Energy-independent optical potentials: Construction and limitations

M. S. Hussein
Instituto de Fisica, Universidade de São Paulo, C.P. 20.516, São Paulo, Brazil

E. J. Moniz
Center for Theoretical Physics, Department of Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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We construct and examine the properties of the energy-independent potential $\overline{U}$ which is wave function equivalent to the usual optical potential $U(E)$. A simple procedure is presented for constructing $\overline{U}$ in the uniform medium, and physical examples are discussed. The general result for the finite system, a recursive expansion in powers of $U(E)$, is used to investigate the multiple scattering expansion of $\overline{U}$. The energy-independent potential is found to have serious shortcomings for direct microscopic construction or phenomenological parametrization. The microscopic theory, as exemplified here by the multiple scattering approach, does not lead to a reliable approximation scheme. Phenomenological approaches to $\overline{U}$ are unattractive because the physics does not guide the parametrization effectively: the structure of the nonlocality is not tied directly to the dynamics; Im $\overline{U}$ changes sign; different elements of the physics, separate in $U(E)$, are completely entangled in $\overline{U}$.

I. INTRODUCTION

Energy dependent optical potentials arise naturally in describing multichannel reaction processes via equivalent one channel theories,[1]

$$U_\alpha(E)=P_\alpha UP_\alpha+P_\alpha UQ_\alpha\frac{1}{E^+-Q_\alpha HQ_\alpha}Q_\alpha UP_\alpha\ , \tag{1}$$

$$P_\alpha=|\Psi_\alpha\rangle\langle\Psi_\alpha|\ ,\quad Q_\alpha=1-P_\alpha\ , \tag{2}$$

where P_α projects onto the channel α. The operator $U_\alpha(E)$ is explicitly energy dependent, complex, and nonlocal because of the intermediate coupling to the other channels. Given $U_\alpha(E)$, the diagonal transition amplitude in the channel α is obtained from solution of a standard one-channel Lippmann-Schwinger equation

$$T_\alpha(E)\equiv P_\alpha T(E)P_\alpha=U_\alpha(E)+U_\alpha(E)G_0(E^+)T_\alpha(E)\ , \tag{3}$$

where $G_0(E)$ describes free propagation in the channel α. The completely off-shell matrix elements of $T_\alpha(E)$ are given correctly by Eqs. (1) and (3). This is not terribly surprising, since a solution for $U_\alpha(E)$ corresponds to solution of the full multichannel problem. The optical potential approach to reaction theory has been very fruitful since, on the one hand, systematic expansions of the elastic channel optical potential have been derived microscopically and, on the other hand, comparatively simple, theoretically motivated phenomenological representations are available.

Microscopic approaches based on the multiple scattering expansion can be organized to build in unitarity constraints at any level of truncation, with each successive level of approximation incorporating reaction processes involving an additional nucleon. For example, the lowest order term,

$$U_0^{(1)}(E)=\sum_i t_i(E)\ , \tag{4}$$

where $t_i(E)$ is the (in-medium) projectile-ith nucleon transition matrix, already generates a good description of intermediate and high-energy hadron-nucleus elastic scattering. The reactive content associated with $U^{(1)}$ is quasifree nucleon knockout, which is the dominant reaction mechanism in the low-density (or peripheral) limit; for central collisions, the strong absorption ("black disc") limit is respected. This simple form [Eq. (4)] then provides the starting point for a comparatively simple yet meaningful phenomenology, with the target geometry and projectile dynamics separated. For example, a static zero-range approximation to Eq. (4) yields the local energy-dependent optical potential

$$\langle\vec{r}\,|\,U_0^{(1)}(E)\,|\,\vec{r}\,'\rangle=\delta(\vec{r}-\vec{r}\,')\rho(r)t(E)\ . \tag{5}$$

Even low-energy nuclear reactions have been described with phenomenological optical potentials tailored according to Eq. (5), with geometrical aspects still described by $\rho(r)$, while the energy-dependent $t(E)$ is replaced by a phenomenological function of energy.

The energy dependence of the microscopic optical potential, though not a serious complication insofar as the calculation of the elastic scattering amplitude is concerned, does pose formal problems when used in a perturbative coupled-channels description of, e.g., intermediate energy inelastic excitation of low lying collective states. The intermediate channel Green's functions, which appear naturally in such perturbative treatments, do not lend themselves to the usual spectral expansion as a result of the nonorthogonality of the scattering states.

