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Liquid-gas phase transitions in finite nuclear matter

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A procedure is developed to calculate the chemical potential of a system of fermions at high temperatures in the independent particle model. This is then used to investigate the occurrence of liquid-gas phase transitions in finite nuclei employing various zero-range nuclear effective interactions. Finite size effects and the Coulomb force are found to lead to a sizeable reduction (~ 8 MeV) in the "critical" temperature as compared to the case of infinite nuclear matter.

I. INTRODUCTION

The occurrence of a first order phase transition in nuclear matter has been the subject of numerous investigations[1–8] that study the transition from the liquidlike phase of ordinary nuclear matter (as encountered at low excitation energies) to a gaseous phase where the average interparticle distance is much larger than the range of the interparticle interaction. Associated with such a phase transition is a critical temperature above which only the gaseous phase can exist. This critical temperature is very interesting in view of experimental results from relativistic heavy-ion[9] and ultrarelativistic proton[10] reactions. The isobaric fragment yields from these reactions obey a power law characteristic of condensation near the critical point as described by the droplet model of Fisher.[11] The calculations,[1–7] however, deal only with infinite nuclear matter and, therefore, ignore surface and Coulomb effects. Evidently, the hot piece of nuclear matter produced in any nuclear collision cannot have more than a few hundred nucleons and so is not adequately described by the properties of infinite nuclear matter. In an earlier paper by the present authors,[8] this point was raised where it was demonstrated, by a rather crude method, that the finite size effects can be quite large and hence cannot be neglected when the equation of state and the critical temperature are calculated. In the present work, we attempt to study liquid-gas phase transitions in finite nuclear systems and to determine the associated critical temperatures.

II. PHASE TRANSITIONS IN FINITE SYSTEMS

A phase transition, strictly speaking, can only occur in the thermodynamic limit in that the critical singularities associated with such a transition appear only for a system with an infinite number of particles. This is simply because the partition function of any finite system must be an analytic function of the temperature and of all the fields entering the Hamiltonian. As an example, the specific heat displays a sharp lambda-type singularity at the critical temperature $T_c(\infty)$ of a liquid-gas transition in an infinite system. The finite system specific heat, on the other hand, does not exhibit such a sharp singularity, but it has a large peak at a temperature $T_{\max}(A)$, which approaches $T_c(\infty)$ as the number of particles $A \to \infty$. The temperature $T_{\max}(A)$ can be regarded as the critical temperature for a finite system of A particles so that the effect of the finite size is in general to shift the critical temperature to a lower value and to turn the associated critical singularities into rounded finite peaks.[12] These effects are illustrated beautifully in the exact and Monte Carlo calculations for finite three-dimensional Ising models with nearest-neighbor interactions carried out by Binder.[13,14] The "critical" temperature for an Ising model system of 54 spins, for instance, is found by calculation to be less than $T_c(\infty)$ by $\sim 40\%$.

III. CALCULATION OF THE CRITICAL POINT

The critical temperature T_c is determined as the temperature at which the isotherm P vs ρ or μ vs ρ (where P is the pressure, ρ is the density, and μ is the chemical potential) has an inflection point

$$\frac{\partial P}{\partial \rho}=\frac{\partial^2 P}{\partial \rho^2}=0 \quad \left[\text{or } \frac{\partial \mu}{\partial \rho}=\frac{\partial^2 \mu}{\partial \rho^2}=0\right] .$$

This inflection point is the critical point. The isotherms themselves must satisfy the inequality

$$\frac{\partial P}{\partial \rho}\geq 0 \quad \left[\text{or } \frac{\partial \mu}{\partial \rho}\geq 0\right]$$

required for thermodynamic stability. However, when the equation of state is calculated in the canonical ensemble with the assumption of a single phase (i.e., uniform density) throughout the volume V occupied by the system, the isotherms corresponding to $T<T_c$ will have Van der Waals loops in which

$$\frac{\partial P}{\partial \rho} \quad \left[\text{or } \frac{\partial \mu}{\partial \rho}\right] < 0 ,$$

and which, therefore, violate the requirements of thermodynamic stability. This unphysical situation, which entails, e.g., a negative compressibility, can be remedied by the usual Maxwell construction. The presence of the loops can be traced back to the (wrong) assumption of a single phase,[15] so that their occurrence is an indication of the coexistence of two phases.

For an infinite system, the P vs ρ and μ vs ρ isotherms are very simply related to each other through the equation

$$P=\rho\mu-f\ , \tag{3.1}$$

where $f=F/V$ is the Helmholtz free energy density of the system, and

$$\mu=\left[\frac{\partial f}{\partial\rho}\right]_T . \tag{3.2}$$

Although the equation of state is customarily given as $P(\rho,T)$, it can be easily converted into $\mu(\rho,T)$ and vice versa. It is, therefore, equally easy to calculate the critical temperature from either the P vs ρ or the μ vs ρ isotherms.

