Approximate projection of physical states in the Dyson boson description of nuclear collective motion

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A new approximate projection method is proposed to treat the spurious state problem in the Dyson boson description of nuclear collective motion. In this method, successive orders of approximate projection are used to construct the various physical states in the Dyson boson representation of the shell model. Even in the lowest orders of approximation, the approximately projected boson states obtained turn out to be quite accurate in producing physical quantities. Furthermore, this approximate projection method is applicable not only to vibrational or transitional nuclei but also to rotational nuclei.

I. INTRODUCTION

The Dyson boson representation of the shell-model pairing and multipole degrees of freedom developed by Janssen *et al.*[1,2] has attracted a great deal of interest in recent years.[3–9] Using this boson representation, important collective features of the nuclear many-body problem can be studied in a very efficient and transparent way. Furthermore, as emphasized in a previous paper[7] (hereafter referred to as I), this boson representation, being non-Hermitian and involving only finite expansions, also provides a distinct advantage for investigating the important question of the microscopic basis of the interacting boson model (IBM) introduced by Arima and Iachello.[10]

As is true for any other type of boson representations, the usefulness of the Dyson boson representation depends to a large extent on how the important task of constructing the complicated physical states in the boson space is actually carried out. Although in I we have been able to use the Dyson boson representation in a very efficient way for the derivation of the seniority-scheme boson mapping based on a monopole pairing interaction, the method employed there for generating the first few relevant components of the complicated physical states relies completely on a special property of the monopole pairing interaction in the non-Hermitian Dyson boson representation and thus cannot be used for more general purposes.

In this paper we present a completely new method of constructing the physical states in the Dyson boson description of nuclear collective motion. In this method, the first few relevant components of the very complicated physical states in the boson space are generated successively by a new approximate projection procedure which has also been applied successfully to treat the number nonconservation problem in the quasiparticle description of nuclear collective motion.[11] As we shall see later on, this new method of constructing the physical states does not depend on any special structure of the shell-model residual interactions and thus should be very useful for general applications of the Dyson boson representation.

The present paper is organized in the following order: In Sec. II, we first give a brief review of the Dyson boson representation of the shell model developed by Janssen *et al.*[1] Then, the approximate projection method for constructing the various physical states in the Dyson boson representation is described in Sec. III. And, to test the accuracy of the present method, in Sec. IV this approximate projection method in its lowest order is applied to a simplified shell model defined by a monopole pairing Hamiltonian. In Sec. V, an approximate construction of rotational physical states in the Dyson boson representation is described. There, the problem of angular momentum conservation is also tackled in an approximate way. The essence of the present method is summarized and discussed in Sec. VI, where further applications and improvements of the method are also briefly mentioned.

II. GENERALIZED DYSON TRANSFORMATION

In the Belyaev-Zelevinsky-Marshalek (BZM) boson expansion framework, corresponding to the fermion pair operators $a^\dagger_\alpha a^\dagger_\beta$, where α and β characterize shell-model single-particle states, one defines a set of antisymmetric boson operators $b^\dagger_{\alpha\beta}=-b^\dagger_{\beta\alpha}$ which satisfy the commutation relations

$$\begin{aligned}&[b_{\alpha\beta},b_{\gamma\delta}]=[b^\dagger_{\alpha\beta},b^\dagger_{\gamma\delta}]=0\ ,\\&[b_{\alpha\beta},b^\dagger_{\gamma\delta}]=\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}\ .\end{aligned}\tag{1}$$

Thus, by using the following mapping for the states:

$$(a^\dagger_{\alpha_1}a^\dagger_{\beta_1})(a^\dagger_{\alpha_2}a^\dagger_{\beta_2})\cdots(a^\dagger_{\alpha_n}a^\dagger_{\beta_n})\,|0\rangle \rightarrow P^\dagger_{\alpha_1\beta_1}P^\dagger_{\alpha_2\beta_2}\cdots P^\dagger_{\alpha_n\beta_n}\,|0)\ ,\tag{2}$$

$$P^\dagger_{\alpha\beta}\equiv b^\dagger_{\alpha\beta}-\sum_{\gamma\delta}b^\dagger_{\alpha\gamma}b^\dagger_{\beta\delta}b_{\gamma\delta}\ ,$$

and

$$\langle 0|(a_{\beta_n}a_{\alpha_n})\cdots(a_{\beta_2}a_{\alpha_2})(a_{\beta_1}a_{\alpha_1}) \rightarrow (0|b_{\alpha_n\beta_n}\cdots b_{\alpha_2\beta_2}b_{\alpha_1\beta_1} , \quad (3)$$

where $|0\rangle$ and $|0)$ are, respectively, the fermion and boson vacuum states, one obtains the generalized Dyson transformation for the fermion pair and multipole operators given by (for even-particle systems)[1]

$$\begin{aligned} (a_\alpha^\dagger a_\beta^\dagger)_D &= P_{\alpha\beta}^\dagger \mathscr{P} = \left[b_{\alpha\beta}^\dagger - \sum_{\gamma\delta} b_{\alpha\gamma}^\dagger b_{\beta\delta}^\dagger b_{\gamma\delta} \right] \mathscr{P} , \\ (a_\beta a_\alpha)_D &= b_{\alpha\beta}\mathscr{P} , \\ (a_\alpha^\dagger a_\beta)_D &= \sum_\gamma b_{\alpha\gamma}^\dagger b_{\beta\gamma}\mathscr{P} , \end{aligned} \quad (4)$$

where $\mathscr{P}$ is the projection operator onto the physical subspace and is given explicitly by

$$\begin{aligned} \mathscr{P} &= \sum_{n=0}^{\infty} \frac{1}{(2n)!(2n-1)!!} \sum_{\substack{\alpha_1\cdots\alpha_n \\ \beta_1\cdots\beta_n}} P_{\alpha_1\beta_1}^\dagger \cdots P_{\alpha_n\beta_n}^\dagger |0)(0| P_{\alpha_n\beta_n}\cdots P_{\alpha_1\beta_1} \\ &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} \sum_{\substack{\alpha_1\cdots\alpha_n \\ \beta_1\cdots\beta_n}} P_{\alpha_1\beta_1}^\dagger \cdots P_{\alpha_n\beta_n}^\dagger |0)(0| b_{\alpha_n\beta_n}\cdots b_{\alpha_1\beta_1} . \end{aligned} \quad (5)$$

Here by physical subspace we mean the boson subspace spanned by the so-called physical states defined by

$$P_{\alpha_1\beta_1}^\dagger P_{\alpha_2\beta_2}^\dagger \cdots P_{\alpha_n\beta_n}^\dagger |0) \sim \mathscr{P} b_{\alpha_1\beta_1}^\dagger b_{\alpha_2\beta_2}^\dagger \cdots b_{\alpha_n\beta_n}^\dagger |0) . \quad (6)$$

Aside from the projection operator $\mathscr{P}$, (4) clearly allows a finite boson representation of the fermion pair and multipole operators. The non-Hermiticity of the above Dyson transformation, i.e., $(a_\alpha^\dagger a_\beta^\dagger)_D \neq (a_\beta a_\alpha)_D^\dagger$, results from the very fact that the mappings (2) and (3), respectively, for the ket and bra states are totally different. As shown in I, however, this non-Hermiticity of the Dyson boson representation is actually not a drawback. In fact, it is a distinct advantage of the Dyson boson representation, in the sense that the boson bra states are often simple in structure and their relations to the original fermion states are very transparent. Furthermore, as long as one works with the physical ket states defined in (6), the projection operator $\mathscr{P}$ appearing in (4) can simply be replaced by unity.