Recently, attempts have been made[2–6] to derive an equivalent energy-independent optical potential which,

when used in the Schrödinger equation, generates the same scattering wave function as that obtained with $U_\alpha(E)$. The idea of an energy-independent potential is actually not new, and a trivial example is the effective mass approximation invoked to treat a linearly energy-dependent potential. More explicitly, given an optical potential of the form [a special case of Eq. (5)]

$$\langle \vec{r} \mid U_0(E) \mid \vec{r}' \rangle = CE\rho(r)\delta(\vec{r}-\vec{r}') , \tag{6}$$

the associated Schrödinger equation,

$$[-\nabla^2 + U_0(E;\vec{r})]\psi_0(\vec{r}) = E\psi_0(\vec{r}) , \tag{7}$$

may be recast (algebraically) into the form

$$(-\nabla^2 + \overline{U}_0)\psi_0(\vec{r}) = E\psi_0(\vec{r}) , \tag{8}$$

with the equivalent energy-independent potential $\overline{U}$ given by

$$\langle \vec{r} \mid \overline{U}_0 \mid \vec{r}' \rangle = \frac{-C\rho(\vec{r})}{1-C\rho(\vec{r})}\nabla^2\delta(\vec{r}-\vec{r}') . \tag{9}$$

This is nonlocal and has a complicated interplay between the potential strength (recall that C is complex) and the target geometry. Whereas geometrical properties and dynamical features of $U(E)$ are clearly separated, this is not so in $\overline{U}$. The complicated coordinate space structure of $\overline{U}$ implies that, even in a rather simple physical situation [as represented by Eq. (6)], energy-independent optical potentials are not amenable to physically motivated direct phenomenological construction. Furthermore, while $U(E)$ respects the previously mentioned "peripheral" and "central" (black disc) limits in a strong absorption situation, these constraints associated with geometry and unitarity are not observed in expansions of $\overline{U}$. Since an effective procedure for microscopically constructing an optical potential must provide a systematic expansion which can be truncated at a low order, the wave function equivalent energy-independent optical potential is rather unattractive as a focus for theoretical effort in many situations of relevance to nuclear physics.

Our paper is intended to be rather pedagogical. Several recent papers have discussed the energy-independent optical potential,[2–6] the majority emphasizing the attractiveness of $\overline{U}$. Our intent is to offer a somewhat different construction procedure for $\overline{U}$, particularly for the uniform medium, and then to expand on the shortcomings of such a potential in a more systematic fashion than done above for the simple effective mass example. We do this both by explicit construction of $\overline{U}$ in cases of physical interest and by developing a multiple scattering expansion. The coupled channel theory of $U(E)$ could equally well be used to demonstrate our arguments, but the example drawn from multiple scattering theory, which has been employed with considerable success at intermediate energy, will be sufficient for our purposes.

In Sec. II, we present a simple method for constructing $\overline{U}$ in the infinite medium and discuss two examples in detail, one a phenomenological local optical potential with quadratic energy dependence, the other a nonlocal optical potential appropriate for intermediate energy pion scattering. In Sec. III, a general derivation of the energy-independent optical potential is presented, culminating in a recursive expansion of $\overline{U}$ in terms of $U(E)$. While our methods are slightly different, the first terms in this expansion have been discussed previously by Ma *et al.*[6] We then go on to use this expansion in a discussion of multiple scattering. The difficulties with reactive content which ensue upon truncating the multiple scattering expansion of $\overline{U}$ are discussed. Section IV contains a summary of our results.

II. ENERGY INDEPENDENT POTENTIAL FOR OPTICAL PROPAGATION IN A UNIFORM MEDIUM

The shortcomings of an equivalent energy-independent optical potential can be seen by constructing $\overline{U}$ for propagation in a uniform medium (i.e., nuclear matter). The Green's function is then diagonal in the initial and final momenta

$$\langle \vec{p} \mid G(E_k) \mid \vec{q} \rangle = (2\pi)^3\delta(\vec{p}-\vec{q})G(p;k) , \quad E \equiv k^2 , \quad 2m \equiv 1 . \tag{10}$$

Translational invariance demands that all scattering states are plane waves, with the wave number determined by the poles of the Green's function,

$$G(p;k)^{-1} = k^2 - p^2 - U(p;k) = 0 . \tag{11}$$

Here, $U(p;k)$ is the usual energy-dependent complex optical potential (or self-energy), with the variation in momentum corresponding to nonlocality. Recall that for scattering, one solves the dispersion equation [Eq. (11)] for the in-medium wave number $p^*(k)$, with $E = k^2$ real. The one-pole approximation for the Green's function can then be written as

$$G(p;k) = \frac{\gamma^*(k)}{p^*(k)^2 - p^2} , \tag{12}$$

$$\gamma^*(k) = \left[1 + \frac{\partial}{\partial p^2}U(p;k)\right]^{-1}_{p=p^*(k)} = \left[\frac{m_p}{m}\right]_{p=p^*(k)} . \tag{13}$$