For a finite system, Eq. (3.1) is no longer valid[16] because of the finite size effects, and there is no simple equation equivalent to it. We, therefore, find it more advantageous to work with the chemical potential rather than the pressure for the following reasons:

(1) Since we will be dealing exclusively with fermions, the chemical potential appears explicitly in the Fermi-Dirac distribution function from the very beginning.

(2) The determination of the pressure in a two-phase system is not an easy one as it depends on the geometry of the interfacial region. Moreover, the macroscopic approaches to this problem break down for very small systems. On the other hand, the chemical potential is well defined and can be determined from the requirement that the total number of particles is constant.

For a finite system, another reason for the existence of a loop for isotherms below the critical temperature comes from the surface effects.[15] This contribution is not of the Van der Waals type in that the loop appears even if variations in the density are allowed for, but it approaches zero as the system becomes larger.

IV. CALCULATING THE CHEMICAL POTENTIAL

In the following, we will attempt to calculate the chemical potential for a system of A nucleons in the independent-particle model. Since we are primarily interested in high temperatures, this model should be adequate. Labeling the single particle energy levels as $\epsilon_1,\epsilon_2,\ldots,\epsilon_n,\ldots$, and specializing, for the moment, to the case $N=Z$ and no Coulomb interactions between the protons, the chemical potential is determined by the equation

$$\frac{A}{g_{S,I}}=\sum_i n_i=\sum_i\{\exp[\beta(\epsilon_i-\mu)]+1\}^{-1}\ , \tag{4.1}$$

where $g_{S,I}$ is the spin, isospin degeneracy factor, n_i is the occupation probability of the ith energy level, and $\beta=1/kT$ is the inverse of the temperature. In what follows we will attempt to invert Eq. (4.1) to obtain the chemical potential μ as a function of A, V, and T.

For high temperatures, the above system of fermions is only partially degenerate and $n_i \ll 1$ for all i. It is then possible to expand about the nondegenerate case by using the approximation

$$\{\exp[\beta(\epsilon_i-\mu)]+1\}^{-1}=e^{-\beta(\epsilon_i-\mu)}-e^{-2\beta(\epsilon_i-\mu)}+\cdots\ . \tag{4.2}$$

Substituting (4.2) into (4.1) we get,

$$\frac{A}{g_{S,I}}=Q(\beta)z-Q(2\beta)z^2+Q(3\beta)z^3-\cdots\ , \tag{4.3}$$

where $Q(\beta)$ is the classical canonical partition function

$$Q(\beta)=\sum_i e^{-\beta\epsilon_i}\ , \tag{4.4}$$

and z is the absolute activity

$$z=e^{\beta\mu}\ . \tag{4.5}$$

Dividing (4.3) by $Q(\beta)$ we get

$$\eta\equiv\frac{A}{g_{S,I}Q(\beta)}=z-S_2(\beta)z^2+S_3(\beta)z^3-\cdots\ , \tag{4.6}$$

where

$$S_n(\beta)=Q(n\beta)/Q(\beta)\ . \tag{4.7}$$

Physically, η is a measure of the degree of degeneracy of the gas.

Equation (4.6) can be inverted algebraically to get the activity z as a function of η,

$$z=\sum_{i=1}^{\infty}a_i\eta^i\ . \tag{4.8}$$

The inversion coefficients, up to a_7, can be found in Ref. 17. Here, we list the first four of them:

$$\begin{aligned}a_1&=1\ ,\\ a_2&=S_2(\beta)\ ,\\ a_3&=2S_2(\beta)^2-S_3(\beta)\ ,\\ a_4&=5S_2(\beta)^3-5S_2(\beta)S_3(\beta)+S_4(\beta)\ .\end{aligned} \tag{4.9}$$

With the use of Eq. (4.5), the chemical potential is then given by

$$\mu=kT\ln z=kT\ln\left[\sum_{i=1}^{\infty}a_i\eta^i\right]\ , \tag{4.10a}$$

$$\mu=kT\ln\eta+kT\ln\left[\sum_{i=1}^{\infty}a_i\eta^{i-1}\right]\ . \tag{4.10b}$$