The physical ket states as defined in (6) are, in general, very complicated in structure. In principle, one can project out from any given boson ket state the desired physical component by directly using the projection operator $\mathscr{P}$. However, in practice, the physical projection operator $\mathscr{P}$ as defined in (5) is very cumbersome to use. Therefore, in the following, we will employ a new approximate projection procedure to construct the physical ket states in the Dyson boson representation.

III. APPROXIMATE PROJECTION OF PHYSICAL STATES

Before describing the approximate projection procedure for constructing the physical states, it is worthwhile to observe a "paradox" in the Dyson boson representation.[12] Namely, when neglecting the physical projection operator $\mathscr{P}$, for any two-body interaction

$$a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma = a_\alpha^\dagger a_\gamma a_\beta^\dagger a_\delta - \delta_{\beta\gamma} a_\alpha^\dagger a_\delta ,$$

(4) gives rise to the following two different transformations:

$$(a_\alpha^\dagger a_\beta^\dagger)(a_\delta a_\gamma) \rightarrow P_{\alpha\beta}^\dagger b_{\gamma\delta} , \quad (7)$$

$$(a_\alpha^\dagger a_\gamma)(a_\beta^\dagger a_\delta) - \delta_{\beta\gamma} a_\alpha^\dagger a_\delta \rightarrow \sum_{\lambda\rho} b_{\alpha\lambda}^\dagger b_{\gamma\lambda} b_{\beta\rho}^\dagger b_{\delta\rho} - \delta_{\beta\gamma} \sum_\lambda b_{\alpha\lambda}^\dagger b_{\delta\lambda} . \quad (8)$$

In general, whereas (7) is a non-Hermitian transformation, (8) is clearly a Hermitian one. In fact, as long as (7) and (8) operate on physical states such as those given in (6), they are actually equivalent. However, because of the neglecting of the physical projection operator $\mathscr{P}$, the transformations (7) and (8) are completely different outside the physical subspace spanned by those states given in (6).

An interesting consequence of (7) and (8) is that, for the product $\hat{N}_F(\hat{N}_F-1)$ of the total fermion number operator $\hat{N}_F = \sum_\alpha a_\alpha^\dagger a_\alpha$, one gets the following two different mappings:

$$\hat{N}_F(\hat{N}_F-1) \rightarrow \begin{cases} \hat{Z} \equiv \sum_{\alpha\beta} P_{\alpha\beta}^\dagger b_{\alpha\beta} \\ 2\hat{n}(2\hat{n}-1) , \end{cases} \quad (9)$$

where $2\hat{n} = \sum_{\alpha\beta} b_{\alpha\beta}^\dagger b_{\alpha\beta}$ is the total boson number operator. It is not difficult to show that the $\hat{Z}$ operator defined in (9) is Hermitian and satisfies the relations[1]

$$\hat{Z}\mathscr{P}|\phi_n) = 2n(2n-1)\mathscr{P}|\phi_n) , \quad (10)$$

$$(\phi_n|\hat{Z}|\phi_n) \le 2n(2n-1) , \quad (11)$$

where $|\phi_n)$ is an arbitrary n-boson state normalized to 1 and the equality in (11) holds only when $|\phi_n)$ is also a physical state, i.e., $\mathscr{P}|\phi_n) = |\phi_n)$. By contrast, $|\phi_n)$ is always an eigenstate of $2\hat{n}(2\hat{n}-1)$ with the eigenvalue $2n(2n-1)$ no matter whether it is a physical state or not. If one uses the transformation (7) or (8) and performs an exact diagonalization of the resulting boson Hamiltonian in a boson basis, Eq. (11) can be used to distinguish between the physical and spurious eigenstates—although exact diagonalizations are usually not practical for systems with many valence particles. For our present purpose of constructing the physical states such as $\mathscr{P}\,|\,\phi_n)$ given in (10), the $\hat{Z}$ operator defined above is particularly useful because what one has to do is simply to find the eigenstates of $\hat{Z}$ with the correct eigenvalue $2n(2n-1)$. Here we will use the following procedure to do the job.

Given an arbitrary n-boson state $|\,\phi_n)$ which is normalized to 1, we write the νth-order approximate eigenstate of the $\hat{Z}$ operator as

$$\mathscr{T}_\nu\,|\,\phi_n)=\sum_{q=0}^{\nu} c_q(n)\hat{Z}^q\,|\,\phi_n)\ , \tag{12}$$

where the coefficients $c_q(n)$ are determined by the following $\nu+1$ conditions ($r=0,1,\ldots,\nu$):

$$(\phi_n\,|\,\hat{Z}^r\mathscr{T}_\nu\,|\,\phi_n)=\sum_{q=0}^{\nu} c_q(n)(\phi_n\,|\,\hat{Z}^{q+r}\,|\,\phi_n)$$
$$=[2n(2n-1)]^r\ . \tag{13}$$

The first condition in (13) with $r=0$ is simply the normalization $(\phi_n\,|\,\mathscr{T}_\nu\,|\,\phi_n)=1$. With this normalization condition, (13) means that we have treated $\mathscr{T}_\nu\,|\,\phi_n)$ as if it is an exact eigenstate of $\hat{Z}$ with the eigenvalue $2n(2n-1)$. Furthermore, by utilizing the expansion

$$|\,0)(0\,|\,=1-\hat{n}+\hat{n}(\hat{n}-1)/2!-\cdots$$

for the boson vacuum projection operator, it is not difficult to show that the physical projection operator $\mathscr{P}$ given in (5) can actually be written as a power series in $\hat{Z}$. Therefore, the operator $\mathscr{T}_\nu$, defined by (12) and (13), can be considered as a truncated version of the exact physical projection operator $\mathscr{P}$.

In what follows, we shall see that even in the lowest order of approximation by taking $\nu=1$ for (12) and (13), the first-order projected states such as

$$\mathscr{T}_1\,|\,\phi_n)=(c_0+c_1\hat{Z})\,|\,\phi_n)\ , \tag{14}$$

obtained from (12) by suppressing the argument n in the $c_q(n)$ coefficients, can be quite accurate in producing physical quantities. In this order of approximation, (13) can be easily solved for c_0 and c_1 to yield

$$c_0=\frac{\langle\hat{Z}^2\rangle-2n(2n-1)\langle\hat{Z}\rangle}{\langle\hat{Z}^2\rangle-\langle\hat{Z}\rangle^2}\ ,$$
$$c_1=\frac{2n(2n-1)-\langle\hat{Z}\rangle}{\langle\hat{Z}^2\rangle-\langle\hat{Z}\rangle^2}\ , \tag{15}$$

for which the notation $\langle\hat{Z}^q\rangle\equiv(\phi_n\,|\,\hat{Z}^q\,|\,\phi_n)$ has been used.

