Electromagnetically induced nuclear beta decay in electric-field gauge

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(Received 29 December 1983)

The theory of the electromagnetic enhancement of nuclear beta decay, previously carried out in Coulomb gauge, is reformulated in electric-field gauge. Final results for the transition probability are identical in the two gauges, but significant differences appear in the two formulations. Whereas field interactions in Coulomb gauge arise from an intimate mixing of nuclear and decay-electron contributions, the entire effect is associated with the beta particle in electric-field gauge. This has the advantage that it is not necessary to assign field-interacting nuclear wave functions in electric-field gauge. The results are a demonstration that physical interpretations can be gauge dependent, even though measurable predictions are gauge invariant.

I. INTRODUCTION

The theory of electromagnetically enhanced nuclear beta decay was given in Refs. 1 and 2 (hereafter referred to as I and II). When forbidden decays are treated, the effects of the applied plane-wave field can be substantial as a result of field-induced angular momentum and/or parity contributions which serve to overcome forbiddenness in the decay. As shown in II, these forbiddenness-removing terms arise from terms transverse in the applied field as well as longitudinal in the applied field. All work in I and II is done entirely in Coulomb gauge. The transverse effects are associated with an intimate mixture of nuclear and beta particle interactions with the applied field, whereas the longitudinal effects come entirely from the interaction of the beta particle with the field.

In this paper, the formalism for electromagnetically enhanced beta decay will be recast in electric-field (EF) gauge terms. EF gauge is defined[3] to be the logical completion of Göppert-Mayer[4] or "$\vec{E}\cdot\vec{r}$" gauge when there is no recourse to long-wavelength approximation nor neglect of magnetic field effects. Explicitly, the scalar and vector potentials for a monochromatic plane-wave field in EF gauge are

$$\phi=-\vec{r}\cdot\vec{E}(t,\vec{r}) \ , \tag{1}$$

$$\vec{A}=-(\vec{k}/\omega)[\vec{r}\cdot\vec{E}(t,\vec{r})] \ , \tag{2}$$

where the electric field vector $\vec{E}$ is a function of $\omega t-\vec{k}\cdot\vec{r}$, $\vec{k}$ is the propagation vector for the field of circular frequency ω (i.e., $|\vec{k}|=\omega$), and c is taken to be unity here (as is also $\hbar$ in subsequent work). When the vector potential is neglected and $\vec{E}$ is taken to be a function of time only [i.e., $\vec{E}(t,\vec{r})\rightarrow\vec{E}(t)$], then Eqs. (1) and (2) reduce to the familiar $\vec{E}\cdot\vec{r}$ gauge. As written, Eqs. (1) and (2) are complete Lorentz potentials which define monochromatic plane-wave electric and magnetic fields without approximations.

When electromagnetically induced beta decay is expressed in EF gauge, it is shown that field interaction with the nucleus can be neglected. The dominant field contributions all come from field interaction with the beta particle. The algebraic expression for the transition amplitude, however, is identical to the expression found in I in Coulomb gauge. Predicted transition probabilities are then precisely the same in the two gauges. This was remarked upon in II.

An important feature of the identity of the results in EF and Coulomb gauges is that the momentum-translation approximation (MTA) employed in Coulomb gauge no longer appears in EF gauge. One can say that the approximation inherent in the MTA in Coulomb gauge is replaced in EF gauge by the approximation of neglecting altogether the interaction of the field with the nucleus. A related feature of the comparison between EF and Coulomb gauges is the significant difference in physical interpretation appropriate in the two gauges. Nuclear interaction with the field plays a significant role in Coulomb gauge, but none at all in EF gauge. There is precedent for this striking change in physical interpretation with gauge in other areas in physics. For example, Bjorken and Drell[5] show that the Compton scattering of a scalar particle can be computed in a gauge in which the entire effect appears to be due to the "seagull" diagram, i.e., to a process confined exclusively to the quadratic e^2A^2 term in the interaction Hamiltonian. Another example is given by Friar and Fallieros.[6] They show that, although the atomic magnetic susceptibility is a physically measurable and gauge invariant quantity, the decomposition of total susceptibility into diamagnetic and paramagnetic components is gauge dependent.

