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Radiative capture estimates via analytic continuation of elastic-scattering data, and the solar-neutrino problem

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Measurements of σ_γ for $^3He(\alpha,\gamma)^7Be$, central to the solar ν problem, disagree. In a direct capture model, the normalization constants $N_{3/2}$ and $N_{1/2}$ of the $P_{3/2}$ and $P_{1/2}$ bound state wave functions of ^7Be at large ^3He-α separations determine σ_γ. $N_{3/2}$ and $N_{1/2}$ are given by (simpler) measurements of σ_γ at a higher energy E, *or*, as here, by analytic continuation of the ^3He-α $p_{3/2}$ and $p_{1/2}$ phase shifts, $\delta(E)$. The method has been successfully tested on calculations of Tang *et al.* Better measurements of $\delta(E)$ are called for.

NUCLEAR REACTIONS $^3He(\alpha,\gamma)^7Be$, $E<300$ keV, $^3He(\alpha,\alpha)^3He$, $E<4$ MeV, effective range function, analytic continuation technique, bound state energies, and normalization.

Current theoretical estimates[1] of the number of solar neutrino units (SNU) differ in the errors assigned; more significantly, they differ by about a factor of 3 from the experimental results of Davis.[2] The discrepancy, presumably in the theoretical analysis, may originate in something as profound as neutrino oscillations or may simply reflect errors in the input data. One of the important pieces of input data is the cross section $\sigma_{34}(E)$ for the $^3He(\alpha,\gamma)^7Be$ reaction, normally expressed in terms of the (slowly varying) cross section factor $S_{34}(E)=E\exp(2\pi\eta)\sigma_{34}(E)$, where $\eta\equiv Z_1Z_2\times e^2/\hbar v\equiv 1/ka$ is the Sommerfeld parameter, with v the relative velocity and E the kinetic energy in the center of mass. The SNU estimate varies roughly[1] as $[S_{34}(0)]^{0.8}$. Measurements by two groups give about the same E dependence for $\sigma_{34}(E)$ but disagree substantially in the overall normalization. In keV b, the latest Caltech group result for $S_{34}(0)$, which seems to have been confirmed, is 0.52 ± 0.03.[3] (The use of the Münster group results[4] of 0.30 ± 0.03 or 0.38 ± 0.03 would reduce the SNU estimate by the very significant factors of 1.6 or 1.3, respectively.)

Many estimates of $S_{34}(0)$ are based on a direct capture model (DCM) in its simplest version: Capture to a given bound state of ^7Be, with an energy $E_B=-\hbar^2\kappa_B^2/2\mu$, is assumed to occur at separations large compared to nuclear dimensions but under the Coulomb barrier.[5] $B=\frac{3}{2}$ and $\frac{1}{2}$ distinguish between the two lowest bound states ($J=\frac{3}{2}$, $L=1$ and $J=\frac{1}{2}$, $L=1$) of ^7Be. (Much of the discussion applies to any bound state of any nucleus.) In the dipole approximation, quite good at the low E involved, the incident channels of interest are therefore the s and d waves. A charged hard core model of radius $r_0=2.8$ fm gives a good approximation to the experimentally determined values of the s- and d-wave phase shifts,[6] and that should be true too for the phase shifts at lower E. At the large separations under consideration the form of the ^7Be bound state wave function Φ_B is known to within a normalization constant N_B while the continuum function is known exactly, and the matrix elements which define $\sigma_{34}(E)$ reduce to the one-body form

$$M_B(E)=N_B\int W_{\rho,\lambda}(2\kappa_B R)Rf(R;E)dR\ ,$$

where the ^3He-α separation R ranges from r_0 to ∞. $W_{\rho,\lambda}$ is the Whittaker function with $\rho=-\eta_B=-1/\kappa_B a$ and $\lambda=L+\frac{1}{2}$, with $L=1$. $f(R;E)$ is an $l=0$ or $l=2$ scattering wave function which describes the relative motion of the nuclei. The unknown constants $N_{3/2}$ and $N_{1/2}$ are obtained by matching the data at a higher E, one at which the contributions to $\sigma_{34}(E)$ due to capture into each of the two ^7Be bound states can be measured. (But at higher E, predictions of the simple DCM should be less accurate.)