The present approximate projection method is particularly useful if it is applied to n-boson states $|\,\phi_n)$ such as

$$|\,\lambda_0^n)=\frac{1}{\sqrt{n!}}(\lambda_0^\dagger)^n\,|\,0)\ ,$$
$$|\,\bar{\lambda}_k\lambda_0^{n-1})=\frac{1}{\sqrt{(n-1)!}}\bar{\lambda}_k^\dagger(\lambda_0^\dagger)^{n-1}\,|\,0)\ , \tag{16}$$

where $\bar{\lambda}_k\equiv\lambda_{k\neq0}$ and the boson operators λ_k satisfy the commutation relations $[\lambda_k,\lambda_l^\dagger]=\delta_{kl}$ which are defined by

$$\lambda_k^\dagger=\frac{1}{\sqrt{2}}\sum_{\alpha\beta}\mathscr{Y}_{\alpha\beta}^{(k)}b_{\alpha\beta}^\dagger\ ,$$
$$\mathscr{Y}_{\alpha\beta}^{(k)}=-\mathscr{Y}_{\beta\alpha}^{(k)},\ \sum_{\alpha\beta}\mathscr{Y}_{\alpha\beta}^{(k)}\mathscr{Y}_{\alpha\beta}^{(l)}=\delta_{kl}\ . \tag{17}$$

By choosing the coefficients $\mathscr{Y}_{\alpha\beta}^{(k)}$ dynamically, for instance, through variational calculations such as

$$\delta(\lambda_0^n\,|\,(H)_D\mathscr{T}_\nu\,|\,\lambda_0^n)=0 \tag{18}$$

subject to appropriate constraints, where $(H)_D$ is the Dyson boson image of some shell-model Hamiltonian in question, the present approximate projection method given by (12)–(15) can be used very efficiently to construct the approximate physical ket states in the boson space such as $\mathscr{T}_\nu\,|\,\lambda_0^n)$, $\mathscr{T}_\nu\,|\,\bar{\lambda}_k\lambda_0^{n-1})$, etc. These approximately projected ket states, together with the unprojected boson bra states $(\lambda_0^n\,|$, $(\bar{\lambda}_k\lambda_0^{n-1}\,|$, etc., may then provide a useful non-Hermitian basis for describing the low-lying collective spectra in not only vibrational nuclei but also rotational nuclei. The transparent relations of the various states in this non-Hermitian basis to the original fermion states can be easily seen from the mappings given in (2) and (3).

When using the non-Hermitian basis such as that described above, there is one important thing to note. Namely, with $|\,\phi_n)$ representing the boson states defined in (16), although we have used the normalization condition $(\phi_n\,|\,\mathscr{T}_\nu\,|\,\phi_n)=1$ in (13), this still leaves the bra and ket states, $(\phi_n\,|$ and $\mathscr{T}_\nu\,|\,\phi_n)$ undetermined up to a constant γ, i.e., $(\phi_n\,|\,\gamma$ and $\gamma^{-1}\mathscr{T}_\nu\,|\,\phi_n)$ also satisfy the same normalization condition. Thus, as stressed in I, in calculating matrix elements, it is necessary to require

$$M=(\phi_n\,|\,\gamma(O)_D\frac{1}{\gamma'}\mathscr{T}_{\nu'}\,|\,\phi'_{n'})$$
$$=(\phi'_{n'}\,|\,\gamma'(O^\dagger)_D\frac{1}{\gamma}\mathscr{T}_\nu\,|\,\phi_n)^*\ , \tag{19}$$

which yields

$$|\,M\,|\,=[(\phi_n\,|\,(O)_D\mathscr{T}_{\nu'}\,|\,\phi'_{n'})(\phi'_{n'}\,|\,(O^\dagger)_D\mathscr{T}_\nu\,|\,\phi_n)^*]^{1/2}\ . \tag{20}$$

In (19) and (20), $(O)_D$ and $(O^\dagger)_D$ are Dyson boson images of some fermion operator O and its Hermitian conjugate $O^\dagger$ in question, and M represents the true matrix element. Note also that, depending on the structures of $(O)_D$ and $(O^\dagger)_D$, the boson numbers n and n' as well as the orders of approximate projection ν and ν' can be different from each other.

IV. APPLICATION TO A SIMPLIFIED SHELL MODEL

As a test of the present method of approximate projection of physical states in the Dyson boson representation, here we apply it to a simplified shell model defined by the following pairing Hamiltonian:

$$H=\sum_a h_a \hat{N}_a - G\sum_{ab}\sqrt{\Omega_a\Omega_b}A^{\dagger}_{00}(aa)A_{00}(bb)\ , \qquad (21)$$

where h_a and $\hat{N}_a$ are, respectively, the single-particle energy and the particle number operator for the ath j shell with pair degeneracy $\Omega_a=j_a+\frac{1}{2}$, and the pair operators are defined by

$$A^{\dagger}_{JM}(ab)=\frac{1}{\sqrt{2}}\sum_{m_a m_b}(j_a m_a j_b m_b\,|\,JM)a^{\dagger}_{\alpha}a^{\dagger}_{\beta}\ . \qquad (22)$$

Note that in (22) the notations $\alpha=(j_a m_a)$, $\beta=(j_b m_b)$, etc., have been used, and $(j_a m_a j_b m_b\,|\,JM)$ are Clebsch-Gordan coefficients for angular momentum coupling.

Corresponding to (22), it is useful to define

$$P^{\dagger}_{JM}(ab)=\frac{1}{\sqrt{2}}\sum_{m_a m_b}(j_a m_a j_b m_b\,|\,JM)P^{\dagger}_{\alpha\beta}\ ,$$
$$B^{\dagger}_{JM}(ab)=\frac{1}{\sqrt{2}}\sum_{m_a m_b}(j_a m_a j_b m_b\,|\,JM)b^{\dagger}_{\alpha\beta}\ . \qquad (23)$$

It follows from (1) that the multipole boson operators $B_{JM}(ab)$ satisfy the commutation relations

$$[B_{JM}(ab),B_{J'M'}(a'b')]=[B^{\dagger}_{JM}(ab),B^{\dagger}_{J'M'}(a'b')]=0\ ,$$
$$[B_{JM}(ab),B^{\dagger}_{J'M'}(a'b')]=\tfrac{1}{2}\delta_{JJ'}\delta_{MM'}[\delta_{aa'}\delta_{bb'}-(-1)^{j_a+j_b+J}\delta_{ab'}\delta_{ba'}]\ . \qquad (24)$$