The EF-gauge interacting lepton wave functions are given in Sec. II below. Interacting nuclear wave functions are examined in Sec. III. This involves an inspection of the order of magnitude of each of the interaction terms in the equation of motion. It is found that the EF-gauge interaction terms can be neglected. The transition amplitude for field-enhanced beta decay in EF gauge is stated in Sec. IV, where it is found to be identical to the Coulomb gauge result. A physical interpretation of the results is discussed in Sec. V.

II. INTERACTING LEPTON WAVE FUNCTIONS

The beta decay electron is described by a wave function for a free, charged particle in the presence of a plane-wave electromagnetic field. Coulomb corrections are neglected. The appropriate wave function is the so-called Volkov solution.[7,8] The result in Coulomb gauge for an electron in a monochromatic, linearly polarized field is given in Eq. (17) of I.

The Volkov solution is normally stated in Coulomb gauge. One application where it is used in $\vec{E}\cdot\vec{r}$ gauge is in a paper by Keldysh,[9] where the Volkov solution is used in a long-wavelength, nonrelativistic version. The result employed in the Keldysh paper follows simply by applying the gauge-transformation factor $\exp[ie\vec{r}\cdot\vec{A}(t)]$, where e is a positive number (that is, the charge of the electron is $-e$), and $\vec{A}(t)$ is the Coulomb-gauge vector potential in the long-wavelength approximation. A simple generalization of the same technique will be applied here. The Volkov solution given in Eq. (17) of I is simply multiplied by the gauge transformation factor[3]

$$\exp[ie\vec{r}\cdot\vec{A}(t,\vec{r})]\ , \tag{3}$$

which carries wave functions from Coulomb gauge to EF gauge. In Eq. (3), $\vec{A}(t,\vec{r})$ is the full space- and time-dependent Coulomb-gauge vector potential for the plane-wave field.

The neutrino wave function does not reflect any interaction with the applied field. The function given in Eq. (13) of I remains appropriate. Change in gauge of the overall lepton part of the four-fermion interaction of beta decay is accomplished with Eq. (3) above.

III. INTERACTING NUCLEAR WAVE FUNCTIONS

The essential angular momentum properties of the nucleus are in terms of internal, i.e., relative coordinates. The nucleus is therefore separated into interacting ("fragment") and noninteracting ("core") parts, and the state of the system is expressed in center-of-mass (c.m.) and relative coordinates. It is appropriate to treat the nuclear wave functions as nonrelativistic, and to employ a long-wavelength approximation for the applied field. The two-body nonrelativistic Schrödinger equation in the presence of a long-wavelength plane-wave electromagnetic field expressed in EF gauge is[3]

$$\begin{aligned} i\partial_t\Psi(t,\vec{r},\vec{R}) = \{ &-e_t\vec{E}\cdot\vec{R} - \tilde{e}\vec{E}\cdot\vec{r} + (1/2m_t)[-i\vec{\nabla}_R + e_t(\vec{k}/\omega)\vec{E}\cdot\vec{R} + \tilde{e}(\vec{k}/\omega)\vec{E}\cdot\vec{r}]^2 \\ &+ (1/2m_r)[-i\vec{\nabla}_r + \tilde{e}(\vec{k}/\omega)\vec{E}\cdot\vec{R} + e_e(\vec{k}/\omega)\vec{E}\cdot\vec{r}]^2 + V(r)\}\Psi(t,\vec{r},\vec{R})\ , \end{aligned} \tag{4}$$

where $\vec{r}$ and $\vec{R}$ are relative and c.m. coordinates, respectively, defined by