The in-medium wave number is complex, with the imaginary part giving the optical damping of the wave function. (Note that the dispersion equation may have multiple eigenmode solutions in general. For simplicity, we shall keep only the optical eigenmode, which is unambiguous for a weak enough potential strength. See Ref. 7 for a discussion of the Green's function with multiple eigenmodes.) The residue at the pole in Eq. (12) defines the effective k mass.[8]

The energy-independent potential $\overline{U}(p)$ is defined to produce the same pole in p in the associated dispersion equation

$$\overline{G}(p;k)^{-1} = k^2 - p^2 - \overline{U}(p) = 0 . \tag{14}$$

The potential is obtained easily as

$$\bar{U}(p)=\bar{k}^2(p)-p^2 , \tag{15}$$

where $\bar{k}(p)$ is the (complex) solution of Eq. (11) for k with p real. This solution is generally applicable to calculations of quasiparticle lifetimes in a Lehmann representation of the Green's function, not to a study of optical propagation. The one-pole approximation for the Green's function has the form

$$G(p;k)=\frac{\bar{\gamma}(p)}{k^2-\bar{k}^2(p)}=\bar{\gamma}(p)\bar{G}(p;k) , \tag{16}$$

$$\bar{\gamma}(p)=\left[1-\frac{\partial}{\partial k^2}U(p;k)\right]^{-1}_{k=\bar{k}(p)}=\left[\frac{m}{m_E}\right]_{E=\bar{k}^2(p^2)} . \tag{17}$$

The Green's functions G and $\bar{G}$ have the same pole, since $\bar{k}^2[p^*(k)]=k^2$, but differ by the effective E mass[8] evaluated at the quasiparticle pole.

We wish to use the formal solution for $\bar{U}(p)$ [Eq. (15)] to study cases of physical relevance. The simple effective mass result discussed in the Introduction is, of course, recovered trivially as

$$\begin{aligned} U(p;k)&=Ck^2\rho , \\ \bar{k}(p)^2&=p^2/(1-C\rho) , \\ \bar{U}(p)&=\frac{C\rho}{1-C\rho}p^2 . \end{aligned} \tag{18}$$

In the infinite medium, p^2 plays the role of $-\nabla^2$, so that Eqs. (18) and (9) agree. A less trivial example is presented by a potential with quadratic energy dependence. Bauer *et al.*[9] have suggested this form for nucleon scattering with $E\lesssim 200$ MeV, together with a theoretical justification related to the effective mass. Ma *et al.*[6] state that the quadratic term, since it is weak, should not modify $\bar{U}$ very much. This statement should be viewed with caution. To show this simply, we take a local potential

$$U(p;k)=U(k)=C_0E_0+C_1E+C_2E^2/E_0 , \quad E=k^2 , \tag{19}$$

where the C_i are dimensionless strength parameters and E_0 is an energy scale parameter. We obtain

$$\bar{U}(p)=E_0\frac{(1-C_1)-[(1-C_1)^2-4C_2(C_0+p^2/E_0)]^{1/2}}{2C_2}-p^2 , \tag{20}$$

$$\underset{C_2\to 0}{\to}\frac{C_0E_0}{1-C_1}\left[1-\frac{C_2C_0}{1-C_1}\right]+\frac{C_1}{1-C_1}\left[1+\frac{2C_2}{C_1(1-C_1)^2}\right]p^2+\frac{C_2}{(1-C_1)^3}\frac{p^4}{E_0}+\cdots . \tag{21}$$

Clearly, there is a very complicated interplay in $\bar{U}$ [Eq. (20)] among the various terms in U [Eq. (19)]. In particular, the energy-independent term C_0 is *not* isolated as a separate term in $\bar{U}$. Even though C_2 may be small, it may produce a large effect in $\bar{U}$ by "interference" with a large energy-dependent potential; for example, the $C_2C_0/(1-C_1)$ coefficient in the local piece of $\bar{U}$ may be significant (compared to one) even for a small C_2. Of course, the quadratic term in U becomes large at sufficiently high energy for small but finite C_2, so that one may not be surprised that $\bar{U}$, which must reproduce the wave function associated with U at all energies, is strongly influenced by C_2. One may attempt to circumvent this problem by introducing arbitrary high energy cutoffs in the original optical potential. However, even in cases where this is allowed by the physics, the fact remains that individual contributions to $U(E)$ which generally have well-defined physical content, are completely entangled in $\bar{U}$; this includes any energy-independent contribution to $U(E)$. Thus, one has no real guidance from microscopic theory for direct phenomenological construction of $\bar{U}$.