Using the expansion

$$\ln(1+\delta)=\delta-\frac{\delta^2}{2}+\frac{\delta^3}{3}-\frac{\delta^4}{4}+\frac{\delta^5}{5}-\frac{\delta^6}{6}+\cdots\ , \tag{4.11}$$

we can expand Eq. (4.10b) in terms of η,

$$\mu=kT\ln\eta+kT\sum_{i=1}^{\infty}b_i\eta^i\ , \tag{4.12}$$

where the first six coefficients are the following:

$$\begin{aligned} b_1&=a_2\ ,\\ b_2&=a_3-a_2^2/2\ ,\\ b_3&=a_4-a_2a_3+a_2^3/3\ ,\\ b_4&=a_5-a_3^2/2-a_2a_4+a_2^2a_3-a_2^4/4\ ,\\ b_5&=a_6-a_2a_5-a_3a_4+a_2a_3^2+a_2^2a_4-a_2^3a_3+\frac{a_2^5}{5}\ ,\\ b_6&=a_7-a_2a_6-a_3a_5-\frac{a_4^2}{2}+a_2^2a_5+2a_2a_3a_4\\ &\quad+\frac{a_3^3}{3}-\tfrac{3}{2}a_2^2a_3^2-a_2^3a_4+a_2^4a_3-\frac{a_2^6}{6}\ . \end{aligned} \tag{4.13}$$

The above procedure is quite general since we have not specified the single particle energies yet. As an illustration, we will calculate the chemical potential for two simple cases. For an infinite ideal gas of nucleons (with $Z=N$), the single particle energies are given by

$$\epsilon_k=\frac{\hbar^2k^2}{2m}\ , \tag{4.14}$$

and the partition function is

$$Q(\beta)=\frac{V}{\lambda_T^3},\quad \lambda_T=\left[\frac{2\pi\hbar^2}{mkT}\right]^{1/2}\ , \tag{4.15a}$$

where λ_T is the thermal wavelength of a nucleon. From (4.6), (4.7), and (4.15a) we then have,

$$\eta=\frac{A}{g_{S,I}Q(\beta)}=\frac{A}{g_{S,I}V}\lambda_T^3=\frac{\rho}{4}\lambda_T^3 \tag{4.15b}$$

and

$$S_n(\beta)=n^{-3/2}\ . \tag{4.15c}$$

Note that in this case the S_n's are independent of the temperature. From (4.9) and (4.13), it then follows that

$$a_2=\frac{1}{2^{3/2}},\quad a_3=\frac{1}{4}-\frac{1}{3^{3/2}},\quad \text{etc.},\ldots, \tag{4.15d}$$

and

$$b_1=\frac{1}{2^{3/2}},\quad b_2=\frac{3}{16}-\frac{1}{3^{3/2}}=\frac{3}{2}\left[\frac{1}{8}-\frac{2}{9\sqrt{3}}\right],\ \text{etc.},\ldots, \tag{4.15e}$$

which are the results derived earlier.[8]

If the Fermi gas is nonideal, then in the effective mass approximation,

$$\epsilon_k=\frac{\hbar^2k^2}{2m^*}+\epsilon_0\ , \tag{4.16a}$$

$$Q(\beta)=\frac{V}{\tilde{\lambda}_T^3}e^{-\beta\epsilon_0},\quad \tilde{\lambda}_T=\left[\frac{2\pi\hbar^2}{m^*kT}\right]^{1/2}\ , \tag{4.16b}$$

$$\eta=\frac{\rho\tilde{\lambda}_T^3}{4}e^{\beta\epsilon_0}\ , \tag{4.16c}$$

$$S_n(\beta)=n^{-3/2}e^{-(n-1)\beta\epsilon_0}\ , \tag{4.16d}$$

and

$$\mu=\epsilon_0+\mu_{\rm id}(m^*)\ , \tag{4.16e}$$

where $\mu_{\rm id}(m^*)$ is the chemical potential of an ideal Fermi gas of nucleons having a mass m^*. These results are identical to those derived earlier.[8] Equation (4.16e), in particular, is correct even at $T=0$, as is the corresponding equation for the pressure [Eq. (3.1) in Ref. 8].

The series (4.12) can be viewed as an expansion in powers of $1/kT$. This is obvious from the above two examples where

$$\eta\sim\lambda_T^3\sim(kT)^{-3/2}\ .$$

Actually, (4.12) is an asymptotic (or semiconvergent) series in that it will converge only in the limit of large kT. Physically, this corresponds to the fact that the Fermi system approaches the classical limit as $T\to\infty$ so that all the terms in (4.12) tend to zero except for the logarithmic (classical) term. We can, therefore, employ the usual techniques[18] in summing asymptotic series. These involve stopping the summation at the term with the lowest magnitude and the averaging of successive terms.[18] Associated with this there will be an unavoidable computational error which tends to zero as the temperature increases. Since we are interested in relatively high temperatures, the errors involved are usually small and often negligible. Usually, only the first few terms in the series (4.12) are needed in the calculations that follow. In any case, the summation was always stopped before the $i=7$ term.