$$\vec{R} = (m_1\vec{r}_1 + m_2\vec{r}_2)/m_t\ ,\quad \vec{r} = \vec{r}_1 - \vec{r}_2\ , \tag{5}$$

with the subscript 1 referring to fragment quantities and the subscript 2 referring to core quantities. The masses and charges appearing in Eqs. (4) and (5) are defined by[1,3]

$$\begin{aligned} &m_t = m_1 + m_2\ ,\quad m_r = m_1m_2/m_t\ , \\ &e_t = e_1 + e_2\ ,\quad \tilde{e} = (e_1m_2 - e_2m_1)/m_t\ , \\ &e_e = (e_1m_2^2 + e_2m_1^2)/m_t^2\ . \end{aligned} \tag{6}$$

As a preliminary to assessing the orders of magnitude of the various interaction terms that appear in Eq. (4), the magnitudes of the quantities in Eqs. (5) and (6) will be estimated. In Eq. (6), the total mass m_t and reduced mass m_r are simply

$$m_t = AM\ ,\quad m_r = O(M)\ , \tag{7}$$

where M is a nucleon mass, and A is the nuclear mass number. The total charge e_t, reduced charge $\tilde{e}$, and effective charge e_e are

$$e_t = Ze\ ,\quad \tilde{e} = O(e)\ ,\quad e_e = O(e)\ , \tag{8}$$

where e is the proton charge and Z is the proton number of the nucleus. The relative coordinate $\vec{r}$ in Eq. (5) is confined by the nuclear binding forces to be of the order of the nuclear radius R_0, and so

$$|\vec{r}| = O(R_0)\ . \tag{9}$$

The magnitude of the c.m. coordinate $\vec{R}$ will be estimated after a few other preliminaries.

Six interaction terms in Eq. (4) will be examined. They are

$$e_t\vec{E}\cdot\vec{R}\ , \tag{10a}$$

$$\tilde{e}\vec{E}\cdot\vec{r}\ , \tag{10b}$$

$$e_t\vec{E}\cdot\vec{R}\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_R}{2m_t}\ , \tag{10c}$$

$$\tilde{e}\vec{E}\cdot\vec{r}\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_R}{2m_t}\ , \tag{10d}$$

$$\tilde{e}\vec{E}\cdot\vec{R}\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_r}{2m_r}\ , \tag{10e}$$

$$e_e\vec{E}\cdot\vec{r}\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_r}{2m_r}\ . \tag{10f}$$

The $\vec{p}_R$ and $\vec{p}_r$ quantities in Eqs. (10c)–(10f) are the operators $(-i\vec{\nabla}_R)$ and $(-i\vec{\nabla}_r)$, respectively. Equation (4) also contains quadratic interaction terms in addition to the linear terms in Eq. (10), but these quadratic contributions can be neglected from the outset.

The magnitude of the electric field will be expressed in terms of the intensity parameter

$$z_f = \frac{e^2\langle \vec{E}^2 \rangle}{m^2\omega^2}, \tag{11}$$

where the angular bracket on $\vec{E}^2$ refers to a time average over a wave period. The mass m in the denominator of Eq. (11) is an electron mass. Although it was concluded in II that values of z_f up to 10 were of interest, the estimates made here will consider z_f values as large as 10^4. In other words, wherever $\vec{E}$ appears, it will be replaced in magnitude by

$$|\vec{E}| = O(z_f^{1/2} m\omega/e)$$
$$\leq O(10^2 m\omega/e). \tag{12}$$

To estimate the magnitude of the c.m. coordinate $\vec{R}$, it is presumed that the c.m. of the nucleus moves freely under the influence of the applied field, and that the relevant magnitude of $\vec{R}$ is given by the change in $\vec{R}$ which occurs in the course of a "beta decay time." This time is given by $1/m$ from the Heisenberg uncertainty relation $\Delta t \Delta E \approx 1$, with $\Delta E \approx m$. The amplitude of motion of a charged system (the nucleus) under the influence of an applied plane-wave field is[10,11]