Another instructive example may be drawn from the multiple scattering theory. We take a first order energy dependent optical potential [as in Eq. (5)] with a resonance-dominated projectile-nucleon transition matrix modeled after that appropriate for intermediate energy pion scattering,

$$U(p;k)=\frac{k_0}{\lambda_R}\frac{\mu\Gamma}{k^2-k_0^2+i\frac{k^2}{k_0^2}\mu\Gamma-\frac{\mu}{M_\Delta}p^2} . \tag{22}$$

Here, k_0 corresponds to the resonant πN c.m. momentum and Γ to the resonance width. We take $k_0=1.25$ fm^{-1}, $\Gamma=0.55$ fm^{-1}, pion mass $\mu=0.7$ fm^{-1}, and resonance mass $M_\Delta=6.2$ fm^{-1} (note that this is not a particularly narrow resonance). The k^2 dependence of the width has been taken for convenience. The p^2 term in the denominator corresponds to intermediate propagation of the resonance and produces a corresponding nonlocality in the energy-dependent potential

$$\begin{aligned} \langle\vec{r}\,|\,U_k\,|\,\vec{r}\,'\rangle=U_k(s)&=\int\frac{d\vec{p}}{(2\pi)^3}e^{i\vec{p}\cdot\vec{s}}U(p;k) \\ &=-\frac{k_0}{\lambda_R}M_\Delta\Gamma\frac{e^{ip_0s}}{4\pi s} , \quad \vec{s}=\vec{r}-\vec{r}\,' , \end{aligned} \tag{23}$$

$$p_0^2\equiv\frac{M_\Delta}{\mu}\left[k^2-k_0^2+i\frac{k^2}{k_0^2}\mu\Gamma\right] . \tag{24}$$

Using Eq. (15), the exact wave function equivalent energy-independent potential is, in momentum representation,

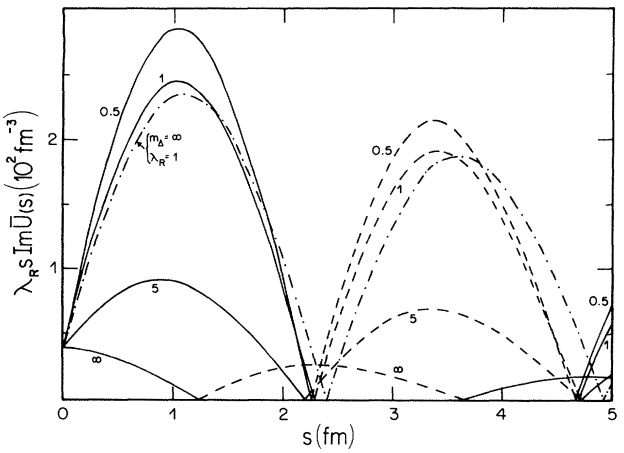

FIG. 1. The imaginary part of the energy-independent optical potential $\bar{U}(s)$ [Eq. (44)] plotted vs s for several values of λ_R (in fm). The dashed lines correspond to negative values of Im $\bar{U}$. The dash-dot line corresponds to $M_\Delta=\infty$ with $\lambda_R=1$ fm.

$$\bar{U}(p)=\tfrac{1}{2}(p^2-Q^2)\left[1+\frac{4\xi}{(p^2-Q^2)^2}\right]^{1/2}-\tfrac{1}{2}(p^2-Q^2)\,, \tag{25}$$

$$Q^2\equiv\frac{k_0^2+\frac{\mu}{M_\Delta}p^2}{1+i\mu\Gamma/k_0^2}\,, \tag{26}$$

$$\xi\equiv\frac{k_0\mu\Gamma/\lambda_R}{1+i\mu\Gamma/k_0^2}\,. \tag{27}$$

The coordinate space representation $\bar{U}(s)$ is shown in Fig. 1 for several values of λ_R. This parameter, which determines the strength of $U(p,k)$, corresponds to the mean free path at resonance for pion propagation in nuclear matter. For normal nuclear matter density, the mean free path is $\lambda_R\approx 1$ fm. We see from Fig. 1 that the range of the nonlocality depends strongly on the strength of the optical potential. This follows from the fact that the exact $\bar{U}$ [Eq. (44)] is nonlinear in the strength parameter ξ. Furthermore, note that the imaginary part of $\bar{U}(s)$ changes sign. These results point out the difficulties with phenomenological forms for $\bar{U}$: Even with a rather simple physical picture, $\bar{U}$ has a complicated structure not obviously related to the underlying physics.

