PHYSICAL REVIEW C VOLUME 29, NUMBER 1 JANUARY 1984

Deuteron electric polarizability

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(Received 28 September 1983)

The deuteron electric polarizability is calculated and the various contributions are analyzed theoretically. Upper limits are constructed from zero-range approximations. These analytic approximations explicate the smallness of the deviation of the exact numerical results from the zero-range approximation using no odd-parity forces and no deuteron D state.

[NUCLEAR STRUCTURE Electric polarizability.]

I. INTRODUCTION

Recently there has been renewed interest in the properties of the deuteron. This interest has been spurred by a variety of novel experiments[1,2] and by theoretical attempts[3–5] to probe the limits of traditional nuclear physics in the simplest nuclei. Traditional nuclear physics can be defined as the following: (1) nonrelativistic nucleons; (2) nucleons only (no nucleon substructure or explicit mesons); and (3) two-nucleon forces only. There now exists excellent evidence[6] for deviation from (2), good evidence[4,8] for deviation from (1), and circumstantial evidence[7] for deviations from (3).

Among those properties of complex systems which can be described as fundamental are the susceptibilities[9,10]: the electric polarizability and the magnetic susceptibility. These quantities are the respective second-order responses to static, uniform electric and magnetic fields. Although a close connection exists between deuteron photodisintegration sum rules and the susceptibilities, which we will discuss in the next section, the first direct experiment to measure the electric polarizability, α_E, was recently performed by scattering low energy deuterons from the intense electric field of a heavy nucleus,[2] and observing deviations from the Rutherford scattering law. The direct and sum rule approaches to obtaining α_E are just consistent, 0.70(5) fm^3 for the former[2] and 0.61(4) fm^3 for the latter,[11] while theoretical calculations[12–18] give results in the range 0.62–0.64 fm^3. A recent calculation[18] has shown that the value of α_E should be strongly correlated to other deuteron observables, [3,19] A_S and η, the deuteron asymptotic S-wave normalization and the asymptotic D to S ratio; in the context of photodeuteron reactions this has been known for a generation.[12]

Our motivation is tripartite: (1) develop methods for the deuteron which can be used *mutatis mutandis* for the much more complicated trinucleon systems; (2) explore the sensitivity of α_E to two-nucleon observables and develop simple analytic approximations and bounds to various contributions; and (3) make pedagogical and practical use of the old perturbation theory techniques.[20–24] Our hope is that additional experimental work will be forthcoming; in particular, new low energy photodeuteron data[25] and more accurate direct experimental values for α_E of the deuteron and, hopefully, the trinucleon systems are needed.

The remainder of the paper is organized as follows: Section II describes the sum rule approach to calculating α_E, and presents results for eight different potential models; Sec. III describes various zero-range approximations and discusses how well they work; Sec. IV describes the formal perturbation theory approach to calculating α_E; Sec. V treats the S-wave part of α_E by means of an effective range expansion; Sec. VI uses the Yamaguchi-Hulthén model of the deuteron to develop alternative effective range expansions; Sec. VII summarizes the results. Various S-wave potential models for α_E are solved analytically in the four Appendices. Throughout, our approach will emphasize analytic results and approximations, zero-range expansions as effective upper bounds to portions of α_E, and the effective "coupling constants" which determine the value of α_E. Our methods in many cases will be applicable in the future to the trinucleon problem, and this is an important motivation.

II. SUM RULES

Low energy photonuclear reactions provide a useful way to "measure" and calculate the susceptibilities.[10–12] At low photon energies ω (<10 MeV), the wavelength of the photon is very large compared to the size of the deuteron and the effective unretarded electric dipole interaction between a photon and the deuteron is the nuclear current given by Siegert's theorem: $\vec{D}_0$, where $\vec{D}_0$ is the nonrelativistic dipole operator. The total photoabsorption cross section $\sigma_\gamma(\omega)$ can be weighted by any function of ω and integrated over all energies to form a sum rule. Of special interest to us is the σ_{-2} sum rule,[11]

$$\bar{\sigma}_{-2}=\frac{1}{2\pi^2}\int_{\omega_0}^{\infty}\sigma_\gamma(\omega)\frac{d\omega}{\omega^2}=\alpha_E+\beta_m+\cdots, \qquad (1)$$

where ω_0 is the threshold for photodisintegration. The additional terms indicated by the ellipsis are small in the deuteron case and of no interest to us here. The unretarded electric-dipole interaction defined above produces α_E, and no other terms. Unfolding the expression for the total cross section and performing the integral one finds

$$\alpha_E=\frac{2\alpha}{3}\sum_{N\neq 0}\frac{\langle 0|\vec{D}_0|N\rangle\cdot\langle N|\vec{D}_0|0\rangle}{E_N-E_0}\ , \tag{2}$$

where α is the fine structure constant, $H_0|N\rangle=E_N|N\rangle$, H_0 is the nuclear Hamiltonian, and $|0\rangle$ is the lowest lying of the states, $|N\rangle$. For the deuteron case, $\vec{D}_0=[\Delta\tau_z/2]\vec{r}/2$, $\vec{r}$ is the vector pointing from neutron to proton, and $\Delta\tau_z=\tau_z(1)-\tau_z(2)$. The appropriate matrix element of the bracketed isospin operator is 1.