The reliability of the above approach is greatly strengthened by the microscopic calculations of Tang and his co-workers.[7] With the two-body nuclear potential V_{nuc} chosen to reproduce many properties of ^3He, ^4He, and ^7Be, they perform a single-channel ($^3He+{}^4He$) resonating group calculation. The merit of the approach is that fully antisymmetrized seven-nucleon wave functions are used, that capture data at higher energies are not required, and that the inner region contribution is not ignored. Calculations by Kim, Izumoto, and Nagatani[8] approximate exchange effects and thereby simplify the analysis. Both Liu *et al.*[7] and Kim *et al.*[8] find the inner region contribution to be small.

Merits of the simple DCM include its simplicity, its applicability at least in principle to systems of any number of particles, and its model independence with regard to V_{nuc} and the forms of ^7Be bound state wave functions and of ^3He-α scattering wave functions at small R. (The microscopic approach must not only specify V_{nuc} but for technical reasons must choose V_{nuc}—and the ^3He and α normalized ground state wave functions $^3\phi$ and $^4\phi$—to be Gaussians, and must assume neutron and proton masses to be equal, but these need not normally be serious drawbacks.) On the other hand, the simple DCM does not include the (presumably small) inner region contributions.

The calculations described above seem eminently reasonable, but the importance of the problem and the experimental discrepancy in the estimate of $S_{34}(0)$ warrant further theoretical study. We suggest the use of the same simple picture which led to $M_B(E)$, but modified to avoid the use of higher E gamma capture data—that data could conceivably be wrong, or the model might be somewhat less accurate at higher E—to determine N_B. Rather, N_B is determined by analytic continuation of the experimentally determined phase shifts $\delta(E)$ for elastic scattering in the incident channel with the same total and orbital angular momentum quantum numbers J and L as the bound state B to which capture occurs. Thus, to evaluate capture to the $L=1$ states of ^{7}Be, we need the s- and d-wave phase shifts to specify the continuum functions; in addition, we need the $p_{3/2}$ and $p_{1/2}$ phase shifts, $\delta(E)$ for this case, to determine $N_{3/2}$ and $N_{1/2}$ *via* analytic continuation.

The starting point in the analyticity approach is the construction of an effective range function $K(k^2)$ which can be analytically continued from $E>0$, where it can be determined from a knowledge of the phase shifts, to E_B. (By convention, the argument of K is k^2 rather than E.) For the technically simpler non-Coulombic one-body problem, the connection between N_B and the $\delta(E)$ has been described in a number of texts,[9] and we give but a brief description, reserving details for a later publication. The hardest part of the present problem is the determination of the appropriate $K(k^2)$ in the presence of Coulomb fields, and that has been done.[10]

Since the scattering channel of present interest and the bound state have the same J and L, the subscripts J and L on $K(k^2)$, on the nuclear T matrix $T(E)$ and the nuclear phase shifts $\delta(E)$ in the presence of a Coulomb field, and on the scattering wave functions $\Psi_c^-(E)$ and $\Psi^+(E)$ to be defined, can be omitted. We restrict our discussion to scattering of a spin one-half system by a spin zero system. With

$$\exp(2i\sigma_L)=\Gamma(L+1+i\eta)/\Gamma(L+1-i\eta)\ ,$$

$\psi(i\eta)$ the digamma function, and H the full Hamiltonian, we have

$$K(k^2)=-\Omega^{-1}(E)T^{-1}(E)+Q(E)\ , \tag{1}$$

$$\Omega^{-1}(E)=k^{2L}e^{2i\sigma_L}C_0^2(\eta)\Pi(\eta)\ , \tag{2}$$

$$\Pi(\eta)=\prod_{m=1}^{L}\left[1+\frac{\eta^2}{m^2}\right],\quad C_0^2(\eta)=\frac{2\pi\eta}{e^{2\pi\eta}-1}\ ,$$