This difficulty is reinforced by examining the $M_\Delta\to\infty$ limit of the model problem. The energy-dependent potential $U_k(s)$ is local in this limit [see Eqs. (23) and (24)]. Surprisingly, the energy-independent potential $\bar{U}(s)$ becomes even more nonlocal, as shown in Fig. 1. In fact, this peculiar behavior can be seen in the lowest order term of $\bar{U}$,

$$\bar{U}^{(1)}(p)=U(p;p)=\frac{\xi}{p^2-Q^2}\,, \tag{28}$$

$$\bar{U}^{(1)}(s)=\frac{k_0\mu\Gamma/\lambda_R e^{-i\alpha s}}{\left[1-\frac{\mu}{M_\Delta}+i\frac{\mu\Gamma}{k_0^2}\right]4\pi s}\,, \tag{29}$$

$$\alpha^2\equiv\frac{k_0^2}{1-\frac{\mu}{M_\Delta}+i\frac{\mu\Gamma}{k_0^2}}\underset{M_\Delta\to\infty}{\to}k_0^2\left\{\frac{1-i\mu\Gamma/k_0^2}{1+(\mu\Gamma/k_0^2)^2}+\frac{\mu}{M_\Delta}\frac{1-(\mu\Gamma/k_0^2)^2-2i(\mu\Gamma/k_0^2)}{[1+(\mu\Gamma/k_0^2)^2]^2}+\cdots\right\}. \tag{30}$$

As M_Δ increases, α decreases, corresponding to greater nonlocality. The energy-independent optical potential $\bar{u}$ has a complicated, nonintuitive structure.

Finally, we comment on the nonlocality structure for the case of rapid energy dependence in $U(E)$. This situation occurs in our simple model in the limit $\Gamma\to 0$. As already noted, the nonlocality in the energy-dependent optical potential is directly associated with the underlying physics. For Eqs. (22) and (23), the nonlocality range is given by the propagation distance of the resonance $d\equiv 2k_0/M_\Delta\Gamma$. For example, in the limit $M_\Delta\to\infty,\Gamma\to 0$ with d fixed, we still have

$$U_{k_0}(x)\sim e^{-s\sqrt{k_0/d}}\,, \tag{31}$$

i.e., the nonlocality is still finite and characterized by the resonance decay length. On the other hand, the rapid energy dependence in $U(p,k)$ generates rapid momentum dependence in $\bar{U}$, and thus a long-range nonlocality not associated with the underlying reaction dynamics. For example, we obtain

$$\bar{U}^{(1)}(s)\sim e^{-s(\mu\Gamma/2k_0)}=e^{-(s/d)(\mu/M_\Delta)}\,, \tag{32}$$

in the limit described above, meaning that the nonlocality becomes infinite in range. These conclusions are clearly independent of the specific model considered here. We note that, even with the less stringent requirement of phase shift equivalence, the energy-independent potential may develop similar pathological momentum dependence.[10]

III. ENERGY INDEPENDENT POTENTIAL FOR FINITE SYSTEMS AND THE MULTIPLE SCATTERING EXPANSION

We start this section with a formal derivation of the wave function equivalent energy independent optical potential, appropriate for scattering from a finite system. Many of the results here are not new,[3–6] although our discussion differs somewhat from those previously given; in particular, the symmetry properties of $\overline{U}$ for scattering with spin are discussed. We present a recursive expansion for $\overline{U}$ in terms of $U(E)$ and then apply this to a discussion of the multiple scattering expansion, demonstrating again the shortcomings of $\overline{U}$ as a vehicle for theoretical studies of hadron-nucleus scattering.

The energy-independent potential equivalent to $U(E)$ for incoming scattering states is defined by the relation

$$\overline{U}_+ \mid \Psi^{(+)}_{\vec{k}} \rangle = U(E_k) \mid \Psi^{(+)}_{\vec{k}} \rangle \ . \tag{33}$$

To solve for $\overline{U}_+$, we introduce the dual states $\langle \tilde{\Psi}^{(+)}_{\vec{q}} \mid$, defined such that

$$\langle \tilde{\Psi}^{(+)}_{\vec{q}} \mid \Psi^{(+)}_{\vec{k}} \rangle = (2\pi)^3 \delta(\vec{q}-\vec{k}) \ . \tag{34}$$

A formal expression for $\overline{U}_+$ follows directly, namely,

$$\overline{U}_+ = \int \frac{d\vec{k}}{(2\pi)^3} U(E_k) \mid \Psi^{(+)}_{\vec{k}} \rangle \langle \tilde{\Psi}^{(+)}_{\vec{k}} \mid \ . \tag{35}$$

We shall assume that there are no bound states (see Ref. 6 for a discussion of $\overline{U}$ in the presence of bound states), so $\overline{U}_+$ is unique by construction. We stress that the dual states $\langle \tilde{\psi}^{(+)}_k \mid$ are not identical with the dual states usually employed in low energy nuclear reaction theories.[11] In these theories, the optical potential is assumed, in conflict with Eq. (1), to be energy independent albeit complex. With this assumption, the dual states are given simply by the solution of the Schrödinger equation with U replaced by U^+ and with incoming boundary conditions. With an energy-dependent potential, the dual states can be obtained only with solution of an integral equation, which we now derive. The scattering parts of $\mid \psi^{(+)}_k \rangle$ and $\langle \tilde{\psi}^{(+)}_k \mid$ are defined by

$$\langle \vec{p} \mid \Psi^{(+)}_{\vec{k}} \rangle \equiv (2\pi)^3 \delta(\vec{p}-\vec{k}) + \phi^{(+)}_{\vec{k}}(\vec{p}) \ , \tag{36}$$

$$\langle \tilde{\Psi}^{(+)}_{\vec{q}} \mid \vec{p} \rangle \equiv (2\pi)^3 \delta(\vec{p}-\vec{q}) - \tilde{\phi}^{(+)}_{\vec{q}}(\vec{p}) \ . \tag{37}$$