The factor of r in $\vec{D}_0$ weights the matrix element in the tail of the wave function, which for the dominant S-wave part (u) behaves as $A_S e^{-\kappa r}$, where $\kappa=\sqrt{2\mu E_B}$, μ is the reduced mass, $2\mu=938.926$ MeV, and $E_B=-E_0=2.224\,575$ MeV (Ref. 19) is the binding energy.[5] The negative parity vector $\vec{D}_0$ determines that $^3P_0$, $^3P_1$, $^3P_2$, and $^3F_2$ intermediate states contribute to α_E. Only the D-wave part (w) of the deuteron wave function connects to the latter state. Equation (2) provides a simple, inelegant method for calculating α_E numerically. Contributions from the various angular momentum states are incoherent, and only the coupling of the last two partial waves by the tensor force provides any difficulty. The contribution of their sum can be shown to be explicitly independent of the mixing parameter and the phase shifts. The net result is that α_E depends only on the radial matrix elements in the eigenphase representation.

Because of its simplicity, we have calculated α_E for eight potential models using the method described above. The results are listed in Table I, including the S-wave zero-range approximation to be discussed later, α_E^0, the S-wave (of the deuteron) approximation, α_E^S, the effect of the potential in odd parity waves, $\Delta\alpha_E^V$, and the effect of D waves when there are no odd-parity forces, $\Delta\alpha_E^D$. We have also scaled the results to a value[3] of $A_S=0.885$ fm$^{-1/2}$ [i.e., $\alpha_E\rightarrow\alpha_E\,(0.885/A_S)^2$]. Although not listed in the table, we have also calculated separately the F-wave contribution ($\sim 0.001\,53$ fm^3) and have verified[18] that the D-wave portion scales approximately as $A_D^2\equiv A_S^2\eta^2$. The Hamada-Johnston (HJ) result was at variance because the binding energy of the deuteron does not correspond to experiment. The Humberston-Wallace modification does have a good binding energy and scales to a "good" value of α_E.

The effect of a potential in odd-parity intermediate states in Eq. (2) was small in every case. Dropping the deuteron D state, the P-wave forces give a positive contribution ~ 0.001 fm^3, while the additional effect with the D state of the deuteron included is ~ -0.003 fm^3, almost entirely from the P-wave intermediate states. We have also calculated the effect of adding the spin-orbit part of the dipole operator[4,8] to Eq. (1). The net result of this relativistic correction is negligible, although it plays a non-negligible role ($\sim 20\%$) in photodisintegration at 0° outgoing proton angle.

III. ZERO-RANGE APPROXIMATIONS

The results of the previous section confirm the conclusions of Refs. 12 and 18 that the only significant contributions come from the S-wave part of the deuteron, that the forces in the odd waves are not very important, and that the binding energy and asymptotic S-wave normalization are the two most relevant deuteron observables. In order to understand the latter dependencies, we perform the simplest possible calculation of α_E. We assume that the deuteron S-wave function is given by its asymptotic form,

$$u(r)=A_S e^{-\kappa r}\ , \tag{3}$$

and that the P-wave radial functions are the same in all partial waves:

$$\phi_P(r)=j_1(kr)+k^3 a n_1(kr)\ . \tag{4}$$

We have assumed the asymptotic form for the P waves and $\tan\delta\rightarrow -k^3 a$; the P-wave scattering volume, a, vanishes for free waves. Using Eq. (1) for α_E one finds

TABLE I. The electric polarizability, α_E, its zero-range approximation, α_E^0, the S-wave approximation without odd-parity potentials, α_E^S, the additional D-wave contribution, $\Delta\alpha_E^D$, in the absence of odd-parity forces, the effect of forces in the odd waves, $\Delta\alpha_E^V$, the asymptotic S-wave normalization, A_S, and $\tilde{\alpha}_E$, the result of scaling α_E by $(0.885/A_S)^2$ for eight different potential models. These potentials are Reid soft core (Ref. 26), Paris (Ref. 27), Hamada-Johnston (Ref. 28), Humberston-Wallace version of the former (Ref. 29), super soft-core version C (Ref. 30), de Tourreil-Rouben-Sprung (Ref. 31), Argonne-V_{14}, (Ref. 32), and OPEP for the deuteron (Ref. 5), but SSC(C) for the odd parity waves. All polarizabilities are in units of fm^3.