$$|\omega\vec{R}|_{\max} \approx (2z_N)^{1/2}, \tag{13}$$

where z_N is the nuclear equivalent of the electron intensity parameter z_f stated in Eq. (11). The only difference is in the squared mass, and so

$$z_N = \left[\frac{m}{AM}\right]^2 z_f \approx 3\times 10^{-11} z_f, \tag{14a}$$

where A is the nuclear mass number, and M/m is the proton-to-electron mass ratio. The numerical estimate in Eq. (14a) follows from taking $A \approx 10^2$ for a medium weight nucleus. With $z_f \leq 10^4$ as assumed above, then

$$z_N \leq 3\times 10^{-7}. \tag{14b}$$

The change in $\vec{R}$ in the fraction of the period of motion $2\pi/\omega$ occupied by the beta decay time $1/m$ is thus

$$|\omega\vec{R}| = O\left[\frac{1/m}{2\pi/\omega}|\omega\vec{R}|_{\max}\right]$$
$$\approx O\left[\frac{\omega}{2\pi m}(6\times 10^{-11} z_f)^{1/2}\right]$$
$$\leq O\left[10^{-4}\frac{\omega}{m}\right]. \tag{15}$$

The c.m. momentum factors that appear in Eq. (10) can be estimated from the same solutions for free-body motion in a plane-wave field that were used to estimate $\vec{R}$. In fact, the result for the maximum $|\vec{p}_R|$ is given by the same expression as found in Eq. (13), i.e.,

$$|\vec{p}_R|/m_t \leq (2z_N)^{1/2} \approx 10^{-5} z_f^{1/2} \leq 10^{-3}, \tag{16}$$

where the numerical factors in Eq. (16) arise from Eqs. (14a) and (14b).

To estimate a value for $\vec{p}_r$, one can assume a single nucleon bound to the rest of the nucleus by a few MeV, in a potential well of depth 50 MeV or so. This gives

$$\frac{|\vec{p}_r|}{2m_r} = O\left[\left[\frac{p_r^2}{2M}\right]^{1/2}\frac{1}{(2M)^{1/2}}\right]$$
$$= O\left[\left[\frac{50\text{ MeV}}{2\times 10^3\text{ MeV}}\right]^{1/2}\right] = O(10^{-1}). \tag{17}$$

All of the above estimates can be brought together to assess the interaction terms in Eq. (10). From Eqs. (8), (12), and (15), the magnitude of Eq. (10a) as compared to a typical beta-decay transition energy ($\Delta E \approx m$) is

$$\frac{|e_t\vec{E}\cdot\vec{R}|}{m} \leq \frac{1}{m}O\left[(Ze)\left[10^2\frac{m\omega}{e}\right]\left[10^{-4}\frac{1}{m}\right]\right]$$
$$\leq O(\omega/m). \tag{18a}$$

The corresponding result for Eq. (10b) is, from Eqs. (8), (12), and (9),

$$\frac{|\tilde{e}\vec{E}\cdot\vec{r}|}{m} \leq \frac{1}{m}O\left[e\left[10^2\frac{m\omega}{e}\right]R_0\right]$$
$$\leq O(\omega/m). \tag{18b}$$

In this result, the value

$$mR_0 \approx 10^{-2},$$

typical of medium-weight nuclei, has been used. To judge the interaction term in Eq. (10c), the results in Eqs. (18a) and (16) can be combined to give

$$\frac{1}{m}\left|e_t\vec{E}\cdot\vec{R}\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_R}{2m_t}\right| \leq O\left[10^{-3}\frac{\omega}{m}\right], \tag{18c}$$

which involves the arbitrary restriction $z_f \leq 10^4$. In like fashion, Eqs. (18b) and (16) lead to the same result

$$\frac{1}{m}\left|\tilde{e}\vec{E}\cdot\vec{r}\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_R}{2m_t}\right| \leq O\left[10^{-3}\frac{\omega}{m}\right]. \tag{18d}$$