The orthogonality relation [Eq. (34)] then yields

$$\tilde{\phi}^{(+)}_{\vec{q}}(\vec{k}) = \phi_{\vec{k}}(\vec{q}) - \int \frac{d\vec{p}}{(2\pi)^3} \phi^{(+)}_{\vec{k}}(\vec{p}) \tilde{\phi}^{(+)}_{\vec{q}}(\vec{p}) \ . \tag{38}$$

Since the scattering part of the incoming scattering wave function is given by

$$\begin{aligned} \phi^{(+)}_{\vec{k}}(\vec{p}) &= \langle \vec{p} \mid G_0(E_k^+) T(E_k^+) \mid \vec{k} \rangle \ , \\ &\equiv \langle \vec{p} \mid \omega(E_k^+) \mid \vec{k} \rangle \ , \end{aligned} \tag{39}$$

we have the formal solution of Eq. (38),

$$\tilde{\phi}^{(+)}_{\vec{q}}(\vec{k}) = \left\langle \vec{q} \left| \frac{1}{1+\omega(E^+_{\vec{\mathscr{P}}})} \omega(E_k) \right| \vec{k} \right\rangle \ . \tag{40}$$

Here, $\vec{\mathscr{P}}$ is the momentum operator acting to the right. This has a clear meaning in a power series expansion

$$\begin{aligned} \tilde{\phi}^{(+)}_{\vec{q}}(\vec{k}) = &\langle \vec{q} \mid \omega(E_k) \mid \vec{k} \rangle \\ &- \int \frac{d\vec{p}}{(2\pi)^3} \langle \vec{q} \mid \omega(E_p) \mid \vec{p} \rangle \langle \vec{p} \mid \omega(E_k) \mid \vec{k} \rangle \\ &+ \cdots \ . \end{aligned} \tag{41}$$

From Eq. (35), we have the matrix elements of $\overline{U}_+$,

$$\begin{aligned} \langle \vec{p} \mid \overline{U}_+ \mid \vec{q} \rangle = &\langle \vec{p} \mid T(E_q) \mid \vec{q} \rangle \\ &- \int \frac{d\vec{k}}{(2\pi)^3} \langle \vec{p} \mid T(E_k) \mid \vec{k} \rangle \tilde{\phi}^{(+)}_{\vec{k}}(\vec{q}) \ . \end{aligned} \tag{42}$$

Unfortunately, the solution of the integral equation for $\tilde{\phi}^{(+)}$ [Eq. (38)] is generally rather difficult, partly because of the peculiar intertwining of energy and momentum arguments seen in the expansion [Eq. (41)]. We shall return to a "Born expansion" of $\overline{U}_+$ below.

We have been careful to state that $\overline{U}_+$ is the equivalent potential only for incoming scattering states. In general, a different potential must be defined for outgoing waves,

$$\langle \Psi^{(-)}_{\vec{k}} \mid \overline{U}_- \equiv \langle \Psi^{(-)}_{\vec{k}} \mid U(E_k) \ . \tag{43}$$

The development given above for $\overline{U}_+$ can be repeated for $\overline{U}_-$, with the result

$$\langle \vec{p} \mid \overline{U}_- \mid \vec{q} \rangle^* = (\langle \vec{q} \mid \overline{U}_+ \mid \vec{p} \rangle)_{U \to U^\dagger} \ . \tag{44}$$

The expression on the right-hand side means that $U(E)$ should be replaced by $U^\dagger(E)$ everywhere on the right-hand side of Eq. (42), i.e., in $T(E)$ and in $\tilde{\phi}^{(+)}$. For scattering of spin-zero particles, the symmetry

$$\langle \vec{p} \mid U(E) \mid \vec{q} \rangle = \langle \vec{q} \mid U(E) \mid \vec{p} \rangle \ ,$$

together with Eq. (44), implies that

$$\langle \vec{p} \mid \overline{U}_- \mid \vec{q} \rangle = \langle \vec{q} \mid \overline{U}_+ \mid \vec{p} \rangle \ .$$

