Extended Siegert theorem

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The extension of Siegert's theorem for the retarded electric multipole field is constructed. Analogously, a form of the electric current is developed which is manifestly conserved. Isolation of those components of the Fourier transform of the current which are constrained by current conservation is shown to be unique. In the new form of the electric multipole fields, the current enters only in the combination $\vec{\mu}(\vec{x})=\frac{1}{2}\vec{x}\times\vec{J}(\vec{x})$.

I. INTRODUCTION

The history of photonuclear physics is the story of Siegert's theorem.[1] At the dawn of quantum mechanics, Schrödinger performed the first modern quantum calculation of an electromagnetic transition, treating electric dipole decays in the hydrogen atom.[2] In the absence of a quantum theory of electrodynamics, Schrödinger made a semiclassical assumption about the coupling of the photon to the atomic currents. Specifically, he assumed that for long wavelength photons the transition was determined by the time rate of change of the electric dipole operator, an assumption which holds both classically and quantum mechanically. The correspondence between the long wavelength current and the time rate of change of the dipole operator depends on current conservation, which has impeccable credentials. Schrödinger later proved current conservation for his equation,[2] assuming "ordinary" forces and currents. However, the specific form of the current is not obvious; indeed, even now it is only partially known in nuclear physics. By the time of the first photonuclear experiment by Chadwick and Goldhaber[3] it was known that the nuclear force contained elements not present in an atom. Not only was this force of short range, but it contained an "exchange" component, which could interchange the positions of a neutron and a proton. The latter obviously generated an electromagnetic current, of uncertain form and origin at that time, which is not present in atoms or classical systems, and is certainly not ordinary. The theoretical work of that era worried about this complication,[4,5] but continued to use Schrödinger's approach.

Clarification of the situation was provided by Siegert in two ways: (1) he showed by the use of current conservation in the long wavelength limit that the electromagnetic current is equivalent to the time rate of change of the multipole fields based on the charge operator; (2) he showed that the nonrelativistic charge density should be an excellent approximation to the complete density, since nucleons are heavy and move more slowly than whatever binds the nucleons together. Corrections to the nonrelativistic density should be of order $1/c^2$. This pair of elements is known as Siegert's theorem; it is actually a nonrelativistic approximation as discussed above and is the foundation stone of photonuclear physics. Simple calculations of low energy nuclear electric dipole processes using the "classical" charged particle current $q\vec{v}/c$, where $\vec{v}$ is the particle velocity and q is its charge, typically are 50—100% at variance with the Siegert form of the current. The defect is caused by the "hidden" exchange currents, which are produced by the motion of charged mesons or non-nucleonic constituents. Since the motion of any charged (virtual) particles in a nucleus will lead to an interaction or exchange current, it is necessary to have a detailed theoretical understanding of the underlying strong interaction processes binding a nucleus together in order to calculate those currents. Thus Siegert's theorem, by providing a calculational framework for electric multipole transitions, made photonuclear experiments interpretable theoretically.

The early theoretical and experimental work was dominated by long wavelength considerations. Modern photonuclear work[6–8] at higher photon energies and electron scattering from nuclei leave that regime, and require consideration of electric multipole processes at short wavelengths. Unfortunately, no variant of Siegert's theorem exists for arbitrary wavelengths. It is our purpose here to provide the framework. We will do this by two different methods. The first method does not use a multipole expansion; it does use several "tricks." The resulting form of the current is *manifestly* conserved. A multipole decomposition is then performed by an unconventional method. The second technique assumes analyticity in the photon frequency and expands the current in a power series in the photon momentum $\vec{q}$ using a Cartesian basis. Such bases are no longer very common in theoretical work[9–11] with most practitioners preferring the more tractable spherical tensor basis. The second technique leads to the same result as the first, but displays in a much more transparent way the uniqueness of our prescription.