The combination of Eqs. (8), (12), (15), and (17) in Eq. (10e) yields

$$\frac{1}{m}\left|\tilde{e}\vec{E}\cdot\vec{R}\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_r}{2m_r}\right| \leq O\left[10^{-3}\frac{\omega}{m}\right]. \tag{18e}$$

Finally, Eq. (10f) becomes

$$\frac{1}{m}\left|e_e\vec{E}\cdot\vec{r}\frac{\vec{k}}{\omega}\cdot\frac{\vec{p}_r}{2m_r}\right| \leq O\left[10\frac{\omega}{m}mR_0\right]$$
$$\leq O\left[10^{-1}\frac{\omega}{m}\right]. \tag{18f}$$

Important conclusions can be drawn from Eqs. (18a)–(18f). Although the numerical coefficients in these results are only rough estimates, the essential fact is that all six interaction terms are proportional to ω/m. As envisioned in I and II, applied fields are best chosen to be of frequencies less than visible, and so ω/m is an extremely small number. Therefore, it is appropriate to neglect all field interaction with the nucleus in EF gauge.

The same conclusions do not hold in Coulomb gauge. Unlike EF gauge, field interaction with the nucleus is important in Coulomb gauge. Were there only an electromagnetic field interaction in the problem, or if the field represented a gamma ray, then the ratio ω/m would be of order unity, and many of the interaction terms in Eq. (18) would be very important. The major distinction between EF and Coulomb gauges, as far as the nuclear wave functions are concerned, arises only because essentially all of the interaction energy is in the weak interaction, not in the electromagnetic field interaction. If the interaction were purely electromagnetic, then gauge equivalence would demand like results in both gauges.

IV. TRANSITION AMPLITUDE IN EF GAUGE

An S matrix was derived in I which describes the beta decay undergone by a nucleus in the presence of an electromagnetic field. The asymptotic states include the full effects of the field, and the weak interaction is treated as a first-order perturbation. This formalism, in its general expression, is entirely independent of gauge. The S matrix is

$$S_{fi}=-i\frac{G}{2^{1/2}}\int d^4x[\overline{\Psi}_f\gamma_\mu(1-\kappa\gamma^5)\Psi_i]\times[\overline{\Psi}^{(e)}\gamma^\mu(1-\gamma^5)\Psi^{(\nu)}]\;, \tag{19}$$

where G is the weak-interaction coupling constant, the γ^μ and γ^5 are Dirac matrices employed with the conventions of, e.g., Bjorken and Drell,[12] κ is the ratio of axial-vector to vector coupling strength, Ψ_f and Ψ_i are final and initial nuclear wave functions in the presence of the field, and $\Psi^{(e)}$ and $\Psi^{(\nu)}$ are electron and neutrino wave functions in the presence of the field.

In EF gauge, as concluded above, Ψ_f and Ψ_i are simply the ordinary nuclear wave functions with no field present. In Coulomb gauge, the product $\overline{\Psi}_f\Psi_i$, which appears in Eq. (19), consists of the ordinary nuclear wave functions and the further factor

$$\exp[-i(\widetilde{e}_f-\widetilde{e}_i)\vec{r}\cdot\vec{A}(t,\vec{r})]\;. \tag{20}$$

In I in Coulomb gauge, the lepton wave functions are specified by the Dirac Volkov wave function for the electron, and a free particle solution for the antineutrino. In EF gauge, the only difference in the lepton functions is the extra factor contributed by Eq. (3), which means that the $\overline{\Psi}^{(e)}\Psi^{(\nu)}$ combination has the extra factor in EF gauge

$$\exp[-ie\vec{r}\cdot\vec{A}(t,\vec{r})] \tag{21}$$

as compared to Coulomb gauge. Therefore, the transition amplitude expressions in the two gauges are identical apart from the difference between Eqs. (20) and (21). However, as shown in Eq. (49) of I,

$$\widetilde{e}_f-\widetilde{e}_i=e\frac{A_{\text{core}}}{A}\approx e\;, \tag{22}$$

where A is the total mass number and A_{core} refers to the number of nucleons in the core alone. The substitution $\widetilde{e}_f-\widetilde{e}_i\to e$ was made in I from Eq. (49) on. This means that the field-enhanced beta decay calculated in Coulomb gauge in I and II is identical to the result as calculated in EF gauge.