	Potential model							
	RSC	Paris	HJ	HW	SSC(C)	TRS	AV_{14}	OPEP
α_E (fm^3)	0.6225	0.634	0.610	0.630	0.649	0.636	0.641	0.6015
α_E^0 (fm^3)	0.627	0.640	0.616	0.636	0.655	0.642	0.648	0.608
α_E^S (fm^3)	0.620	0.632	0.607	0.627	0.647	0.634	0.639	0.600
$\Delta\alpha_E^D$ (fm^3)	0.0047	0.0047	0.0048	0.0048	0.0045	0.0047	0.0049	0.0047
$\Delta\alpha_E^V$ (fm^2)	−0.0020	−0.0024	−0.0015	−0.0015	−0.0025	−0.0026	−0.0028	−0.0035
A_S (fm$^{1/2}$)	0.8776	0.8869	0.8919	0.8852	0.8969	0.8883	0.8920	0.8642
$\tilde{\alpha}_E$ (fm^3)	0.633	0.631	0.601	0.630	0.632	0.631	0.631	0.631

$$\alpha_E=\frac{4\alpha A_S^2\mu}{3\pi\kappa^5}\int_0^\infty \frac{dx\,x^{3/2}}{(1+x)^5}[1-y(1+3x)]^2 \tag{5a}$$

$$=\frac{\alpha\mu A_S^2}{32\kappa^5}(1-12y+116y^2)\cong\frac{\alpha\mu A_S^2}{32\kappa^5}-\frac{3\alpha\mu A_S^2 a}{16\kappa^2}$$

$$\equiv\alpha_E^0+\Delta\alpha_E^{sc}\ , \tag{5b}$$

where $y=\kappa^3 a/2$. We also note that

$$A_S^2=2\kappa/[1-\kappa\rho(-E_B,-E_B)] \tag{5c}$$

relates the deuteron-based[19] effective range $\rho(-E_B,-E_B)$ to A_S.

Momentarily setting a to zero, we evaluate the zero-range approximation to the electric polarizability, α_E^0. Using[3] $\rho=1.764(6)$ fm and $E_B=2.224\,575$ MeV we find $A_S=0.885(1)\ \text{fm}^{-1/2}$. Thus

$$\alpha_E^0=0.638(2)\ \text{fm}^3\ . \tag{6}$$

Clearly, this simple approximation accounts for most of the polarizability. Moreover, we see that the "average" effect of the P-wave forces expressed in terms of the P-wave scattering volume enters as $\kappa^3 a$. The factor of κ^3 is 0.0124 fm^{-3}, while the appropriate a, to be discussed in Sec. IV, is also numerically small ($\sim 0.1\ \text{fm}^3$) both experimentally and for various potential models. Thus the S-wave part of the potential effect, $\Delta\alpha_E^V$, must be small. We note that the zero range estimate we have made is typically larger than a better estimate, and will be an effective (nonrigorous) upper limit. Note also that $\alpha_E^0\sim E_B^{-2}/(1-\kappa\rho)$, and a poor binding energy for the deuteron could have a deleterious effect on α_E^0. This accounts for the HJ result being much too low.

IV. PERTURBATION THEORY

The techniques of perturbation theory provide the most direct method for calculating α_E, particularly in systems more complex than the deuteron.[22] The expression for α_E obviously arises as a part of a second-order energy shift. This shift for a perturbation ΔH can be expressed as

$$\Delta E^{(2)}=\sum_{N\neq 0}\frac{\langle 0|\Delta H|N\rangle\langle N|\Delta H|0\rangle}{E_0-E_N}\ , \tag{7}$$

with a corresponding *first*-order shift in the wave function, $\Delta\psi$, given by

$$|\Delta\psi\rangle=\sum_{N\neq 0}\frac{|N\rangle\langle N|\Delta H|0\rangle}{E_0-E_N}\ , \tag{8}$$

which satisfies the inhomogeneous differential equation

$$(E_0-H)|\Delta\psi\rangle=\Delta H|0\rangle-|0\rangle\langle 0|\Delta H|0\rangle\ . \tag{9}$$

We have written $\psi_0=|0\rangle$. This allows us to write

$$\Delta E^{(2)}=\langle 0|\Delta H|\Delta\psi\rangle\ . \tag{10}$$

Thus, solving for $\Delta\psi$ allows construction of $\Delta E^{(2)}$ by a simple quadrature. This can be applied to α_E, where the perturbation is $-e\vec{\mathrm{E}}\cdot\vec{\mathrm{D}}_0$, corresponding to a uniform electric field $\vec{\mathrm{E}}$. The function $|\Delta\psi\rangle$ is first split into components which arise from four contributing isospin triplet partial waves: ${}^3P_0$, ${}^3P_1$, ${}^2P_2$, and ${}^3F_2$. Since $|0\rangle$ contains both S- and D-state parts, $\Delta H|0\rangle$ will also contain the same partial waves. We can perform the isospin matrix elements, yielding an effective interaction $-e\vec{\mathrm{E}}\cdot\vec{\mathrm{r}}/2$. We write the deuteron intrinsic spin wave function as $\chi\cdot\hat{e}$, where $\hat{e}$ is a unit vector and find

$$\Delta\psi=-2\mu r\sum_{J,l}\frac{f_J^l}{r}\frac{[Y_l\otimes\chi_1]_J}{\sqrt{3}}(E_1\otimes\hat{e}_1)_J\ , \tag{11}$$

where

$$\psi_0=\left[\frac{u}{r}+\frac{S_{12}}{\sqrt{8}}\frac{w}{r}\right]\frac{\chi\cdot\hat{e}}{\sqrt{4\pi}} \tag{12a}$$

and

$$\vec{\mathrm{E}}\cdot\vec{\mathrm{r}}\psi_0=\sum_{J,l}g_J^l\frac{[Y_l\otimes\chi_1]_J}{\sqrt{3}}(E_1\otimes\hat{e}_1)_J\ . \tag{12b}$$