$$Q(E)=2\eta k^{2L+1}\Pi(\eta)h(i\eta)\ , \tag{3}$$

$$h(i\eta)=\psi(i\eta)+(1/2i\eta)-\ln(i\eta)\ .$$

$T(E)$ is given by $-k^{-1}\exp i(2\sigma_L+\delta)\sin\delta$ for $E>0$, or by

$$T(E)=(2\mu/\hbar^2)\langle[H-E]\Psi_c^-(E)|\Psi^+(E)\rangle\ .$$

We have $\Psi_c^{\pm}(E)={}^3\phi^4\phi F_L\mathscr{Y}\exp(\pm i\sigma_L)$, where $kRF_L(R;E)$ is the standard regular Coulomb function and $\mathscr{Y}$ is the appropriate angular momentum factor; $\Psi^+(E)$ is the antisymmetrized partial-wave "outgoing" scattering wave function with Coulomb and nuclear effects fully accounted for. The essential point is that $K(k^2)$ is real analytic in the neighborhood of $k^2=0$. The branch cuts of Ω, T, and Q along the positive energy axis cancel, by construction, in forming $K(k^2)$ in Eq. (1). Left-hand cuts appear due to various exchange effects. The branch point of $K(k^2)$ in the complex k-plane closest to the origin, associated with single pion exchange, is defined by $2ik_{\rm bp}=-m_\pi c/\hbar$. The associated branch point in the complex E plane is then $E_{\rm bp}=-m_\pi^2c^2/8\mu$. For ^{3}He-$\alpha$ or ^{3}H-α, we have $\mu\approx(12/7)M_{\rm prot}$ and $E_{\rm bp}\approx-1.5$ MeV. For the ^{3}He-α case this may be sufficiently far from the $P_{1/2}$ excited state energy $E_{1/2}=-1.16$ MeV for the Padé approximant approach (see below) to be effective, but for the $P_{3/2}$ ground state of energy $E_{3/2}=-1.59$ MeV it may be necessary to explicitly extract the pion exchange cut, whose form is more or less known, before using a Padé approximant. Since the ratio of capture to the $P_{3/2}$ and $P_{1/2}$ states seems to be well known, it might be sufficient to study only capture to the $P_{1/2}$ state. [For ^{3}H$(\alpha,\gamma)^7$Li, a reaction of interest in big bang studies of E's rather larger than those relevant to solar neutrino studies but still low, $E_{\rm bp}$ lies above both the $P_{3/2}$ and $P_{1/2}$ states.] There will also be poles in $K(k^2)$ at zeros of $T(E)$ (other than the zero at $E=0$), but the nature of the $p_{1/2}$ and $p_{3/2}$ phase shifts—and numerical studies of $K(k^2)$—give no evidence of any poles near the origin.

For real $E>0$, $K(k^2)/[k^{2L+1}\Pi(\eta)]$ becomes

$$C_0^2(\eta)\cot\delta+2\eta[\mathrm{Re}\psi(i\eta)-\ln\eta]\ .$$

The experimental values of $\delta(E)$ determine the values of $K(k^2>0)$, and analytic continuation to $K(k^2<0)$ is trivial if we can fit $K(k^2>0)$ to Padé approximants. For an elastic-scattering process $\sigma\sim1/k$ and $K\sim k^{2I+2}$ for large k. This suggests the form

$$K(k^2)=\sum\alpha_ik^{2i}/\sum{}'(1+\beta_ik^{2i})\ , \tag{4}$$

where $\sum$ and $\sum'$ are sums from 0 to I and from 1 to $I-L-1$, and α_i and β_i are adjustable parameters.

N_B is related to the residue $\mathscr{R}$ of $\Omega(E)T(E)$ at $E=E_B$. $\Omega(E)\Psi_c^{-*}(E)\Psi_c^+(E)$, analytic in E, can be continued from $E>0$ to E_B, and one finds—we have expressed $\Psi^+(E)$ in terms of the full Green's function—that $\frac{1}{2}\hbar^2\mathscr{R}/\mu$ is the $E\to E_B$ of

$$\Omega(E)\langle(H-E)\Psi_c^-(E)|\Phi_B\rangle\langle\Phi_B|[H-E]\Psi_c^+(E)\rangle\ .$$

Rewritten as a surface term, $\mathscr{R}$ becomes

$$\mathscr{R}=(\hbar^2/2\mu)[\gamma N_B/\kappa_B^L]^2\ , \tag{5}$$

where $\gamma=\Gamma(L+1)/\Gamma(L+1+\eta_B)$. An application of the definition of $\mathscr{R}$ to Eq. (1) gives a second form for $\mathscr{R}$. Comparison of the two forms gives

$$N_B^2=-\frac{\kappa_B^{2L}\Gamma^2(L+1+\eta_B)}{(L!)^2d[K(k^2)-Q(E)]/dk^2}\ ,$$

where the presence of a derivative (to be evaluated at $k=i\kappa_B$) suggests, unfortunately, that the δ's will have to be reasonably accurately known.