V. CALCULATIONS FOR FINITE SYSTEMS

As a model for a finite system of nucleons, we take a box of volume V and surface area S containing A nucleons (with $N=Z$). For the interaction between the nucleons, we take a zero-range Skyrme-type force

$$v_{12}=-t_0\delta(\vec{r}_1-\vec{r}_2)+\frac{t_3}{6}\rho^\sigma\left[\frac{\vec{r}_1+\vec{r}_2}{2}\right]\delta(\vec{r}_1-\vec{r}_2)\ . \tag{5.1}$$

The values of t_0, t_3, and σ used are listed in Table I. These values are taken from Ref. 8 where they were obtained by fitting the binding energy and density of infinite nuclear matter. Assuming that the nuclear density ρ is uniform inside the volume V, the single particle energies of the nucleons are,

$$\begin{aligned} \epsilon_k&=\frac{\hbar^2k^2}{2m}+\epsilon_0\ ,\\ \epsilon_0&=-\tfrac{3}{4}t_0\rho+\tfrac{3}{24}t_3\left[1+\frac{\sigma}{2}\right]\rho^{1+\sigma}\ . \end{aligned} \tag{5.2}$$

TABLE I. The parameters of the Skyrme interactions used in calculations.

Force	σ	t_0 (MeV fm^3)	t_3 (MeV fm$^{3(1+\sigma)}$)
ZR1	1	1003.9	13287.2
ZR2	$\frac{2}{3}$	1192.2	11041.0
ZR3	0.1	4392.2	26967.3

For such a system, the number of states with wave number between k and $k+dk$ is given approximately by a formula due to Hill and Wheeler,[19] which includes the effects of the finite size of the system (see the Appendix),

$$dN = V\frac{k^2dk}{2\pi^2} - S\frac{k\,dk}{8\pi} + L\frac{dk}{8\pi} \,. \tag{5.3}$$

This formula was originally derived for a rectangular parallelpiped, of dimensions a,b,c, for which $L=2a+2b+2c$ is a measure of the average linear size of the system. It can, however, be applied to any box whose shape is not too irregular.[19] For a sphere of radius R, L is equal to $2\pi R$, the circumference.

With the use of (5.2) and (5.3), the partition function for the system is found to be

$$Q(\beta) = \frac{Ve^{-\beta\epsilon_0}}{\lambda_T^3}\left[1 - \frac{\lambda_T}{4}\frac{S}{V} + \frac{\lambda_T^2}{8}\frac{L}{V}\right] , \tag{5.4}$$

where λ_T is defined in Eq. (4.15a). For a sphere,

$$\frac{S}{V} = \frac{3}{R},\quad \frac{L}{V} = \frac{3}{2}\frac{1}{R^2} = \frac{1}{6}\left[\frac{S}{V}\right]^2 ,$$

so that

$$Q(\beta) = \frac{Ve^{-\beta\epsilon_0}}{\lambda_T^3}\left[1 - \frac{3}{4}\frac{\lambda_T}{R} + \frac{3}{16}\frac{\lambda_T^2}{R^2}\right] . \tag{5.5}$$

It is obvious that the last term in (5.4) or (5.5) is a second order surface correction.

Using this partition function, the coefficients (4.9) and (4.13) are evaluated and the series (4.12) is summed as explained in Sec. IV. In particular, we stop at the term of lowest magnitude and take the average of successive terms. This means that only half the last term is added to the sum. The error in the summation is taken as half the magnitude of this term. Figure 1 is a plot of several isotherms calculated for a system of 50 neutrons and 50 protons with the force ZR1 from Table I. The isotherms indicate that the critical temperature is ≈ 18 MeV and the critical density $\approx 0.38\rho_0$, where ρ_0 is the density of infinite nuclear matter which we take to be 0.17 nucleons/fm^3. Table II shows the effect of finite size on the critical point of $N=Z$ systems with no Coulomb force, using the forces listed in Table I. The errors quoted for the critical temperature result from the uncertainty in determining the chemical potential because of the computational errors resulting from summing the series (4.12). These errors appear for the smallest systems only because their critical temperatures are low enough that these errors begin to become non-negligible. Moreover, for a given temperature, the errors get larger as A decreases. The situation is further illustrated by the error bars on the isotherms of Fig. 1. The results given in Table II show a sizeable drop of about 6 MeV in the critical temperature for the smallest system considered. This is in agreement with the results obtained earlier[8] for a finite system. The inclusion of the Coulomb force and surface diffuseness effects, as well as