Thus, from (4), one obtains the following Dyson transformation:[7]

$$[A^{\dagger}_{JM}(ab)]_D=P^{\dagger}_{JM}(ab)$$
$$=B^{\dagger}_{JM}(ab)-2\sum_{\substack{cdL\\K_1K_2K_3}}\hat{K}_1\hat{K}_2\hat{K}_3\hat{L}\begin{Bmatrix}K_2 & K_3 & L\\ j_c & j_b & j_d\end{Bmatrix}\begin{Bmatrix}K_1 & L & J\\ j_b & j_a & j_c\end{Bmatrix}[B^{\dagger}_{K_1}(ca)\times[B^{\dagger}_{K_2}(db)\times\tilde{B}_{K_3}(cd)]^{(L)}]^{(J)}_M\ ,$$
$$[A_{JM}(ab)]_D=B_{JM}(ab)\ , \qquad (25)$$
$$(\hat{N}_a)_D=2\sum_{JMb}B^{\dagger}_{JM}(ab)B_{JM}(ab)\ ,$$

where $[\ \times\ ]^{(J)}_M$, etc., represent standard angular momentum couplings, the curly brackets denote 6-j symbols, and the notations $\hat{K}=\sqrt{2K+1}$, $\tilde{B}_{K\mu}=(-1)^{K-\mu}B_{K,-\mu}$, etc., have been used. Since we are going to use the approximate physical ket states given in (12)–(15), the physical projection operator $\mathscr{P}$ associated with the above transformation can be neglected. Furthermore, under the above transformation, the Dyson boson image of the pairing Hamiltonian (21) is readily obtained as

$$(H)_D=2\sum_{abJM}h_aB^{\dagger}_{JM}(ab)B_{JM}(ab)$$
$$-G\sum_{ab}\sqrt{\Omega_a\Omega_b}P^{\dagger}_{00}(aa)B_{00}(bb)\ , \qquad (26)$$

where $P^{\dagger}_{00}(aa)$ is given explicitly in (25). Due to the fact that

$$P^{\dagger}_{00}(aa)B_{00}(bb)\neq[P^{\dagger}_{00}(bb)B_{00}(aa)]^{\dagger}\ ,$$

the above boson Hamiltonian is non-Hermitian.

Next, as a special case of (17), we introduce the following unitary transformation:

$$S^{\dagger}_k=\sum_a\alpha_{ak}B^{\dagger}_{00}(aa)\ ,$$
$$\sum_a\alpha_{ak}\alpha_{al}=\delta_{kl},\ \sum_k\alpha_{ak}\alpha_{bk}=\delta_{ab}\ . \qquad (27)$$

By choosing the coefficients α_{ak} through variational calculations such as (32) below, one can use $S^{\dagger}_0$ to represent the Cooper pair of the pairing correlations in nuclei. Then, the ground state of the pairing Hamiltonian (21) in the Dyson boson representation can be approximated by the boson condensate bra state

$$(S^n_0\,|=(0\,|\frac{S^n_0}{\sqrt{n!}} \qquad (28)$$

and the corresponding physical ket state $\mathscr{P}\,|S^n_0)$. Using the first-order projection result (14) and (15), the physical ket state $\mathscr{P}\,|S^n_0)$ can be constructed approximately as

$$\mathscr{T}_1\,|S^n_0)=\left[c_0+2c_1\sum_{abJM}P^{\dagger}_{JM}(ab)B_{JM}(ab)\right]|S^n_0)\ ,$$
$$c_0=1-\frac{n}{2R_n}[1-(n-1)t_2]\ , \qquad (29)$$
$$c_1=\frac{1}{4R_n}\ ,$$

where

$$\hat{Z}=2\sum_{abJM}P^{\dagger}_{JM}(ab)B_{JM}(ab)$$

has been used, and

$$R_n=1-t_2+(n-2)(t_4-t_2^2)/(1+\tfrac{1}{2}t_2)\ ,$$
$$t_q=\sum_a\alpha^2_{a0}\left[\frac{\alpha_{a0}}{\sqrt{\Omega_a}}\right]^q\ . \qquad (30)$$

Writing out (29) explicitly, we obtain

$$\mathcal{T}_1|S_0^n)=|S^n)-\frac{\sqrt{n(n-1)}}{R_n}\left[\sqrt{n-1}\sum_{ak}{}'\frac{\alpha_{a0}^3\alpha_{ak}}{\Omega_a}S_k^\dagger|S_0^{n-1})+\tfrac{1}{2}\sum_{akl}{}'\frac{\alpha_{a0}^2\alpha_{ak}\alpha_{al}}{\Omega_a}S_k^\dagger S_l^\dagger|S_0^{n-2})\right.$$
$$\left.+\tfrac{1}{2}\sum_{\substack{ab\\K\mu}}{}'\frac{\alpha_{a0}\alpha_{b0}}{\sqrt{\Omega_a\Omega_b}}B_{K\mu}^\dagger(ab)\tilde{B}_{K\mu}^\dagger(ab)|S_0^{n-2})\right], \tag{31}$$

where $\tilde{B}_{K\mu}=(-1)^{K-\mu}B_{k,-\mu}$, and $\sum'$ sums only over nonvanishing values of k, l, or K. In the degenerate limit where $h_a=h$ and $\alpha_{a0}=\sqrt{\Omega_a/\Omega}$ with $\Omega=\sum\Omega_a$, it is not difficult to check that (31) actually gives exactly the first few (and relevant) components of the exact physical state $\mathcal{P}|S_0^n)$, [cf. Eq. (14) of I]. For nondegenerate multiple j shells, although the first few components of $\mathcal{P}|S_0^n)$ given in (31) are no longer exact, in the following we shall see that they are still quite accurate in producing physical quantities. Similar approximate projections can also be used to construct physical states such as $\mathcal{P}B_{2\mu}^\dagger(ab)|S_0^{n-1})$, etc.

Having obtained the approximate physical ket state $\mathcal{T}_1|S_0^n)$, one can evaluate the approximate ground state energy

$$E_0(n)=(S_0^n|(H)_D\mathcal{T}_1|S_0^n)$$

by the variational principle

$$\frac{\partial}{\partial\alpha_{a0}}(S_0^n|(H)_D\mathcal{T}_1|S_0^n)=0\ , \tag{32}$$

subject to the constraint $\sum_a\alpha_{a0}^2=1$. In (32), $(H)_D$ is the Dyson boson image of the pairing Hamiltonian given explicitly in (26). Alternatively, one can first use (27)—(31) to obtain an unitary boson mapping for the monopole pair operators $A_{00}^\dagger(aa)$ and then apply the resulting boson mapping to the ground state calculation for the pairing Hamiltonian (21). This latter approach is described below.