However, this does not apply in general. For example, the optical potential for scattering spin-$\frac{1}{2}$ and spin zero particles can be written as

$$\begin{aligned} \langle \vec{q} \mid U(E) \mid \vec{p} \rangle \equiv &\ V_0(q^2,p^2,\vec{q}\cdot\vec{p};E) \\ &+ V_1(q^2,p^2,\vec{q}\cdot\vec{p};E) i\hat{q}\times\hat{p}\cdot\vec{\sigma} \ . \end{aligned} \tag{45}$$

The relation given by Eq. (44) can be seen already in the first order contribution to $\overline{U}_\pm$ in an expansion in powers of $U(E)$,

$$\begin{aligned} \langle \vec{p} \mid \overline{U}^{(1)}_+ \mid \vec{q} \rangle = &\ V_0(q^2,p^2,\vec{q}\cdot\vec{p};E_q) \\ &+ V_1(q^2,p^2,\vec{q}\cdot\vec{p};E_q) i\hat{p}\times\hat{q}\cdot\vec{\sigma} \ , \\ \langle \vec{q} \mid \overline{U}^{(1)}_- \mid \vec{p} \rangle = &\ V_0(q^2,p^2,\vec{q}\cdot\vec{p};E_q) \\ &+ V_1(q^2,p^2,\vec{q}\cdot\vec{p};E_q) i\hat{q}\times\hat{p}\cdot\vec{\sigma} \ . \end{aligned} \tag{46}$$

This relationship between central and spin-orbit terms in $\overline{U}_{\pm}$ persists to all orders, as implied by Eq. (44), although it must be stressed that the central (spin-orbit) term in the full $\overline{U}$ will include contributions from V_1 (V_0) in higher order. We conclude that $\overline{U}_+$, constructed from Eq. (35), is unique[6] (in the absence of bound states[12]) but not symmetric. This is in contrast with phase-shift equivalent energy-independent potentials, which are symmetric but nonunique.

Our thrust will now be to understand the relationship of $\overline{U}$ to a microscopic theory of the optical potential. We shall use the multiple scattering theory, which is highly developed and quite successful. For this purpose, and to show more directly the relationship of $\overline{U}$ and $U(E)$, it is useful to expand Eqs. (42), (38), and (39) and $T(E)$ in powers of $U(E)$. We obtain a recursive expression

$$\langle \vec{p} \mid \overline{U}_+ \mid \vec{q} \rangle = \int \frac{d\vec{q}\,'}{(2\pi)^3} \langle \vec{p} \mid \sum_{n=0}^{\infty} \Delta_n(q,q') \mid \vec{q}\,' \rangle \langle \vec{q}\,' \mid U(E_q) \mid \vec{q} \rangle , \tag{47}$$

$$\Delta_0(q,q') = 1 , \tag{48}$$

$$\Delta_n(q,q') = \frac{\Delta_{n-1}(q,\vec{\mathscr{P}})U(E_q) - \Delta_{n-1}(q',\vec{\mathscr{P}})U(E_{q'})}{E_q - E_{q'}} , \quad n > 0 . \tag{49}$$

The operators Δ_n, for $n > 0$, play the role of "fluctuation" operators as they explicitly vanish when the optical potential is energy independent. The first three terms ($n < 2$) in this expansion have been explicitly written already by Ma *et al.*[6] Note that there is no singularity in $\Delta_n(q,q')$ since the numerator in Eq. (49) vanishes when $q = q'$. The correctness of the energy fluctuation expansion [Eqs. (47)—(49)] can be verified directly for the trivial effective mass example. Using Eq. (6) for the energy-dependent potential, we immediately obtain $\Delta_n = (c\rho)^n$. Equation (47) is then a simple geometric series which sums to give Eq. (9).

The success of the multiple scattering theory rests upon its provision of a rapidly convergent expansion of the optical potential. Successive terms in the expansion correspond to direct reaction processes involving more and more target nucleons. For algebraic simplicity, we work with the optical potential in the fixed-scatterer, large-A limit (the full optical potential, without approximation, could equally well be used). The first two terms in an expansion in powers of the projectile-nucleon t matrix are then[13]

$$\langle \vec{p} \mid U(E) \mid \vec{q} \rangle = A\rho(\vec{p} - \vec{q})t(\vec{p},\vec{q};E) + A^2 \int \frac{d\vec{q}\,'}{(2\pi)^3} C^{(2)}(\vec{p} - \vec{q}\,', \vec{q}\,' - \vec{q}) \times \frac{t(\vec{p},\vec{q}\,';E)t(\vec{q}\,',\vec{q};E)}{E^+ - E_{q'}} + \cdots , \tag{50}$$

$$C^{(2)}(\vec{p},\vec{q}) \equiv \int d\vec{x}\, d\vec{y} e^{i\vec{p}\cdot\vec{x} + i\vec{q}\cdot\vec{y}} C^{(2)}(\vec{x},\vec{y}) , \tag{51}$$

where the nuclear two-body correlation function is defined through the two-body density $\rho^{(2)}$ as

$$C^{(2)}(\vec{x},\vec{y}) = \rho^{(2)}(\vec{x},\vec{y}) - \rho(\vec{x})\rho(\vec{y}) , \qquad \int d\vec{y}\,\rho^{(2)}(\vec{x},\vec{y}) \equiv \rho(\vec{x}) . \tag{52}$$