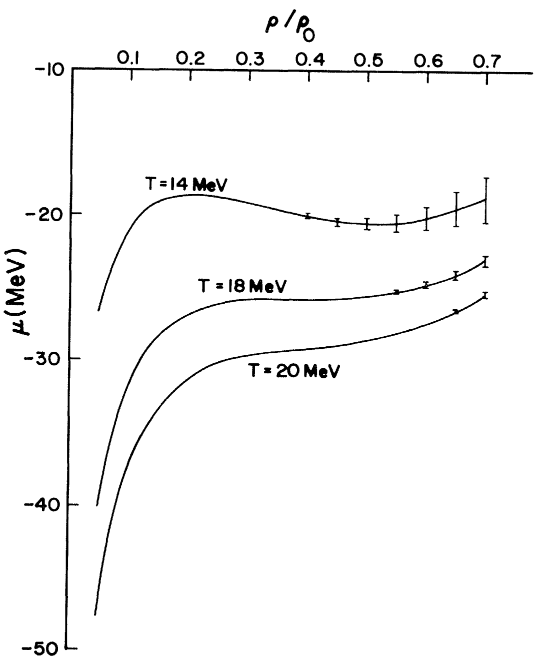

FIG. 1. Isotherms for a system of 50 protons and 50 neutrons interacting via the zero-range Skyrme force ZR1. Error bars indicate computational errors involved in summing the asymptotic series (4.12).

TABLE II. Effect of finite size on the critical point of nuclear matter for $N=Z$ systems with no Coulomb force. Calculations are made for the various forces shown in Table I. The errors are explained in the text. When no errors are reported they are ≤ 0.05 MeV.

No. of Nucleons	Force ZR1 T_c (MeV)	ρ_c/ρ_0	Force ZR2 T_c (MeV)	ρ_c/ρ_0	Force ZR3 T_c (MeV)	ρ_c/ρ_0
∞	22.90	0.40	20.5	0.375	16.00	0.325
10^6	22.80	0.40	20.4	0.375	15.80	0.325
10^4	22.00	0.39	19.7	0.375	15.00	0.325
10^3	20.80	0.39	18.6	0.37	14.00	0.30
500	20.20	0.38	17.9	0.35	13.50	0.30
200	19.10	0.38	16.9	0.33	12.6±0.1	0.28
100	18.1±0.1	0.38	15.9±0.1	0.33	11.6±0.3	0.25
50	16.5±0.5	0.34	14.5±0.5	0.30	10.5±1.0	0.23
Previous result From Ref. 8	13.4	0.39	11.9	0.36	8.5	0.29

the use of a finite-range term in the Skyrme interaction, are expected to lower the critical temperature still further and so bring it into better agreement with the results of Ref. 8. Some of these effects will be considered in the following sections.

VI. INCLUSION OF THE COULOMB FORCE

Once the Coulomb force between the protons is taken into account, the isospin symmetry which we have utilized in the preceding sections will be broken. In particular, the protons and neutrons will have different energy levels and chemical potentials (μ_p and μ_n, respectively), and Z will no longer be equal to N. In this case, separate calculations must be carried out for μ_p and μ_n. For this purpose, we use a more general form of the zero-range Skyrme force

$$v_{12}=-t_0(1+x_0P^\sigma)\delta(\vec{r}_1-\vec{r}_2) + \frac{t_3}{6}(1+x_3P^\sigma)\rho\left[\frac{\vec{r}_1+\vec{r}_2}{2}\right]\delta(\vec{r}_1-\vec{r}_2)\ , \qquad (6.1)$$

with the same values for t_0 and t_3 as the ZR1 force in Table I. The values of x_0 and x_3 do not enter in the calculations for the case of $N=Z$ systems, and so the force (6.1) is identical to the ZR1 force for these systems. The value of x_3 is taken to be 1, while x_0 is treated as a parameter. With this interaction, the single-nucleon energy levels are given by

$$\epsilon_{kt}=\frac{\hbar^2k^2}{2m}-t_0\left[\left[1+\frac{x_0}{2}\right]\rho-\left[x_0+\frac{1}{2}\right]\rho_t\right] + \tfrac{1}{4}t_3(\rho^2-\rho_t^2)+\delta_{t,1/2}V_C\ , \qquad (6.2)$$

where $t=+\frac{1}{2}$ for a proton and $t=-\frac{1}{2}$ for a neutron. The Coulomb term V_C is taken to be the average Coulomb potential energy per proton in a uniformly charged sphere

$$V_C=\frac{6}{5}\frac{Ze^2}{R}\left[1-5\left[\frac{3}{16\pi Z}\right]^{2/3}-\frac{1}{Z}\right]\ , \qquad (6.3)$$

where the $Z^{-2/3}$ term is the exchange contribution and the Z^{-1} term subtracts the interaction of the proton with itself.[20]