With the Dyson boson images of $A_{00}^\dagger(aa)$ and $A_{00}(aa)$ given explicitly in (25), (28), and (31) enable one to evaluate straightforwardly

$$(S_0^{n+1}|[A_{00}^\dagger(aa)]_D\mathcal{T}_1|S_0^n)$$
$$=\sqrt{n+1}\alpha_{a0}\left[1-\frac{n\alpha_{a0}^2}{\Omega_a}Y_a(n)\right] \tag{33}$$

and

$$(S_0^n|[A_{00}(aa)]_D\mathcal{T}_1|S_0^{n+1})=\sqrt{n+1}\alpha_{a0}Y_a(n+1)\ , \tag{34}$$

where

$$Y_a(n)=1-\frac{n-1}{R_n}\left[\frac{\alpha_{a0}^2}{\Omega_a}-t_2\right], \tag{35}$$

with R_n and t_2 defined in (30). Thus, following (19), we take the geometric mean of (33) and (34) to obtain the true matrix elements

$$(S_0^{n+1}|\gamma_0(n+1)[A_{00}^\dagger(aa)]_D\frac{1}{\gamma_0(n)}\mathcal{T}_1|S_0^n)=\sqrt{n+1}\alpha_{a0}\left\{Y_a(n+1)\left[1-\frac{n\alpha_{a0}^2}{\Omega_a}Y_a(n)\right]\right\}^{1/2}$$
$$\equiv\sqrt{n+1}f_{a0}(n)\ . \tag{36}$$

Similarly, we obtain for $k\neq0$

$$(S_0^{n+1}|\gamma_0(n+1)[A_{00}^\dagger(aa)]_D\frac{1}{\gamma_k(n)}\mathcal{T}_1|S_0^{n-1}\bar{S}_k)=-\sqrt{n(n+1)}\frac{\alpha_{a0}^2\alpha_{ak}}{\Omega_a\sqrt{R_{n+1}}}\equiv\sqrt{n(n+1)}g_{ak}(n)\ , \tag{37}$$

where

$$|S_0^{n-1}\bar{S}_k)\equiv S_{k\neq0}^\dagger|S_0^{n-1})\ .$$

In obtaining (37), we have set $\sum_a\alpha_{a0}^3\alpha_{ak}/\Omega_a$ with $k\neq0$ to zero since they vanish exactly in the degenerate limit and are usually quite small even in the case of nondegenerate j shells. Under this approximation, there is no need to explicitly construct the approximate physical states $\mathcal{T}_1|S_0^{n-1}\bar{S}_k)$ because only the leading terms $|S_0^{n-1}\bar{S}_k)$ in them contribute to the result given in (37). Finally, for the ground state expectation values of the fermion number operators $\hat{N}_a$, we get

$$(S_0^n|\gamma_0(n)(\hat{N}_a)_D\frac{1}{\gamma_0(n)}\mathcal{T}_1|S_0^n)$$
$$=(S_0^n|(\hat{N}_a)_D\mathcal{T}_1|S_0^n)=2n\alpha_{a0}^2Y_a(n)\ . \tag{38}$$

It is understood here that the γ_0 and γ_k factors in (36)—(38) are used to ensure that $(S_0^n|\gamma_0^{(n)}$ and $\gamma_0^{-1}(n)\mathcal{T}_1|S_0^n)$, etc., represent the correct non-Hermitian basis in the Dyson boson representation.

Now, to get unitary boson mappings for $A_{00}^\dagger(aa)$ and $\hat{N}_a$, we first make the following mapping for the states[7]

$$\frac{1}{\gamma_0(n)}\mathcal{T}_1 |S_0^n) \to |s_0^n) ,$$
$$\frac{1}{\gamma_k(n)}\mathcal{T}_1 |S_0^{n-1}\bar{S}_k) \to |s_0^{n-1}\bar{s}_k), \text{ etc. }, \tag{39}$$

and

$$(S_0^n|\gamma_0(n) \to (s_0^n| ,$$
$$(S_0^{n-1}\bar{S}_k|\gamma_k(n) \to (s_0^{n-1}\bar{s}_k|, \text{ etc. }, \tag{40}$$

where s_0 and $\bar{s}_k \equiv s_{k\neq 0}$ are new s-boson operators satisfying the commutation relation $[s_k, s_l^\dagger]=\delta_{kl}$, and the boson states on the right-hand sides of (38) and (39) are defined in exactly the same way as their counterparts appearing on the left-hand sides. Then, by the method of equating the corresponding matrix elements in the old and new basis,[13] it is straightforward to obtain

$$A_{00}^\dagger(aa) = \sum_k s_k^\dagger f_{ak}(\hat{n}) + \sum_{k\neq 0} s_0^\dagger s_0^\dagger s_k g_{ak}(\hat{n}) + \cdots$$
$$= [A_{00}(aa)]^\dagger , \tag{41}$$

$$\hat{N}_a = 2\alpha_{a0}^2 s_0^\dagger s_0 Y_a(\hat{n}) + \cdots , \tag{42}$$

where $\hat{n}$ is the total boson operator $\sum_k s_k^\dagger s_k$, and f_{a0}, g_{ak}, and Y_a are functions of $n \to \hat{n}$ defined in (35)–(37). Since $s_k^\dagger$ terms with $k\neq 0$ in (41) will not contribute to the following ground state calculation, the functions f_{ak} with $k\neq 0$ are not given explicitly. Furthermore, from the way they are derived, the above unitary boson mappings involve nonperturbative instead of perturbative expansions. Thus, in forming the products $A_{00}^\dagger(aa)A_{00}(bb)$ to obtain the unitary boson mapping for the pairing interaction given in (21), we will keep only those terms explicitly shown in (41). Therefore, for the collective subspace consisting only of the state $|s_0^n)$, the boson image of the pairing Hamiltonian (21) is given approximately by

$$H_B = 2\sum_a h_a \alpha_{a0}^2 Y_a(\hat{n}) s_0^\dagger s_0 - G\sum_{ab}\sqrt{\Omega_a\Omega_b}\, s_0^\dagger [f_{a0}(\hat{n}) f_{b0}(\hat{n}) + s_0^\dagger s_0 \sum_{k\neq 0} g_{ak}(\hat{n}) g_{bk}(\hat{n})] s_0 , \tag{43}$$

where $\hat{n}$ is equivalent to $s_0^\dagger s_0$ since, after appropriate normal orderings if necessary, the non-s_0 bosons have been dropped in arriving at (43). The above boson Hamiltonian is clearly Hermitian and depends on the parameters α_{a0} (but not α_{ak} with $k\neq 0$ due to the summation over nonvanishing k), through the functions Y_a, f_{a0}, and g_{ak} defined in (35)–(37).

