$$+\tfrac{1}{2}\sum_{\mu\nu}\left\langle\frac{\partial^2 V}{\partial Q_{2\mu}\partial Q_{2\nu}}\right\rangle(Q_{2\mu}-\langle Q_{2\mu}\rangle)(Q_{2\nu}-\langle Q_{2\nu}\rangle)\ . \tag{40}$$

From Eqs. (27) and (31), it follows that $V^{(0)}+V^{(1)}=U_H$, so that

$$V=U_H+V^{(2)}+\cdots\ , \tag{41}$$

giving a series in which the first term is the Hartree potential arising from V, and $V^{(2)}$ is an effective, essentially two-body residual interaction. Since V is a homogeneous function of degree one, its first partial derivatives are homogeneous of degree zero. If one applies Euler's theorem to these first derivatives, followed by Hartree factorization, the result is

$$\begin{aligned}&\left\langle\frac{\partial^2 V}{\partial(R^2)^2}\right\rangle\langle R^2\rangle+\sum_\mu\left\langle\frac{\partial^2 V}{\partial Q_{2\mu}\partial R^2}\right\rangle\langle Q_{2\mu}\rangle=0\ ,\\&\left\langle\frac{\partial^2 V}{\partial R^2\partial Q_{2\mu}}\right\rangle\langle R^2\rangle+\sum_\nu\left\langle\frac{\partial^2 V}{\partial Q_{2\mu}\partial Q_{2\nu}}\right\rangle\langle Q_{2\nu}\rangle=0\ .\end{aligned} \tag{42}$$

With the aid of these identities, $V^{(2)}$ may be simplified to

$$V^{(2)}=\tfrac{1}{2}\sum_{\mu\nu}\left\langle\frac{\partial^2 V}{\partial Q_{2\mu}\partial Q_{2\nu}}\right\rangle\left[Q_{2\mu}-\frac{\langle Q_{2\mu}\rangle}{\langle R^2\rangle}R^2\right]\times\left[Q_{2\nu}-\frac{\langle Q_{2\nu}\rangle}{\langle R^2\rangle}R^2\right]\ . \tag{43}$$

The evaluation of (43) is straightforward. With the assumption that the equilibrium shape is axially symmetric, so that

$$\langle Q_{2\mu}\rangle=\delta_{\mu,0}\left[\frac{5}{4\pi}\right]^{1/2}\langle R^2\rangle\sigma_0\ ,$$

(43) takes the form

$$V^{(2)}=-\tfrac{1}{2}\sum_{\mu=-2}^{2}\chi_{2\mu}\left[Q_{2\mu}^\dagger-\delta_{\mu,0}\left[\frac{5}{4\pi}\right]^{1/2}\sigma_0R^2\right]\times\left[Q_{2\mu}-\delta_{\mu,0}\left[\frac{5}{4\pi}\right]^{1/2}\sigma_0R^2\right]\ , \tag{44a}$$

where the coupling constants $\chi_{2\mu}$ are given by

$$\begin{aligned}\chi_{20}&=\frac{4\pi}{5}\frac{m\omega_0^2(\sigma_0,0)}{\langle R^2\rangle}\frac{1}{1+2\sigma_0}\ ,\\ \chi_{21}=\chi_{2-1}&=\frac{4\pi}{5}\frac{m\omega_0^2(\sigma_0,0)}{\langle R^2\rangle}(1-\sigma_0)\ ,\\ \chi_{22}=\chi_{2-2}&=\frac{4\pi}{5}\frac{m\omega_0^2(\sigma_0,0)}{\langle R^2\rangle}(1+2\sigma_0)\ .\end{aligned} \tag{45a}$$

In order to facilitate comparison with previous work employing different definitions of the deformation parameter, it is convenient to express (44) and (45) directly in terms of the volume-conserving oscillator frequencies $\omega_x=\omega_y\equiv\omega_\perp$ ($\sigma_2=0$) and ω_z, and to use in place of $\langle R^2\rangle$ the deformation-independent radius $\langle R^2\rangle_0$ [Eq. (8)], related to $\langle R^2\rangle$ by the last of Eqs. (20). Equation (44a) can then be written as

$$V^{(2)}=-\tfrac{1}{2}\chi_{20}\left[Q_{20}-\left[\frac{5}{4\pi}\right]^{1/2}\left[\frac{\omega_\perp^2-\omega_z^2}{\omega_\perp^2+2\omega_z^2}\right]R^2\right]^2-\tfrac{1}{2}\sum_{\mu\neq0}\chi_{2\mu}Q_{2\mu}^\dagger Q_{2\mu}\ , \tag{44b}$$

where

$$\begin{aligned}&\chi_{20}=\chi_0\left[\frac{\omega_\perp^2+2\omega_z^2}{3\mathring{\omega}_0^2}\right]^2,\quad \chi_0=\frac{4\pi}{5}\frac{m\mathring{\omega}_0^2}{\langle R^2\rangle_0}\ ,\\&\chi_{21}=\chi_{2-1}=\chi_0\left[\frac{\omega_\perp\omega_z}{\mathring{\omega}_0^2}\right]^2,\quad \chi_{22}=\chi_{2-2}=\chi_0\left[\frac{\omega_\perp}{\mathring{\omega}_0}\right]^4.\end{aligned} \tag{45b}$$

Equations (44) and (45) completely agree with those obtained from the VPM (Ref. 15) and more recently by Suzuki and Rowe[18] and Kurasawa.[17]

If in place of the interaction V, one uses V', then a similar treatment gives for axially symmetric deformed nuclei the effective interaction

$$V'^{(2)}=-\tfrac{1}{2}\sum_{\mu=-2}^{2}\chi_{2\mu}Q_{2\mu}^\dagger Q_{2\mu}\ , \tag{46}$$

in which the coupling constants are still given by Eqs. (45a). Absent is the monopole term in the $\mu=0$ channel, which affects only $K=0$ excitations (β vibrations). An interaction of the form (46) was derived by Bochnacki and co-workers using a variant of the Landau theory.[19] Although their value of χ_{22} agrees with the above, that of χ_{20} is, for unclear reasons, a little different, while χ_{21} was never explicitly calculated.