Before trying to calculate separate proton and neutron chemical potentials with the use of the energy levels given by (6.2), it is possible to get an approximate idea of the effect of the Coulomb force without losing the symmetry exploited in the preceding sections. The Coulomb force raises the single-proton energies by the amount V_C and favors making $Z<N$. The effect of such an asymmetry is seen from Eq. (6.2) to lead to raising the single-neutron energy levels at the low densities of interest in the present calculations. We can, therefore, use the following approximate single-particle energy levels for both protons and neutrons and keep $N=Z$

$$\epsilon_k=\frac{\hbar^2k^2}{2m}-\tfrac{3}{4}t_0\rho+\tfrac{3}{24}t_3\left[1+\frac{\sigma}{2}\right]\rho^{1+\sigma}+\frac{Z}{A}V_C\ , \qquad (6.4)$$

where V_C is given by (6.3). The effect of this approximate treatment of the Coulomb force is shown in Table III for $A=50$, 100, and 200, where it is seen that the Coulomb force lowers the critical temperature by a further 1—3 MeV, depending on the size of the system. The calculations are carried out with the ZR1 force of Table I.

On the other hand, if the Coulomb force is to be treated correctly, then the protons and neutrons must be treated separately and have different energy levels and chemical potentials. Here it must be noted that the protons and neutrons in hot nuclear matter produced in a relativistic collision are not in chemical equilibrium although they may be in thermal equilibrium (as we have been assuming). As a result, their chemical potentials are not related to each other. The situation is, therefore, completely different from that found in supernova matter where there is chemical equilibrium (or β equilibrium) between the neutrons, protons, electrons, and neutrinos.[5] However, since β equilibrium is achieved via the weak force, it cannot be realized in a nuclear collision.

As a result of having different chemical potentials and different isotherms, the protons and neutrons will also appear to have different critical temperatures, T_{cp} and T_{cn}, respectively (assume that $T_{cp}<T_{cn}$ for the sake of argument). This, of course, makes no sense physically and it results from the fact that, while calculating the chemical potential for the neutrons, we are assuming the protons are still present [see, e.g., Eq. (6.2)] which is obviously wrong if $T>T_{cp}$. In addition, it makes no sense for the neutrons to stick together after all the protons have boiled off, and vice versa if $T_{cn}<T_{cp}$. It is, therefore, obvious that the correct critical temperature in such a situation is given by the smaller of T_{cp} and T_{cn}.

The chemical potentials for the protons and neutrons are now calculated from Eq. (4.12) where η is defined in Eq. (4.6), except that A is replaced by Z or N and $g_{S,I}$ is now equal to 2. The single-particle energies used are those

TABLE III. Effect of the Coulomb force on the critical temperature of finite nuclear systems. The nuclear force used is ZR1. Columns 2 and 4 give the critical temperature and density when no Coulomb effects are included (i.e., the same as Table II), while columns 3 and 5 give T_c and ρ_c with the approximate inclusion of the Coulomb force as given by Eq. (6.4).

No. of Nucleons	T_c (MeV) (no Coulomb)	T_c (MeV) (with Coulomb)	ρ_c/ρ_0 (no Coulomb)	ρ_c/ρ_0 (with Coulomb)
50	16.5±0.5	15.5±0.5	0.34	0.32
100	18.1±0.1	16.2±0.3	0.38	0.37
200	19.1	15.7	0.38	0.37

given by (6.2). The critical temperatures T_{cp} and T_{cn} are listed in Table IV for a system of 40 protons and 60 neutrons (first three rows) and for a system of 44 protons and 56 neutrons (second three rows). For each system, we let x_0 [see Eq. (6.1)] take the values 0, 0.2, and 0.5. The effect of increasing x_0 is to increase T_{cp} and lower T_{cn}, as can be seen from Table IV. The values of x_0 used in the literature vary, although a choice around 0.5 is quite popular. We notice that for $x_0=0.2$ and 0.5, the results in Table IV indicate that systems with a larger neutron-proton asymmetry will have lower critical temperatures. Notice also that in most cases $T_{cp} > T_{cn}$, which reflects the fact that a neutron has less protons to interact with while the proton has more neutrons with which it can interact as compared to the $N=Z$ case. For comparison, we give in the last row of Table IV the results obtained earlier in this section with an approximate treatment of the Coulomb force.