Next, with $|s_0^n)$ representing the ground state of the pairing Hamiltonian (21) for the $2n$ particle system, (42) and (43) yield the following approximate ground state energy and occupation probabilities:

$$E_0(n) = (s_0^n|H_B|s_0^n)$$
$$= 2n\sum_a h_a \alpha_{a0}^2 Y_a(n) - nG\left\{\sum_a \alpha_{a0} Y_a^{1/2}(n)[\Omega_a - (n-1)\alpha_{a0}^2 Y_a(n-1)]^{1/2}\right\}^2 - n(n-1)G(t_2 - t_1^2)/R_n , \tag{44}$$

$$v_a^2(n) = \frac{1}{2\Omega_a}(s_0^n|\hat{N}_a|s_0^n) = n\alpha_{a0}^2 Y_a(n)/\Omega_a , \tag{45}$$

where R_n, t_1, and t_2 are given in (30). As for the parameters α_{a0} appearing in (44) and (45), they are to be determined by minimizing the ground state energy (44), i.e., by

$$\frac{\partial}{\partial \alpha_{a0}} E_0(n) = 0 , \tag{46}$$

subject to the constraint

$$\sum_a \alpha_{a0}^2 = 1 . \tag{47}$$

Note that if $Y_a(n)$ and $Y_a(n-1)$ defined by (35) are set to unity, (44)–(47) reduce to the one-boson approximation used by Klein *et al.*[14] for a Holstein-Primakoff–type boson mapping of the pairing Hamiltonian (21). In Tables I and II, this one-boson approximation (denoted as $1-b$) together with the present calculation with (44)–(47) (denoted as PW) are applied to some even tin and nickel isotopes. There, results of exact diagonalizations of the pairing Hamiltonian (21) are also given for comparison. It can be seen from the tables that the present calculation yields very accurate ground state energies as well as occupation probabilities for all the tin and nickel isotopes shown, although only the first-order projection of physical states (14) and (15) has been used. For the tin isotopes, the one-boson approximation of Klein *et al.* also gives reasonably accurate results. In the case of nickel isotopes, because of the small effective pair degeneracy, e.g.,

$$\Omega_{\text{eff}}^{-1} = \sum_a \alpha_{a0}^4/\Omega_a \sim 0.3 ,$$

this one-boson approximation breaks down for $n \gtrsim 3$. Even so, overall speaking, it is still more accurate than the number operator approximation (NOA) of Otsuka and Arima,[15] of which a detailed account has been given in Ref. 16.

Since the accuracy of the seniority-scheme boson mapping with Pauli reduction factors derived in I is essentially at the same level as that of the one-boson approximation of Klein *et al.*, the present first-order projection of physical states (14) and (15) can be considered as an improvement over the method used in I. However, what is more important is that, whereas the method used in I is applicable only near the seniority-scheme limit, the present approximate projection method can be applied to not only vibrational but also rotational nuclei. The main purpose of the application to the pairing Hamiltonian (21) made here is to illustrate as well as to test the accuracy of the present approximate projection method described in (12)–(20).

It should be emphasized here that the present first-

TABLE I. Values of the ground state energies and occupation probabilities for the tin isotopes. The parameters used for the pairing Hamiltonian are the following: $h_a=0$, 0.22, 1.90, 2.20, 2.80 MeV; $G=0.187$ MeV; $\Omega_a=3$, 2, 1, 2, 6.

n	E_0	$v^2_{d_{5/2}}$	$v^2_{g_{7/2}}$	$v^2_{s_{1/2}}$	$v^2_{d_{3/2}}$	$v^2_{h_{11/2}}$	
	−2.626	0.3270	0.2126	0.0287	0.0230	0.0157	$1-b$
2	−2.623	0.3249	0.2143	0.0287	0.0230	0.0156	PW
	−2.624	0.3252	0.2140	0.0287	0.0230	0.0156	Exact
	−3.11	0.732	0.592	0.080	0.061	0.0386	$1-b$
5	−3.085	0.712	0.609	0.0789	0.0604	0.0382	PW
	−3.084	0.7153	0.6070	0.0784	0.0599	0.0380	Exact
	−0.807	0.952	0.876	0.133	0.095	0.053	$1-b$
7	−0.741	0.913	0.912	0.127	0.090	0.051	PW
	−0.700	0.9361	0.9095	0.1200	0.0850	0.0481	Exact

order calculation with (44)—(47) slightly violates the Pauli principle for $n>2$, owing to the fact that the relevant physical states in the boson space are constructed approximately with the first-order projection. This violation of the Pauli principle can become very serious when one goes beyond the middle of the major shell. To remedy this, one can switch to the hole formalism and use the boson representation for the hole pairs instead of the particle pairs. Furthermore, one can also improve the present first-order calculation by using higher-order projections to construct the physical states. In fact, we have performed the second-order projection by taking $\nu=2$ in (12) and (13) and have obtained improved results for the pairing Hamiltonian (21). Another point to note is that if one uses (28)—(32) directly for the ground state calculation for the pairing Hamiltonian, the ground state energies and occupation probabilities thus obtained are only slightly worse than those obtained from using (44)—(47). Therefore, (28)—(32) and (44)—(47) can be considered as essentially the same. The purpose of using (44)—(47) here is mainly to illustrate how the present approximate projection method in the non-Hermitian Dyson boson representation can be applied to derive an IBM-type Hamiltonian from the shell model.

TABLE II. Values of the ground state energies and occupation probabilities for the nickel isotopes. The parameters used for the pairing Hamiltonian are the following: $h_a=0$, 0.78, 1.56, 4.52 MeV; $G=0.331$ MeV; $\Omega_a=2$, 3, 1, 5.

n	E_0	$v^2_{p_{3/2}}$	$v^2_{f_{5/2}}$	$v^2_{p_{1/2}}$	$v^2_{g_{9/2}}$	
	−2.12	0.635	0.194	0.081	0.014	$1-b$
2	−2.09	0.624	0.201	0.081	0.013	PW
	−2.10	0.629	0.198	0.081	0.013	Exact
	−1.87	0.847	0.354	0.144	0.020	$1-b$
3	−1.77	0.801	0.383	0.148	0.020	PW
	−1.75	0.764	0.404	0.153	0.021	Exact
	1.44	1.06	0.784	0.379	0.031	$1-b$
5	1.72	0.925	0.866	0.395	0.031	PW
	1.70	0.934	0.856	0.408	0.031	Exact