We obtain upon projection

$$\alpha_E=\frac{-\alpha\mu}{3}\sum_{J,l}\int dr\,r^2 g_J^l f_J^l\frac{(2J+1)}{9}\ , \tag{13a}$$

$$g_J^l=u+\lambda_J^l w/\sqrt{2}\ , \tag{13b}$$

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{2}{r^2}-\kappa^2-2\mu V_j^1\right]f_J^1-2\mu V_2^{13}f_J^3\delta_{J,2}$$

$$=u+\lambda_J^1 w/\sqrt{2}\ , \tag{13c}$$

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{12}{r^2}-\kappa^2-2\mu V_2^3\right]f_2^3-2\mu V_2^{13}f_2^1$$

$$=\lambda_2^3 w/\sqrt{2}\ , \tag{13d}$$

where $\lambda_0^1=-2$, $\lambda_1^1=1$, $\lambda_2^1=-\frac{1}{5}$, and $\lambda_2^3=3\sqrt{6}/5$. In Eq. (13) V_J^l is the diagonal potential in the (l,J) partial wave and V_2^{13} is the coupling potential between the P and F waves for $J=2$. Using (13b) we can rewrite the expression for α_E:

$$\alpha_E=\frac{-\alpha\mu}{3}\int dr\,r^2(uf+\tfrac{2}{5}wg+\tfrac{3}{5}wh)\ , \tag{14a}$$

$$f=\tfrac{1}{9}\sum_J f_J^1(2J+1)\ , \tag{14b}$$

$$g=\frac{1}{9}\frac{5}{2\sqrt{2}}\sum_J(2J+1)\lambda_J^1 f_J^1\ , \tag{14c}$$

$$h=\frac{5}{3\sqrt{3}}f_2^3\ . \tag{14d}$$

This compact representation for α_E has been chosen so that the S- and D-wave ground state dependencies are manifest; the latter has been separated into P-wave (g) and F-wave (h) intermediate state parts.

In order to see the underlying structure better, we resort to first-order perturbation theory in the nuclear potentials, V_j^l. We define f_0, g_0, and h_0 to be the solutions to Eq. (13) in the absence of a nuclear potential, in which case

the f_J^1 are simple linear combinations of two independent solutions, easily obtained using Green's functions. The function f_0 corresponds to a right-hand side in Eq. (13c) of u, while g_0 and h_0 correspond to w on the right-hand sides of (13c) and (13d). We find after some manipulation,

$$\alpha_E \cong \frac{-\alpha\mu}{3}\int dr\, r^2(uf_0+\tfrac{2}{5}wg_0+\tfrac{3}{5}wh_0)-\frac{2\alpha\mu^2}{3}\int dr\, r^2(h_0V_2^{13}(f_0-g_0/5\sqrt{2})+f_0^2V_C+\sqrt{2}f_0g_0V_T+g_0^2V_u/2+h_0^2V_F)\,, \tag{15}$$

where

$$V_C=\tfrac{1}{9}\sum_J(2J+1)V_J^1\,, \tag{16a}$$

$$V_T=\tfrac{1}{9}\sum_J(2J+1)(\lambda_J^1)V_J^1\,, \tag{16b}$$

$$V_u=\tfrac{1}{9}\sum_J(2J+1)(\lambda_J^1)^2V_J^1\,, \tag{16c}$$

and

$$V_F=\tfrac{3}{5}V_2^3\,. \tag{16d}$$

Nominally, we expect the parts of α_E in Eq. (15) that are proportional to f_0^2 (i.e., u^2) to dominate. This produces

$$\Delta\alpha_E^V\cong\frac{-2\alpha\mu^2}{3}\int dr\, r^2f_0^2V_C\,. \tag{17}$$

Using the results of Sec. VI, Eq. (24a), for the zero-range approximation to f_0, $\bar{f}$, we find

$$\Delta\alpha_E^V\cong\frac{-2\alpha\mu A_S^2}{48\kappa^2}\int_0^\infty dr\, r^4e^{-2\kappa r}V_C \tag{18}$$

$$=-6\alpha_E^0\kappa^3a_C^B\left[\int_0^\infty dr\, r^4e^{-2\kappa r}V_C\right]\Big/\left[\int_0^\infty dr\, r^4V_C\right] \tag{19}$$

since the Born approximation to a_C, generated by the central combination of forces, V_C, is proportional to the denominator integral in Eq. (19). To the extent that one can drop the exponential in Eq. (19), we have reproduced Eq. (5b). Also, Eq. (19) for finite κ is smaller than our zero-range approximation, Eq. (5b). This confirms the smallness of $\Delta\alpha_E^V$ since a_C is small experimentally,[33,34] and for "realistic" potential models. A recent study of pp scattering[34] produced a Coulomb-corrected value of $a_C=-0.12\pm0.13$ fm^3. The tensor combination was not small, however: $a_T=1.70\pm0.25$ fm^3, all assuming uncorrelated errors. Thus, the normally smaller contributions from the D waves dominate $\Delta\alpha_E^V$; both are small, however.