VII. CALCULATIONS WITH SERR's INTERACTIONS (Sks)

The Sks interaction[21] has three density-dependent terms and, for the present calculation, it can be written as follows:

$$v(\vec{r}_{12})=-t_0\delta(\vec{r}_{12})+\frac{t_1}{2}[k^2\delta(\vec{r}_{12})+\delta(\vec{r}_{12})k^2]+t_2\vec{k}\delta(\vec{r}_{12})\vec{k}+\frac{t_3}{6}\rho\left[\frac{\vec{r}_1+\vec{r}_2}{2}\right]\delta(\vec{r}_{12})$$
$$+\frac{t_4}{6}\rho^2\left[\frac{\vec{r}_1+\vec{r}_2}{2}\right]\delta(\vec{r}_{12})+\frac{t_5}{6}\rho^3\left[\frac{\vec{r}_1+\vec{r}_2}{2}\right]\delta(\vec{r}_{12})\;. \tag{7.1}$$

This force is interesting since it can reproduce the properties of the giant monopole resonance (GMR), although it has an incompressibility $K=400$ MeV that is much higher than the generally accepted value[22] which is derived from an analysis of the available experimental results pertaining to the GMR. The Sks incompressibility is, however, comparable to that obtained with regular Skyrme forces (like the ZR1 force above) that have a linear density dependence.

The parameters of the Sks force are shown in the first column of Table V together with other relevant quantities including the incompressibility K, the effective mass m^*/m, and the "surface compressibility parameter" η (Ref. 21), which serves as an indicator of the surface compressibility. We also show in the first column the critical temperature $T_c(\infty)$ and density $\rho_c(\infty)$ for infinite nuclear matter calculated with the Sks interaction. These values of $T_c(\infty)$ and $\rho_c(\infty)$ are much lower than the corresponding values given by the usual Skyrme-type interactions (see the first row of Table II, for example). In the second column of Table V, we show the parameters of another force denoted Skj, which is very similar to Sks except that it has an effective mass of 1. For this force, $T_c(\infty)$ and $\rho_c(\infty)$ are somewhat higher than for Sks, which is expected from the value of the effective mass.[8] However, these values are still much lower than the corresponding values in the first row of Table II, except for the very soft ($K=190$ MeV) ZR3 force [with $\sigma=0.1$ in Eq. (5.1)] which gives a comparable $T_c(\infty)$, but a still higher value of $\rho_c(\infty)$. Also included in the second column of Table V are T_c and ρ_c for a system of 100 nucleons (with $Z=N$) with the Coulomb interaction included according to the approximation (6.4). Again we notice that, for the Skj force, the critical temperature is reduced by about 5 MeV because of surface and Coulomb effects, which is roughly in agreement with the results obtained in the preceding sections with regular Skyrme-type forces. For comparison we include in the third and fourth columns of Table V similar calculations with two Skyrme-type forces with linear density dependence ($\sigma=1$). The force called Skg has an effective mass of 0.8 (same as Sks), whereas the ZR1 force has an effective mass of 1 (same as Skj). The incompressibilities of all four forces are very close, but Skg and ZR1 differ from Sks and Skj in the value of the surface compressibility parameter η. For a Skyrme force, η and K are related to each other and cannot be varied independently, unlike the Sks and Skj forces. The smaller values of η for the latter indicate that these forces are quite soft at densities $\lesssim\rho_0/2$, although they have a high incompressibility at nuclear matter density. This accounts for the fact that T_c and ρ_c obtained with these forces are closer to those obtained from the soft Skyrme forces ($\sigma\approx 0$).

TABLE IV. The proton and neutron "critical" temperatures and "critical" densities with the correct inclusion of Coulomb and symmetry energies. Note that $\rho_c^{(p)}$ (or $\rho_c^{(n)}$) is the *total* nuclear density and not the proton (or neutron) density.

Z	N	x_0	T_{cp} (MeV)	$\rho_c^{(p)}/\rho_0$	T_{cn} (MeV)	$\rho_c^{(n)}/\rho_0$
44	56	0.0	15.7±0.3	0.36	17.1±0.3	0.39
44	56	0.2	16.6±0.2	0.36	16.3±0.3	0.38
44	56	0.5	17.9±0.1	0.37	15.0±0.4	0.35
40	60	0.0	17.2±0.1	0.36	16.5±0.3	0.37
40	60	0.2	18.6±0.1	0.37	14.6±0.4	0.38
40	60	0.5	20.7	0.39	12.5±0.5	0.32
50	50		16.2±0.3	0.37	16.2±0.3	0.37

TABLE V. Parameters for the Sks, Skj, Skg, and ZR1 forces together with some relevant quantities calculated with these forces as explained in the text.