The terms in the expansion of $U(E)$ have well-defined physical content: the first term corresponds to quasifree nucleon knockout; the second term includes a correction for projectile scattering from a correlated pair. It is the dominance of the intermediate energy hadron-nucleus reaction cross section by quasifree scattering that dictates an expansion of the optical potential in terms of $t(E)$. In the absence of more complicated reaction mechanisms, such as those associated with correlations, the expansion will naturally truncate. The unitarity constraints respected by the first order term lead to reasonable results even in the strong absorption limit, where multinucleon knockout becomes important.

In contrast, the expansion of $\overline{U}$ has none of these attractive features. Using Eqs. (47)—(52), we have the expansion through second order in t,

$$\langle \vec{p} \mid \overline{U}_+ \mid \vec{q} \rangle = A\rho(\vec{p} - \vec{q})t(\vec{p},\vec{q};E_q) + A^2 \int \frac{d\vec{q}\,'}{(2\pi)^3} [\rho^{(2)}(\vec{p} - \vec{q}\,', \vec{q}\,' - \vec{q})t(\vec{p},\vec{q}\,';E_q) - \rho(\vec{p} - \vec{q}\,')\rho(\vec{q}\,' - \vec{q})t(\vec{p},\vec{q}\,';E_{q'})] \otimes \frac{t(\vec{q}\,',\vec{q};E_q)}{E_q^+ - E_{q'}} + \cdots . \tag{53}$$

In the absence of correlations,

$$\rho^{(2)}(\vec{p},\vec{q})=\rho(p)\rho(q)\ , \tag{54}$$

the second term in $\overline{U}$ does not vanish; clearly, similar problems are present in higher order. A truncated $\overline{U}$ has no interpretation in terms of specific reaction mechanisms. It provides no reliable guidance for phenomenological construction of an energy-independent optical potential.

IV. CONCLUSIONS

We have discussed the construction and properties of the wave function equivalent energy-independent optical potential $\overline{U}$, considered recently by several authors.[2–6] A central question has been whether energy-independent potentials provide an effective approach to direct microscopic or phenomenological construction, recognizing that these two concerns are not independent. For example, the multiple scattering approach to the optical potential, in building in specific reaction mechanisms in a unitary way, provides a rapidly convergent scheme and thereby guidance for an effective phenomenology based upon the target geometry and elementary interaction parameters. The equivalent energy-independent potential $\overline{U}$ does not share these advantages.

The multiple scattering approach, used here as an effective example of microscopic construction of $U(E)$, leads to no systematic, reliable truncation scheme for $\overline{U}$. Different physical processes become entangled. For example, the inclusion of target correlations, which would directly lead to adding a term to the first order $U(E)$ with a specific geometry (i.e., ρ^2), has no special signature in $\overline{U}$. Thus, phenomenological incorporation of corrections to the basic theory are very difficult.

The nonlocality associated with specific physical mechanisms in $U(E)$ is reflected in a completely nonintuitive way in $\overline{U}$. We saw this in the example of the first order $U(E)$ drawn from the multiple scattering approach to intermediate energy pion-nucleus scattering. The nonlocality is then just the propagation or decay distance for the intermediate resonance. By contrast, decreasing this propagation distance actually led to an increase in the nonlocality of $\overline{U}$ and, in the limit of small resonance decay width with fixed propagation distance, the nonlocality range became infinite. Since this nonlocality structure is not obviously tied to the physics, with $\mathrm{Im}\,\overline{U}$ changing sign, phenomenological approaches are again basically without guidance.

Even for cases where a phenomenological local $U(E)$ is appropriate, the associated $\overline{U}$ may be uncomfortably complicated. For example, one might hope that, for a $U(E)$ with separate energy-independent and energy-dependent terms, the former would remain isolated in $\overline{U}$. This is not the case, as shown by the example of a local $U(E)$ with up to quadratic energy dependence. The energy-independent potential is nonlocal and has "interferences" between the energy-independent and energy-dependent terms of $U(E)$. Since one may expect that the different energy dependences in $U(E)$ arise from different physical processes, this is again an example of how the physics get mixed up in going to $\overline{U}$. Finally, we discussed the symmetry of $\overline{U}$. In general, different potentials are needed for incoming and outgoing scattering states.

In conclusion, while the concept of an energy-independent optical potential may be of use in limited contexts [such as when $U(E)$ has a very weak energy dependence], $\overline{U}$ has serious shortcomings in not reflecting the underlying physics understandably.

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