V. EFFECTIVE RANGE CORRECTION

In order to understand the smallness of the deviation of α_E from its zero-range approximation, we resort to another type of effective range approximation. Ignoring D-wave terms and forces in the P waves, the expression for α_E becomes

$$\alpha_E\cong\frac{-\alpha\mu}{3}\int_0^\infty dr\, r^2a(r)f(r)\,, \tag{20a}$$

$$f(r)=\frac{-1}{4\pi}\int_0^\infty\frac{e^{-\kappa|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}xu(r')d^3r'\,, \tag{20b}$$

where $x=\hat{r}\cdot\hat{r}'$. We write $u(r)$ as the zero-range approximation minus a defect function $\Delta(r)$, and keep linear terms in Δ; the zero-range approximation to α_E was previously calculated. One finds the defect contribution

$$\Delta\alpha_E^\Delta=\frac{2\alpha\mu}{3}\int_0^\infty dr\, r^2\Delta(r)\bar{f}(r)e^{-\kappa r}$$

$$=\frac{-\alpha\mu A_S}{6\kappa}\int_0^\infty dr\, r^3e^{-\kappa r}\Delta(r)$$

$$=\frac{-16\kappa^4}{3}\alpha_E^0\int_0^\infty dr\, r^3\left[\frac{\Delta(r)}{A_S}\right]e^{-\kappa r}\,. \tag{21}$$

We see that this correction to α_E^0 behaves as $(\kappa R)^4$, if we write

$$\int dr\, r^3[\Delta(r)/A_S]\sim R^4\sim 1\text{ fm}^4$$

and is correspondingly small. For the Reid soft core (RSC) model, the calculated correction is more than 90% of the difference of the exact and zero-range S-wave results. Moreover, the integrand in Eq. (21) is heavily weighted toward the tail of the defect wave function, which is dominated by the one-pion exchange potential (OPEP); fifty percent of the integral arises from r greater than 2 fm in the integrand, and over 90% from r greater than 0.75 fm. Consequently, the correction should be similar for all realistic potential models.

VI. SEPARABLE MODEL

The separable potential Yamaguchi model[35] provides a good, albeit simplified, form of the two-nucleon scattering amplitude and bound state. The essential simplification is the replacement of the left-hand cut in the scattering amplitude by a pole or sum of poles, which allows analytic reconstruction of the wave functions. These wave functions possess many properties of realistic wave functions, and we will use them to further explore the electric polarizability.

The standard Yamaguchi S-wave weight function $g(p)=(\beta^2+p^2)^{-1}$ leads to

$$u(r)=A_S(e^{-\kappa r}-e^{-\beta r})\,, \tag{22a}$$

while the D-wave weight function $T(p)=tp^2/(\gamma^2+p^2)^3$ leads to

$$w(r)=A_S\eta\left\{\frac{H_2(\kappa r)-H_2(\gamma r)}{\kappa^2 r^2}-e^{-\gamma r}\frac{(\gamma^2-\kappa^2)}{2\kappa^2}\times\left[1+\gamma r+\frac{(\gamma^2-\kappa^2)}{4}r^2\right]\right\}, \tag{22b}$$

where $H_2(z)=z^3h_2^{(1)}(iz)$ and t, γ, and β are constants. We note that the original Yamaguchi $T(p)$ was slightly different from ours and led to a $w(r)$ which behaved as r^2 near the origin, rather than the proper r^3 behavior of (22b). Given the functions (22), we can use the results of Sec. IV without potentials to calculate α_E for the Yamaguchi model, and, incidentally, a heuristic upper bound for the D-wave contributions.

We have to solve the equations

$$G''+\frac{2G'}{r}-\frac{l(l+1)G}{r^2}-\kappa^2G=\phi_0(r)\,, \tag{23}$$

with $l=1$ and 3 and $\phi_0=u$ or w. We apply the usual boundary conditions: G vanishes at $r=\infty$, and must be finite at $r=0$. The easiest method is to write

$$G=[e^{-\kappa r}A(r)+e^{-\beta r}B(r)]/r^{l+1}$$

and solve for the polynomials A and B: nonpolynomial behavior leads to the unacceptable form $A\sim e^{2\kappa r}$, etc. For the S-wave case we find[13–16]