We emphasize that for a conserved current our new electric multipole formulae are exactly equivalent to any existing forms. It is only for the (physical) case where the current is not entirely known and therefore not includable in a detailed calculation that there will be a difference. The basis of our argument is that those components of the current which are in principle determined or constrained by current conservation can be identified, extracted, and rearranged as multipole fields based on the charge density, just as Siegert's result was. The remaining components of the current $\vec{J}(\vec{x})$ are determined by a single functional form,

$$\vec{\mu}(\vec{x})=\tfrac{1}{2}\vec{x}\times\vec{J}(\vec{x}) , \tag{4c}$$

the magnetic moment density. Thus the non-Siegert-type parts of the electric multipole fields are determined by the same quantity, $\vec{\mu}(\vec{x})$, which specifies the magnetic multipoles. It is now well known that the magnetic density is sensitive to exchange currents, and that magnetic multipoles are unconstrained by current conservation and are consequently the place to look for interaction currents.[12,13]

Finally, we note that even in cases where meson-exchange currents are missing we expect our new multipole formulae, the extension of Siegert's original long wavelength work, to be useful. Current conservation is difficult to enforce in many-body calculations, because one develops wave functions which are only approximate eigenfunctions of the original many-body Hamiltonian. Thus, even in these cases different multipole formulae will lead to different numerical results. We will argue later that our forms are more suitable and stable than others, and should be used. Thus, this work is also applicable to atomic physics, for example.

The rest of the paper is organized as follows: (1) Section II develops the new representation for the current; (2) Sec. III performs a multipole decomposition of this current; (3) Sec. IV provides an alternative derivation; (4) Sec. V discusses the results; and (5) the Appendix details the properties of auxiliary functions which arise and replace spherical Bessel functions. The casual reader may skip Secs. III and IV, which are tedious. Our primary results are listed below and identified by the appropriate equation numbers in the text. The Jth electric multipole field $T^{\rm el}_{JM}$ is given by

$$T^{\rm el}_{JM}=-\frac{q^{J-1}[(J+1)/J]^{1/2}}{(2J+1)!!}\times[H_0,\int d^3x\, x^J Y_{JM}(\hat{x})g_J(qx)\rho(\vec{x})]+\frac{2q^{J+1}}{(J+2)(2J+1)!!}\int d^3x\, x^J\vec{Y}^{M}_{JJ}\cdot\vec{\mu}(\vec{x})h_J(qx) , \tag{55'}$$

where

$$g_J(z)=\frac{J(2J+1)!!}{z^J}\int_0^z dz'\frac{j_J(z')}{z'} \tag{51'}$$

and

$$h_J(z)=-\frac{(J+2)}{Jz}\frac{d}{dz}z^{-2J}\frac{d}{dz}[z^{2J+1}g_J(z)] . \tag{52'}$$

The Fourier transform of the current $\vec{J}(\vec{x})$ can be rewritten in the form

$$\vec{J}(\vec{q})\equiv\int d^3x\, e^{i\vec{q}\cdot\vec{x}}\vec{J}(\vec{x}) \tag{2b'}$$

$$=i[H_0,\int d^3x\, \vec{x}\alpha(\vec{q}\cdot\vec{x})\rho(\vec{x})]-i\vec{q}\times\int d^3x\, \vec{\mu}(\vec{x})\beta(\vec{q}\cdot\vec{x}) , \tag{6'}$$

where

$$\alpha(z)=\frac{e^{iz}-1}{iz}=\int_0^1 d\lambda\, e^{i\lambda z} \tag{7a'}$$

and

$$\beta(z)=\frac{2}{z^2}[e^{iz}(1-iz)-1]=2\int_0^1 d\lambda\,\lambda e^{i\lambda z} . \tag{7b'}$$