Interaction	Sks	Skj	Skg	ZR1
t_0 (MeV fm^3)	1532.5	1441.4	1035.3	1003.9
$3t_1+5t_2$ (MeV fm^5)	497	0	486.75	0
t_3 (MeV fm^6)	68 022	60 100	11 073	13 287
t_4 (MeV fm^9)	−347 548	−278 264	0	0
t_5 (MeV fm^{12})	708 079	551 292	0	0
m^*/m	0.8	1	0.8	1
K (MeV)	400	400	372	384
η (MeV fm^3)	−250	−250	−352.7	−329.4
T_c (∞)	12.6 MeV	15.0 MeV	20.3 MeV	22.9 MeV
ρ_c (∞)	0.19 ρ_0	0.25 ρ_0	0.4 ρ_0	0.4 ρ_0
T_c ($A=100$)		9.9±0.5 MeV		16.2±0.3 MeV
ρ_c ($A=100$)		0.18 ρ_0		0.37 ρ_0

VIII. CONCLUSION

The effect of finite size and the Coulomb force have been found to lead to a sizeable reduction in the critical temperature of nuclear matter. Surface effects can reduce the temperature by 5—6 MeV while the Coulomb force is responsible for a further reduction of a few MeV. In general, the simple estimates obtained in Ref. 8 are found to be fairly good. By analogy, we expect that similar conclusions will hold for the case of supernova matter and, therefore, the neglect of surface and Coulomb effects in supernova calculations[5] is not justified.

In the present calculations, however, several effects have not been included that may play a role in determining the critical point. For one thing, the use of a finite-range force instead of the zero-range forces used here would have an effect on the nuclear equation of state. By examining the results of Ref. 8 it can be inferred that this could lead to a reduction in the critical temperature of 2—3 MeV. The diffuseness of the nuclear surface should also be taken into account. This will lead to a further reduction in the critical temperature because of the reduction in the binding energy of the surface nucleons as compared with the uniform density model used here. A third effect that has not been included is fluctuation in the occupation probability of the single-particle energy levels [see Eq. (4.1)]. Fluctuations in general tend to reduce the critical temperature and their importance grows as the system's size decreases. The net results of these effects would be to reduce the values of the critical temperature below the results obtained in the present work, so that these results can be regarded as upper limits.

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APPENDIX

Here we give a derivation of the asymptotic formula for the number of states with wave number between k and $k+dk$ in the two-dimensional case. A generalization to the three-dimensional case is straightforward and can be found in the original work of Hill and Wheeler.[19]

Consider a particle confined in a two-dimensional infinite rectangular well of dimensions a,b in the x-y plane, with one corner at the origin. The wave functions for the possible states of this particle are solutions of the equation $(\nabla^2+k^2)\psi=0$, with $\psi=0$ at the surface, and are given by

$$\psi_{lm}=\sin\frac{l\pi}{a}x\,\sin\frac{m\pi}{b}y\ .$$

Each of these solutions corresponds to a lattice point in the two-dimensional k space,

$$k_x=\frac{l\pi}{a},\quad k_y=\frac{m\pi}{b}\ ,$$

with which is associated a characteristic rectangle of area $(\pi/a)(\pi/b)$ (see Fig. 2).

$$dN=\text{number of states between } k \text{ and } k+dk\ ,$$

$$=\frac{\text{area of circular shell}-\text{shaded end regions}}{\text{area per state}}\ ,$$

$$=\frac{\dfrac{\pi}{2}k\,dk-\left[\dfrac{\pi}{2b}+\dfrac{\pi}{2a}\right]dk}{\pi^2/ab}=\frac{ab}{2\pi}k\,dk-\frac{a+b}{2\pi}dk\ ,$$

$$=\frac{S}{2\pi}k\,dk-\frac{L}{4\pi}dk\ ,$$

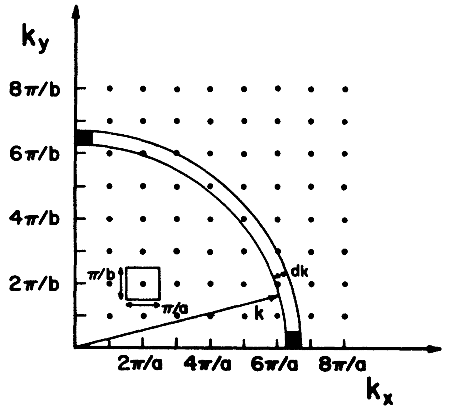

FIG. 2. k space for a particle in a two-dimensional infinite rectangular well.

where $S=ab$ is the area of the well and $L=2a+2b$ is its perimeter. Note that for an infinite system, a and $b\rightarrow\infty$ and the k-space lattice goes into a continuum so that the shaded regions in Fig. 2 vanish. The correction to the value of dN found above essentially contains the effect of zero-point motion and the quantization of the wave number in a finite system.

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