$$f=\frac{2A_S}{(\beta^2-\kappa^2)^2r^2}\left\{e^{-\kappa r}\left[1+\kappa r-\frac{(\beta^2-\kappa^2)^2r^3}{8\kappa}\right]-e^{-\beta r}\left[1+\beta r+\frac{(\beta^2-\kappa^2)r^2}{2}\right]\right\},$$

$$\to-\frac{A_S}{4\kappa}re^{-\kappa r}\,, \tag{24a}$$

and

$$\alpha_E=\frac{\alpha\mu A_S^2(\beta-\kappa)^2}{96\kappa^5\beta^3(\beta+\kappa)^4}(3\beta^5+18\beta^4\kappa+51\beta^3\kappa^2+96\beta^2\kappa^3+48\beta\kappa^4+8\kappa^5)\,,$$

$$\to\frac{\alpha\mu A_S^2}{32\kappa^5}(1-32\kappa^4/\beta^4+\cdots)\,, \tag{24b}$$

where the latter relations are the zero-range approximation ($\beta\to\infty$). We note that the approximate treatment of Sec. V leads to $\Delta=A_Se^{-\beta r}$ and $\Delta\alpha_E^{\Delta}\cong-\alpha_E^0 32\kappa^4/\beta^4$ in agreement with (24b). Indeed, virtually all of the next term in the expansion indicated by the ellipsis is also reproduced.

The D-wave parts are also easily calculated in principle, although somewhat messy in practice. One finds

$$g=\frac{A_S\eta}{\kappa^2r^2}\left[e^{-\kappa r}\left[(1+\kappa r)\lambda-\frac{3r^2}{2}-\frac{\kappa r^3}{4}\right]-e^{-\gamma r}\left\{(1+\gamma r)\lambda+\left[\lambda\frac{(\gamma^2-\kappa^2)}{2}-\frac{3}{2}\right]r^2+\frac{5\gamma r^3}{4}+\left[\frac{\gamma^2-\kappa^2}{8}\right]r^4\right\}\right]$$

$$\to-\frac{A_S\eta}{\kappa^2}e^{-\kappa r}\left[\frac{3}{2}+\frac{\kappa r}{4}\right], \tag{25}$$

where

$$\lambda=3(5\gamma^2-\kappa^2)/(\gamma^2-\kappa^2)^2\,.$$

A tedious integral produces

$$\alpha_E^{D,P}=18\alpha_E^0\eta^2\left[\frac{(\gamma-\kappa)^6(240\gamma^5+585\gamma^4\kappa+490\gamma^3\kappa^2+214\gamma^2\kappa^3+50\gamma\kappa^4+5\kappa^5)}{240\gamma^7(\gamma+\kappa)^4}\right]$$

$$\to18\eta^2\alpha_E^0\left[1-\frac{121}{16}\frac{\kappa}{\gamma}+\cdots\right]. \tag{26}$$

The F-wave part is obtained similarly

$$h=\frac{\lambda'}{r^4}\left[\left[\left[1+\kappa r+2\frac{\kappa^2r^2}{5}+\frac{\kappa^3r^3}{15}\right]e^{-\kappa r}-e^{-\gamma r}\left\{1+\gamma r+\left[\frac{2\gamma^2}{5}+\left[\frac{\gamma^2-\kappa^2}{10}\right]\right]r^2+\left[\frac{\gamma^3}{15}+\beta\frac{(\gamma^2-\kappa^2)}{10}\right]r^3+\frac{(\gamma^2-\kappa^2)}{120}r^4(5\gamma^2-\kappa^2)+\frac{\beta(\gamma^2-\kappa^2)}{120}r^5\right\}\right]A_S\eta/x^2\right.$$

$$\left.+\left\{\frac{e^{-\gamma r}}{4}[1+\gamma r-(\gamma^2-\kappa^2)r^2/2]-\frac{e^{-\kappa r}}{4}(1+\kappa r)\right\}A_S\eta/x^2\right]$$

$$\to-\frac{A_S\eta}{4x^2}e^{-\kappa r}(1+\kappa r)\,, \tag{27}$$

where $\lambda'=180/(\gamma^2-\kappa^2)^2$. The zero-range limits correspond to $\gamma\to\infty$. We also obtain

$$\alpha_E^{D,F}=7\eta^2\alpha_E^0\left[\frac{(\gamma-\kappa)^6(560\gamma^5+3125\gamma^4\kappa+3810\gamma^3\kappa^2+1866\gamma^2\kappa^3+450\gamma\kappa^4+45\kappa^5)}{560\gamma^7(\gamma+\kappa)^4}\right]$$
$$\rightarrow 7\eta^2\alpha_E^0\left[1-\frac{495}{112}\frac{\kappa}{\gamma}+\cdots\right]. \tag{28}$$

The net zero-range D-wave contribution is

$$\alpha_E^D\rightarrow 25\eta^2\alpha_E^0=0.018\alpha_E^0=0.012\ \text{fm}^3 \tag{29}$$

while Eqs. (26) plus (28) gives 0.0042 fm^3, with $\gamma=7\kappa$. The zero-range approximation is essentially an upper limit; the additional pieces of the wave function are required to regularize (render finite) the wave function at the origin. For nonpathological cases with nonoscillatory behavior the additional pieces will be negative and reduce α_E. The large reduction factor (0.36) is typical of higher angular momenta, unlike the small correction for S waves. We note that for the Yamaguchi model

$$A_S^2=\frac{\beta(\beta+\kappa)}{(\beta-\kappa)^2} \tag{30a}$$

and

$$\eta=\frac{t\kappa^2}{\gamma^2-\kappa^2}\ . \tag{30b}$$

The singularities in (30) when $\beta,\gamma\rightarrow\kappa$ are responsible for the corresponding vanishing of the various expressions for α_E.