Since $T(E_B)=\infty$ and $\cot\delta(E_B)=i$, Eq. (1) reduces at $E=E_B$ to $K(-\kappa_B^2)=Q(E_B)$, a very useful check on the numerical accuracy of the representation of $K(k^2)$ given by Eq. (4). If a check is obtained, the (known) value of E_B will be treated as input data, along with the phase shifts.

We have tested our method using the model calculations of Liu *et al.*[7] of the $p_{3/2}$ and $p_{1/2}$ phase shifts. We find excellent agreement with the values of $E_{3/2}$ and $E_{1/2}$ (values which their potential parameters were adjusted to reproduce). For the associated normalization constants, we obtain $N_{3/2}=4.59\pm0.02$ and $N_{1/2}=3.93\pm0.01$, in very good agreement with values extracted from their numerical results for the relative coordinate wave function. (The effectiveness of the analytic continuation technique over an interval of about 2 MeV should be noted. Ordinary effective range theory would not have been sufficient here.) Our associated reduced widths are in agreement with those of Liu *et al.*[7] but somewhat larger than those found in analyses of the experimental data,[6] in about the same ratio as the value of S_{34} deduced by Liu *et al.* to the value of S_{34} deduced from the experimental data. By using the Bessel function expansion of continuum Coulomb functions as Williams and Koonin[11] did, we also reproduce the low-energy expansion of S_{34} obtained by Liu *et al.*[7] Our calculations are somewhat simpler than would be the case with the use of (comparably accurate) experimental phase shifts, since Gaussian potentials do not generate pion exchange cuts.

The analytic continuation technique (ACT) might have a great advantage over the approach in which N_B is determined by measurement of the radiative capture cross section σ_γ at a higher E for reactions such as $d(\alpha,\gamma)^6$Li for which dipole radiation is forbidden so that the experiment remains difficult even at higher E. For this case, $\mu\approx(4/3)M_{\rm prot}$ and $E_{\rm bp}\approx-1.9$ MeV, while the energy of ^{6}Li relative to $d+\alpha$ is -1.47 MeV. The ACT serves as a check on the accuracy of scattering data by its estimate of E_B, and can be used to determine resonance parameters from phase shifts. An alternative to the ACT is a many-level R-matrix theory. An R-matrix approach was used[6] to analyze the $\delta(E)$ data; however, the one-level approximation was used, the WKB approximation was invoked, the E_B was put in by hand. On the basis of our ACT studies, we believe the one-level approximation to be inadequate. Furthermore, it is very useful to allow E_B to be an open parameter, for this provides a strong check on the accuracy of the data; *if* the estimate of E_B is good, one might wish to put in the exact value of E_B in estimating the reduced width. (We note, incidentally, that the R-matrix approach has the nice feature—not shared by the ACT form we used—that it builds in a characteristic nuclear dimension. It is possible to recast the ACT so that the effective range function is not expressed as a ratio of polynomials but, building in the Coulomb interaction *and* the hard core, is expressed in terms of the analytic Coulomb functions and the R matrix; in this form, a one- or two-level approximation to the R matrix might well be adequate.) We intend to elaborate on these matters in a future publication, which would also include a discussion of the accuracy required for experimental $\delta(E)$'s to be amenable to the ACT. As a preliminary comment, we note that the use in the ACT of the most recent experimental data on $\delta(E)$, by Boykin *et al.*,[6] does *not* give the correct values of the E_B's; if we fix the E_B's at their correct values, the values of the N_B^2 obtained from the ACT are about three times smaller than the values required to obtain, via a direct capture model calculation, the value of $S_{34}(0)$ generally accepted. We suspect that the true values of the $\delta(E)$'s lie outside the range suggested,[6] roughly one degree below the bottom of that range. As based on our analysis of model calculations,[7] the ACT might well be useful in the analysis of experimental data if the $\delta(E)$'s can be measured to about one degree down to perhaps 800 keV. This accuracy may be attainable for the $\delta(E)$'s—and for the s-wave shifts—which are of order a few degrees at these energies. (The d-wave phase shifts are very small and can be neglected. See Boykin *et al.*[6])

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