V. APPROXIMATE CONSTRUCTION OF DEFORMED STATES

In the description of the low-lying collective spectra of deformed nuclei using the Dyson boson representation, it is useful to consider the boson states $|\lambda_0^n)$, $|\bar{\lambda}_k\lambda_0^{n-1})$, etc., defined in (16) and (17). In particular, in the case of the boson condensate

$$|\lambda_0^n)=\frac{1}{\sqrt{n!2^n}}\left[\sum_{\alpha\beta}\mathscr{Y}_{\alpha\beta}b^\dagger_{\alpha\beta}\right]^n|0) , \tag{48}$$

where $\mathscr{Y}_{\alpha\beta}\equiv\mathscr{Y}^{(0)}_{\alpha\beta}$ are defined in (17), the first-order projection (14) and (15) readily yields the following approximate physical state:

$$\mathscr{T}_1|\lambda_0^n)=|\lambda_0^n)-2c_1\sqrt{n(n-1)}\times\left[\sum_{\alpha\beta\gamma\delta}\mathscr{Y}_{\alpha\beta}\mathscr{Y}_{\gamma\delta}b^\dagger_{\alpha\gamma}b^\dagger_{\beta\delta}|\lambda_0^{n-2})-2X_1|\lambda_0^n)\right] , \tag{49}$$

where

$$c_1=\frac{1+X_1/\sqrt{n(n-1)}}{4[X_2-(X_1)^2]} , \tag{50a}$$

$$X_1=\sqrt{n(n-1)}\sum_{\alpha\beta\gamma\delta}\mathscr{Y}_{\alpha\beta}\mathscr{Y}_{\gamma\delta}\mathscr{Y}_{\alpha\gamma}\mathscr{Y}_{\beta\delta} , \tag{50b}$$

$$X_2=1-\sum_{\alpha\beta\gamma\delta}\mathscr{Y}_{\alpha\beta}\mathscr{Y}_{\gamma\delta}\mathscr{Y}_{\alpha\gamma}\mathscr{Y}_{\beta\delta}+4(n-2)\sum_{\alpha\beta\gamma\delta\lambda\rho}\mathscr{Y}_{\alpha\gamma}\mathscr{Y}_{\beta\delta}\mathscr{Y}_{\gamma\delta}\mathscr{Y}_{\alpha\lambda}\mathscr{Y}_{\beta\rho}\mathscr{Y}_{\lambda\rho}+(n-2)(n-3)\left[\sum_{\alpha\beta\gamma\delta}\mathscr{Y}_{\alpha\beta}\mathscr{Y}_{\gamma\delta}\mathscr{Y}_{\alpha\gamma}\mathscr{Y}_{\beta\delta}\right]^2 . \tag{50c}$$

It is not difficult to see that in the spherical limit with

$$\mathscr{Y}_{\alpha\beta}=\frac{\alpha_{a0}}{\sqrt{2\Omega_a}}\delta_{\tilde{\alpha}\beta}=(-1)^{j_a-m_a}\delta_{j_aj_b}\delta_{-m_a,m_b}\frac{\alpha_{a0}}{\sqrt{2\Omega_a}} , \tag{51}$$

$(\lambda_0^n|$ and $\mathcal{T}_1|\lambda_0^n)$ given above reduce to the spherical states $(S_0^n|$ and $\mathcal{T}_1|S_0^n)$ given in (28) and (31), respectively. However, by choosing more general coefficients $\mathcal{Y}_{\alpha\beta}$ than those given in (51), $(\lambda_0^n|$ and $\mathcal{T}_1|\lambda_0^n)$ can also be used to represent the intrinsic state associated with the rotational band built on the ground state of a well-deformed nucleus. For instance, in terms of the multipole boson operators $B_{JM}^\dagger(ab)$ defined in (23), (48) can be rewritten as

$$|\lambda_0^n)=\frac{1}{\sqrt{n!}}\left[\sum_{abJM}\xi_{JM}(ab)B_{JM}^\dagger(ab)\right]^n|0)\ , \qquad (52)$$

where

$$\xi_{JM}(ab)=\sum_{m_a m_b}(j_a m_a j_b m_b\,|\,JM)\mathcal{Y}_{\alpha\beta}\ , \qquad (53)$$

and thus

$$\xi_{JM}(ab)=-(-1)^{j_a+j_b+J}\xi_{JM}(ba)\ ,$$
$$\sum_{abJM}\xi_{JM}(ab)\xi_{JM}(ab)=1\ . \qquad (54)$$

If axial symmetry is assumed, the original $\mathcal{Y}_{\alpha\beta}$ coefficients given in (48)—(50) have to be chosen in such a way that $\xi_{JM}(ab)$ vanish for $M\neq 0$. Moreover, if one further assumes the S-D dominance, then $\xi_{00}(aa)$ and $\xi_{20}(ab)$ are the only nonvanishing coefficients appearing in (52)—(54). Here, we will only assume the axial symmetry.

Because of the inclusion of the quadrupole and higher multipole bosons in (52), the deformed boson condensate $(\lambda_0^n|$ and the corresponding approximate physical state $\mathcal{T}_1|\lambda_0^n)$ clearly do not possess good angular momentum. Namely, e.g.,

$$(\hat{L}^2)_D\mathcal{T}_1|\lambda_0^n)\neq I(I+1)\mathcal{T}_1|\lambda_0^n)\ , \qquad (55)$$

where $(\hat{L}^2)_D$ is the Dyson boson image of the angular momentum operator $\hat{L}^2$, which yields the eigenvalue $I(I+1)$ when acting on states with good total angular momentum I. In the following, we will describe an approximate method for constructing deformed physical states with good angular momentum.

First of all, we note that in the original fermion space the angular momentum operator $\hat{L}^2$ is given as

$$\hat{L}^2=\tfrac{1}{3}\sum_{ab\mu}\hat{j}_a^{(3)}\hat{j}_b^{(3)}U_{1\mu}^\dagger(aa)U_{1\mu}(bb)\ , \qquad (56)$$

where

$$\hat{j}^{(3)}\equiv[j(j+1)(2j+1)]^{1/2}$$

and

$$U_{1\mu}^\dagger(aa)=\sum_{m_a}(j_a m_a j_a \mu-m_a\,|\,1\mu)(-1)^{j_a-m_a+\mu}a_{j_a m_a}^\dagger a_{j_a,m_a-\mu}\ . \qquad (57)$$

By rewriting (56) as

$$\hat{L}^2=\sum_{abJM}[J(J+1)-2j_a(j_a+1)]A_{JM}^\dagger(ab)A_{JM}(ab)+\sum_a j_a(j_a+1)\hat{N}_a \qquad (58)$$

and using the non-Hermitian Dyson transformation (25) for the fermion pair and number operators, we obtain the following boson image of the angular momentum operator $\hat{L}^2$:

$$\hat{\mathcal{L}}^2\equiv(\hat{L}^2)_D$$
$$=\sum_{abJM}[J(J+1)-2j_a(j_b+1)]P_{JM}^\dagger(ab)B_{JM}(ab)+2\sum_{abJM}j_a(j_a+1)B_{JM}^\dagger(ab)B_{JM}(ab)\ , \qquad (59a)$$

or

$$\hat{\mathcal{L}}^2=\sum_{abJM}J(J+1)B_{JM}^\dagger(ab)B_{JM}(ab)-2\sum_{\substack{abcdJM\\K_1K_2K_3K_4}}[J(J+1)-2j_a(j_a+1)]\hat{K}_1\hat{K}_2\hat{K}_3\hat{K}_4\begin{Bmatrix}K_2 & K_3 & K_4\\ j_c & j_b & j_d\end{Bmatrix}\begin{Bmatrix}K_1 & K_4 & J\\ j_b & j_a & j_c\end{Bmatrix}$$
$$\times[B_{K_1}^\dagger(ca)\times[B_{K_2}^\dagger(db)\times\tilde{B}_{K_3}(cd)]^{(K_4)}]_M^{(J)}B_{JM}(ab)\ . \qquad (59b)$$