One can now see clearly how the rotationally noninvariant interactions (44) or (46) can arise from the rotationally invariant interactions V or V', respectively, as a consequence of an expansion about a deformed Hartree solution. The value of χ_{21} guarantees that in the RPA the Goldstone mode, which is associated with the $K=1+$ branch of excitations, stays at zero energy.[18] However, if one were to attempt to include anharmonic corrections to the RPA using $V^{(2)}$ within the framework of, say, perturbative boson expansions[20] or nuclear field theory,[21] this would no longer be true. The solution to this problem is simple: One must continue expansion (41) to include the three-body, etc., terms so as to always keep the Goldstone mode at zero energy. This follows from the fact that V or V' are scalars even if the successive terms in the Taylor expansion about a deformed state are not, and also from the fact that the effective expansion parameter is of the order of the expansion parameter in the perturbative boson or nuclear field-theory expansion.

One may hope to test the reality of the monopole terms in V by comparing the effects of $V^{(2)}$ and of $V'^{(2)}$ in deformed nuclei. It has been shown that in the RPA for the pure oscillator model, interaction (44) accounts well for the splitting of the giant quadrupole resonance in deformed nuclei.[16,18] To first order in the deformation parameter σ_0, one obtains the excitation energies

$$\hbar\omega \cong \sqrt{2}\hbar\mathring{\omega}_0(1-\tfrac{1}{2}\sigma_0) ,$$

$$\hbar\omega \cong \sqrt{2}\hbar\mathring{\omega}_0(1-\tfrac{1}{4}\sigma_0) ,$$

and

$$\hbar\omega \cong \sqrt{2}\hbar\mathring{\omega}_0(1+\tfrac{1}{2}\sigma_0)$$

for the $K=0$, 1, and 2 components, respectively [note from Eq. (22) that $\sigma_0 \approx 2\delta/3$]. However, interaction (46) gives exactly the same expressions to first order if self-consistent wave functions are used. Therefore, the splitting of the giant resonance is not a sufficient criterion for distinguishing between the two interactions.

Another property which has been checked is the Inglis cranking-model moment of inertia. Bohr and Mottelson proved long ago that in the pure harmonic-oscillator model with the VC condition, equivalent to the use of V, this moment of inertia is identical to the rigid-body moment of inertia.[12] On the other hand, the use of V', together with Hartree self-consistency, gives a cranking moment of inertia which is not identical to the rigid-body value. However, the moment of inertia one obtains,

$$\mathscr{I} \cong \frac{2m}{3}(1+\tfrac{1}{2}\sigma_0)\langle R^2\rangle ,$$

does agree with the rigid-body value through first order in σ_0. Since there is no general theorem requiring exactly the rigid-body value of $\mathscr{I}$ for finite nuclei, this property does not clearly distinguish between V and V'.

Finally, it should be mentioned that the earliest microscopic calculations of low-lying β vibrations in deformed nuclei within the VPM (Ref. 15) did include the monopole term of $V^{(2)}$, but unfortunately, no attempt was made to separate out the Goldstone mode associated with particle number, arising from pairing effects, a problem not appreciated at the time. In addition, because of computer limitations, only the asymptotic approximations to the full Nilsson wave functions were used. Hence, the results were not definitive. The possibility of redoing the calculations properly, using both $V^{(2)}$ and $V'^{(2)}$, which is rather trivial with present computing facilities, is being contemplated.

VI. SUMMARY AND CONCLUSIONS

Some light has been shed on the arcane but very successful prescription for calculating equilibrium deformations in the Nilsson model by relating it to a Hartree approximation applied to nucleons interacting via many-body forces. Unlike previous treatments, no restriction to small deformations was required. Two interactions V and V' were discussed, both having Taylor expansions beginning with the familiar quadrupole-quadrupole interaction.

On esthetic grounds, V seems preferable to V', since by starting with the Nilsson prescription with volume conservation, one is led to V (although not uniquely), which, in the Hartree approximation, exactly regenerates the Nilsson model, thus providing a nice consistency. Such a consistency holds only approximately for V', which allows for an approximate connection between the Nilsson model and the Hartree method by way of a generalization of an old interpretation given by Moszkowski. These interactions, particularly V, naturally account for the nonrotationally invariant effective two-body interactions used in RPA calculations for deformed nuclei. In principle, higher-order expansions of V for deformed nuclei, involving many-body forces, should allow one to maintain the rotational Goldstone mode at zero-energy. The exploration of such many-body forces in both spherical and deformed nuclei may be very important for anharmonic corrections to RPA calculations. Finally, it would be very interesting to derive the effective interactions corresponding to other more realistic single-particle potentials, such as the Woods-Saxon potential.

This work was supported in part by the National Science Foundation.

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