V. PHYSICAL INTERPRETATION

There is nothing surprising in the fact that a calculation of electromagnetic-field-enhanced transition probabilities in nuclear beta decay gives the same results in Coulomb gauge and in EF gauge. It is reassuring that gauge invariance is confirmed for the physically observable transition probability. There are several qualitative differences in the two calculations, however, which are worthy of note. These differences are the following: the very different roles of the nuclear interaction with the field in the two gauges, the consequent ability to avoid the MTA in EF gauge, and the significant difference in apparent physical mechanisms that occur in the two gauges.

In Coulomb gauge, a significant portion of the transverse component of the transition amplitude arises from field interaction with the nucleus. In EF gauge, field interaction with the nucleus can be neglected altogether. It is important to note that interaction with the electromagnetic field is only one component of the two-interaction field-enhanced beta decay transition. The other part—the weak interaction—is the part which carries with it the overwhelming proportion of the total transition energy when low-frequency applied fields are considered. As has been pointed out by Grynberg and Giacobino[13] in a different context, comparisons of gauges in multi-interaction processes can be very misleading unless all of the interactions are considered.

Because the effect of the applied field on the nuclear wave functions can be entirely neglected in EF gauge when the field frequency is low ($\omega/m\ll1$), there is no need to stipulate a field-interacting nuclear wave function in EF gauge. This circumvents altogether the MTA that was employed in Coulomb gauge. Although the MTA is very simple to apply, it is nevertheless advantageous to avoid entirely any need to specify an approximate field-interacting nuclear wave function. The approximation of completely ignoring field-nucleus interaction in EF gauge is tantamount to the approximation involved in the MTA in Coulomb gauge. However, it is clearly more straightforward to use EF gauge in this problem.

The notion of what physical mechanisms are at work in field-enhanced beta decay differs in EF gauge from what it is in Coulomb gauge. With particular reference to enhancement of forbidden beta decays, the necessary additional angular momentum and/or parity from the field arises entirely from interaction with the decay electron in EF gauge. In Coulomb gauge, the additional angular momentum and/or change in parity can arise from field interactions with all charged systems that appear—the nuclear fragment as well as the beta particle. There is a certain resemblance between this gauge disparity in physical interpretation and that which occurs in Compton scattering of a charged pion.[5] Normally, the Compton scattering process arises (in a Feynman diagram sense) from the single-vertex $A \cdot p$ interaction. This is certainly true in the Compton scattering of a charged fermion, and it is largely true for charged bosons as well. However, there is a gauge in which the entire transition probability for pion Compton scattering comes from the double-vertex or "seagull" diagram which is associated with the $A \cdot A$ term. Clearly, caution is needed in ascribing physical significance to $A \cdot p$ vis-a-vis A^2 interactions. Perhaps even more striking is the example of atomic magnetic susceptibility.[6] For the case of a single spinless particle in a spherically symmetric binding potential in interaction with a uniform static magnetic field, the decomposition of magnetic susceptibility into diamagnetic and paramagnetic components is strongly gauge dependent. For the example cited, there exists one gauge in which the susceptibility is wholly diamagnetic, with zero paramagnetic component. Yet there is another gauge in which the diamagnetic component is twice as large as the total susceptibility, with half of that contribution cancelled by a paramagnetic portion equal in magnitude to the total susceptibility. This radical change in "physical" interpretation in different gauges emphasizes the danger of ascribing a physical meaning to any quantity which is not measurable.

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