VII. CONCLUSIONS

The deuteron electric polarizability has been separated into five pieces, plus a remainder indicated by the ellipsis:

$$\alpha_E\cong\alpha_E^0+\Delta\alpha_E^\Delta+\Delta\alpha_E^V+\alpha_E^{D,P}+\alpha_E^{D,F}+\cdots\ . \tag{31}$$

The zero range approximation is sensitive to the deuteron S-wave asymptotic normalization A_S and to the binding energy and overestimates the complete result by a few percent. The S-wave effective range correction, $\Delta\alpha_E^\Delta$, lowers the zero-range result by about one percent, and is primarily determined by the exterior part of the nucleon-nucleon potential; thus it is similar for all realistic-force models. It has the schematic form $\sim(\kappa R)^4$, and is small because κR is small. The potential correction $\Delta\alpha_E^V$ is small and lowers the result by roughly $\frac{1}{3}$ of one percent. This correction for S waves has the form $\kappa^3 a_C$, where a_C is the central combination of P-wave scattering volumes and is very small ($\sim\pm0.1$ fm^3); thus the potential correction is slightly dominated by tensor terms and should be quite sensitive to the long-range part of the tensor force. The deuteron D wave increases the result by nearly one percent. An effective upper (zero-range) limit can be derived for this contribution, which is 2.5 times larger than the values actually found. Approximately 30 percent of the D-wave contribution comes in combination with F-wave intermediate states.

Table II lists the contributions to α_E, beginning with the zero-range approximation. The corrections are estimates based on Table I and on uncertainties we have investigated using the approximations developed earlier. Error estimates are subjective. We note that our final result has a very small error, which is unattainable experimentally by any known method. This does not mean that we would be shocked if the experimental value settled outside the prediction of Table II. Such an occurrence would be interesting and would indicate that the physics used in our present analysis was incomplete or that experimental values of our parameters were incorrect. The current experimental uncertainty should be improved. It would be very interesting also to extend the experimental work to the trinucleon system.

ACKNOWLEDGMENTS

The work of J. L. F. was performed under the auspices of the U.S. Department of Energy. The work of S. F. was supported in part by the U.S. Department of Energy. We would like to thank E. G. Fuller, D. Skopik, and E. L. Tomusiak for useful conversations.

APPENDIX A: HYDROGEN ATOM

The archetype of the use of special perturbation theory techniques is the electric polarizability of the hydrogen atom. Calculated originally in 1926 as part of an exact treatment[36] of the Stark effect this quantity was also calculated by Podolski[20] using an inhomogeneous differential equation in his treatment of dispersion. The method has been generalized by Dalgarno and Lewis,[23] and is treated in textbooks.[21,22] For purposes of completeness we briefly repeat the derivation.

Using a perturbation $-e\vec{\mathrm{E}}\cdot\vec{r}=-e\vec{\mathrm{E}}\cdot\vec{\mathrm{D}}_0$, writing

TABLE II. Contributions to the deuteron electric polarizability starting with the zero-range approximation based on $A_S=0.885(1)$ fm$^{-1/2}$.

α_E^0	$\Delta\alpha_E^\Delta$	$\Delta\alpha_E^D$	$\Delta\alpha_E^V$	α_E
0.6376(15)	−0.008(1)	0.005	−0.0025(10)	0.632(3)

$$\psi(\vec{r})=-2e\mu\vec{E}\cdot\vec{D}_0 g(r)/r \ ,$$

and the hydrogen-atom ground state as $Ne^{-r/a_0}/\sqrt{4\pi}$, with $N=2a_0^{-3/2}$ and the Bohr radius $a_0=(Z\alpha\mu)^{-1}$, one obtains

$$\alpha_E=-\frac{4\alpha\mu N}{3}\int dr\, r^3 e^{-r/a_0} g(r) \ , \tag{A1}$$

where

$$g''+2g'/r-2g/r^2+2g/a_0 r-g/a_0^2=Nre^{-r/a_0} \ . \tag{A2}$$

Solving this equation subject to the finiteness boundary condition leads to

$$g(r)=-Nre^{-r/a_0}\left[\frac{a_0^2}{2}+\frac{a_0 r}{4}\right] , \tag{A3}$$

and subsequently to the well-known result

$$\alpha_E=\frac{9a_0^3}{2Z} \ . \tag{A4}$$

APPENDIX B: HARMONIC OSCILLATOR

Somewhat more relevant to nuclear physics is the harmonic oscillator problem.[16] Writing