This boson image of the angular momentum operator $\hat{L}^2$ is very useful since it has the following properties:

$$\hat{\mathcal{L}}^2\mathcal{P}\,|\phi_n;Iq)=I(I+1)\mathcal{P}\,|\phi_n;Iq)\ , \qquad (60)$$

but, in general,

$$\hat{\mathcal{L}}^2\,|\phi_n;Iq)\neq I(I+1)\,|\phi_n;Iq)\ , \qquad (61)$$

where $|\phi_n;Iq)$ denotes an arbitrary boson state with good angular momentum I, and $\mathcal{P}\,|\phi_n;Iq)$ the corresponding physical state. Thus, the construction of a physical state with good angular momentum I simply amounts to finding a simultaneous eigenstate of $\hat{\mathcal{L}}^2$ and

$$\hat{Z}=2\sum_{abJM}P_{JM}^\dagger(ab)B_{JM}(ab)$$

with the desired eigenvalues $I(I+1)$ and $2n(2n-1)$, respectively, for $\hat{\mathcal{L}}^2$ and $\hat{Z}$.

Therefore, for a given deformed n-boson state $|\phi_n)$ which is axially symmetric and normalized to unity, the corresponding physical state with good angular momentum I can be approximately constructed as

$$\mathscr{T}_\nu^{(I)} | \phi_n) = \sum_{p=0}^{\nu} \sum_{q=0}^{p} c_{pq} (\hat{\mathscr{L}}^2)^{p-q} \hat{Z}^q | \phi_n) , \tag{62}$$

where the coefficients c_{pq} are determined by the conditions

$$(\phi_n | (\hat{\mathscr{L}}^2)^r \hat{Z}^s \mathscr{T}_\nu^{(I)} | \phi_n) = [I(I+1)]^r [2n(2n-1)]^s , \quad 0 \le r+s \le \nu . \tag{63}$$

Here, because of the inclusion of the approximate projection of angular momentum, (62) and (63) are clearly generalizations of the previous approximate projection method described in (12) and (13). In the first order of approximation by taking $\nu=1$, (62) and (63) yield the following approximate physical state with angular momentum I:

$$\mathscr{T}_1^{(I)} | \phi_n) = (c_{00} + c_{10} \hat{\mathscr{L}}^2 + c_{11} \hat{Z}) | \phi_n) , \tag{64}$$

with the coefficients c_{00}, c_{10}, and c_{11} to be solved from the equations

$$\begin{aligned} (\phi_n | \mathscr{T}_1^{(I)} | \phi_n) &= 1 , \\ (\phi_n | \hat{\mathscr{L}}^2 \mathscr{T}_1^{(I)} | \phi_n) &= I(I+1) , \\ (\phi_n | \hat{Z} \mathscr{T}_1^{(I)} | \phi_n) &= 2n(2n-1) . \end{aligned} \tag{65}$$

The above first-order projection of both physical states and angular momenta can be easily applied to the deformed boson state defined in (52). The resulting approximate physical state $\mathscr{T}_1^{(I)} | \lambda_0^n)$ can then be used for various calculations such as the ground state variational calculation

$$\delta (\lambda_0^n | (H)_D \mathscr{T}_1^{(I)} | \lambda_0^n) = 0 , \tag{66}$$

subject to the constraints given in (54). Because of the assumption of axial symmetry, only $\xi_{JM}(ab)$ with $M=0$ are kept in (52)–(54). Furthermore, $(H)_D$ appearing in (66) is the Dyson boson image of some shell-model Hamiltonian such as the pairing-plus-quadrupole interaction given in the pairing expansion form, i.e.,

$$(H)_D = \sum_a h_a (\hat{N}_a)_D + \sum_{\substack{abcd \\ JM}} V_J(abcd) [A_{JM}^\dagger(ab)]_D [A_{JM}(cd)]_D . \tag{67}$$

The reason for using the non-Hermitian Hamiltonian boson image (67) instead of the alternative Hermitian version implied by (8) is that the former has a less serious problem with the spurious states in the Dyson boson space, and thus yields better results for actual calculations with the various approximate physical states obtained by the present projection method. The advantage of using the non-Hermitian Hamiltonian boson image in the Dyson boson representation has also been emphasized in Ref. 5.

Before making further applications of the present projection method given by (62)–(65) for deformed nuclei, it is important to test, for instance, the accuracy of the ground state variational calculation described in (66). This will be done in a future publication. There, the problem of deriving an IBM-type boson Hamiltonian for deformed nuclei will also be attacked. However, we have been able to apply the present projection method at its first order, i.e., (64) and (65), together with (66) to a much simpler problem:[17] namely, the SU(3) boson model for deformed nuclei. For this boson model, only the angular momentum projection is relevant and thus one sets the coefficient c_{11} to zero in (64) and uses only the first two equations in (65). It is not too difficult to show that this kind of first-order angular momentum projection actually yields the exact result for the ground state band of the SU(3) boson model. Therefore, there is good reason to believe that the present projection method in lower orders may well be adequate for the Dyson boson description of deformed nuclei.

VI. SUMMARY AND DISCUSSION

In this paper we have developed a new method of approximate projection to treat the spurious state problem in the Dyson boson description of nuclear collective motion. In this method, successive orders of approximate projection are used to generate the first few relevant components of the various physical states in the non-Hermitian Dyson boson representation of the shell model. An important feature of the present approximate projection method is that it is applicable not only to spherical nuclei but also to deformed nuclei. And, in the case of deformed nuclei, the present approximate projection method has also been generalized to include the approximate projection of angular momentum for deformed states. In general, depending on what physical quantities one is calculating as well as what physical system one is studying, higher orders of approximate projection of physical states may have to be used. However, for low-lying collective states, usually the first few orders of the present approximate projection method should be adequate, as has been clearly shown by the application to the ground state calculation for the multi-j pairing model done in Sec. IV. Further numerical calculations still have to be done in order to see if the approximate projection method generalized in Sec. V at its lowest order will be adequate for the ground state band of deformed nuclei.

Although in this paper we have restricted ourselves to the Dyson boson representation of even-mass nuclear systems, with only minor modifications, the present approximate projection method of physical states can also be applied to the Dyson boson-fermion representation of odd-mass nuclear systems.[2] Furthermore, as has been shown in the application to the multi-j pairing model, the present method is also very useful for using the Dyson boson representation as an intermediate step to derive an IBM-type boson Hamiltonian for the original shell model fermion problem. Further applications in this direction, especially to the case of deformed nuclei, will be undertaken in a future publication. Suffice it to say here that the approximate projection method described in the text has provided the necessary tool for tackling the problems.

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