$$\psi(r)=-2e\mu\vec{E}\cdot\vec{D}_0 g(r)/r \ ,$$

$$\psi_0=Ne^{-r^2/2b^2}/\sqrt{4\pi} \ ,$$

with $b^2=(\mu k)^{-1/2}$, $\omega_0=\sqrt{k/\mu}$, $E_0=\frac{3}{2}\hbar\omega_0$, and $N^2=4/\pi^{1/2}b^3$, we obtain with $\vec{D}_0=\vec{r}$

$$\alpha_E=-\frac{4\alpha\mu N}{3}\int dr\, r^3 e^{-r^2/2b^2} g(r) \ , \tag{B1}$$

where

$$g''+2g'/r-2g/r^2-r^2 g/b^4+g/b^2=rNe^{-r^2/2b^2} \ , \tag{B2}$$

whose solution is

$$g(r)=Ne^{-r^2/2b^2}(-rb^2/2) \ . \tag{B3}$$

The quadrature is simple:

$$\alpha_E=\alpha\mu b^4=\alpha b^2/\omega_0 \ . \tag{B4}$$

Clearly, the latter result is more simply obtained by using the usual dipole selection rule for harmonic oscillators; only the $1\hbar\omega_0$ state contributes, with strength $b/\sqrt{2}$ per matrix element of D_z, which follows from the virial theorem.

APPENDIX C: DELTA-SHELL INTERACTION

A particle of mass μ interacting with a fixed potential, $V(r)=-\lambda\delta(r-R)$ can have S-wave bound states described by a wave function $u(r)/r$:

$$u(r)=Ne^{-\kappa r} \quad r\geq R \tag{C1}$$

or

$$B\frac{(e^{\kappa r}-e^{-\kappa r})}{2} \quad r\leq R \ ,$$

where the two components of u are constrained by requiring continuity at $r=R$ and a discontinuity of u' at $r=R$ determined by λ. These two conditions and the normalization condition determine N, B, and κ. Again using $\vec{D}_0=\vec{r}$ and

$$\psi(r)=-2e\mu\vec{E}\cdot\vec{D}_0 g(r)/r$$

we can write

$$\alpha_E=-\frac{4\alpha\mu}{3}\int_0^\infty dr\, r^2 u(r) g(r) \ , \tag{C2}$$

where

$$g''(r)+2g'/r-2g/r^2+2\mu\lambda\delta(r-R)-\kappa^2 g=u(r) \ . \tag{C3}$$

One obtains

$$g(r)=-\frac{Nre^{-\kappa r}}{4\kappa}+\frac{\beta Ne^{-\kappa r}}{\kappa^2}\left[\frac{1}{\kappa r}+\frac{1}{(kr)^2}\right] \quad r\geq R$$

$$=\gamma i_1(\kappa r)+\frac{Br}{4\kappa}\cosh(\kappa r) \quad r\leq R \ , \tag{C4}$$

where the β and γ terms are the homogeneous solutions. Their amounts are obtained by the continuity and derivative discontinuity conditions at $r=R$. These coefficients are messy and we abstain from listing them. We have defined the functions

$$i_N(z)=(-i)^N j_N(iz) \ . \tag{C5}$$

APPENDIX D: SQUARE WELL

A particle of mass μ interacting with a square-well potential $V(r)=-V_0\theta(R-r)$ can have S-wave bound states described by a wave function $u(r)/r$ given by

$$u(r)=Ne^{-\kappa r} \quad r\geq R$$

$$=B\sin(\alpha r) \quad r\leq R \ , \tag{D1}$$

where $\alpha^2=2\mu V_0-\kappa^2$. Continuity conditions at $r=R$ determine B and κ.

The electric polarizability is given by

$$\alpha_E=-\frac{4\alpha\mu}{3}\int_0^\infty dr\, r^2 u(r) g(r) \ , \tag{D2}$$

with g determined from

$$g''+2g'/r-2g/r^2+(\alpha^2+\kappa^2)\theta(R-r)g-\kappa^2 g=u(r) \ . \tag{D3}$$

Ignoring the potential in (D3), we find

$$g(r)=-\frac{Ne^{-\kappa r}}{4\kappa}+\frac{\beta Ne^{-\kappa r}}{\kappa^2}\left[\frac{1}{\kappa r}+\frac{1}{(\kappa r)^2}\right] \quad r\geq R$$

$$=\gamma i_1(\kappa r)+B\left[\frac{2\alpha^2 j_1(\alpha r)}{(\alpha^2+\kappa^2)^2}-\frac{\sin(\alpha r)}{(\alpha^2+\kappa^2)}\right] \quad r\geq R \ . \quad \text{(D4)}$$

As in Appendix C the β and γ terms are homogeneous solutions determined by the continuity conditions at $r=R$ and are too messy to warrant listing.

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