These are the principal results of this paper. Note that $\vec{J}$ in Eq. (6′) is manifestly conserved:

$$\vec{q}\cdot\vec{J}(\vec{q})=[H_0,\rho(\vec{q})] , \tag{1'}$$

where

$$\rho(\vec{q})=\int d^3x\, e^{i\vec{q}\cdot\vec{x}}\rho(\vec{x}) . \tag{2a'}$$

II. SIEGERT CURRENT

Our aim, in this section, is the derivation of Eqs. (6) below, which casts the current into a form which is manifestly conserved. The basic input in this derivation will be the continuity equation in momentum space:

$$\vec{q}\cdot\vec{J}(\vec{q})=-i\dot{\rho}(\vec{q})\equiv[H_0,\rho(\vec{q})] , \tag{1}$$

where H_0 is the nuclear Hamiltonian, while the charge-density and current-density operators are given in terms of their corresponding expressions in ordinary space by the standard Fourier transforms

$$\rho(\vec{q})=\int d^3x\, \rho(\vec{x})e^{i\vec{q}\cdot\vec{x}} \tag{2a}$$

and

$$\vec{J}(\vec{q})=\int d^3x\, \vec{J}(\vec{x})e^{i\vec{q}\cdot\vec{x}} . \tag{2b}$$

We wish to express the full current $\vec{J}(\vec{q})$ in any direction in terms of quantities that are both convenient and physically meaningful. The continuity equation will help us show that one of these quantities should be the charge density itself. That there is no difficulty in obtaining information about the transverse current from an expression such as Eq. (1), which deals only with its longitudinal component, is well demonstrated by the original form of Siegert's theorem[1]

$$\vec{J}(0)=i[H_0,\int d^3x\, \vec{x}\rho(\vec{x})] , \tag{3}$$

which is easily seen to be a simple relation between the slopes of the two sides of Eq. (1) at $\vec{q}=0$. The point is, of course, that the quantity $\vec{q}$ is not a fixed vector in space, but a variable, and it can be eliminated by differentiation. It is also clear that this conclusion should not be overextended in the opposite direction to an expectation that repeated differentiation will give all the derivatives of $\vec{J}(\vec{q})$ at $\vec{q}=0$ and hence the quantity itself at any $\vec{q}$. The right-hand side of Eq. (1) is a scalar and all its derivatives with respect to the Cartesian components q_μ, q_ν, etc., will give a function symmetric in μ, ν, etc. The same must be true for the left-hand side and, therefore, only symmetric expressions such as

$$\frac{\partial}{\partial q_\mu}J_\nu(0)+\frac{\partial}{\partial q_\nu}J_\mu(0)=-i\frac{\partial^2}{\partial q_\mu \partial q_\nu}\dot{\rho}(0)$$

can be obtained from this procedure. What is significant for our purpose is the observation that the antisymmetric expression, which remains completely undetermined, is simply the curl of $\vec{J}$, which in turn gives the definition of the magnetization density operator:

$$\vec{\nabla}_q \times \vec{J}(\vec{q})=2i\vec{\mu}(\vec{q}) , \tag{4a}$$

with

$$\vec{\mu}(\vec{q})=\int d^3x\, \vec{\mu}(\vec{x})e^{i\vec{q}\cdot\vec{x}} \tag{4b}$$

and

$$\vec{\mu}(\vec{x})=\tfrac{1}{2}\vec{x}\times\vec{J}(\vec{x}) . \tag{4c}$$

Use of the standard identity

$$\vec{\nabla}_q[\vec{q}\cdot\vec{J}(\vec{q})]=(\vec{q}\cdot\vec{\nabla}_q)\vec{J}(\vec{q})+[\vec{J}(\vec{q})\cdot\vec{\nabla}_q]\vec{q}+\vec{q}\times[\vec{\nabla}_q\times\vec{J}(\vec{q})]$$

together with $[\vec{J}(\vec{q})\cdot\vec{\nabla}_q]\vec{q}=\vec{J}(\vec{q})$ and Eq. (4a) leads to

$$(1+\vec{q}\cdot\vec{\nabla}_q)\vec{J}(\vec{q})=-i\vec{\nabla}_q\dot{\rho}(\vec{q})-2i\vec{q}\times\vec{\mu}(\vec{q}) , \tag{5}$$

which is almost the result that we wish to obtain. What remains is the isolation of the current $\vec{J}(\vec{q})$, which can be accomplished formally through division by the operator $1+\vec{q}\cdot\vec{\nabla}_q$. In accomplishing the latter we assume analyticity in q and disregard any singular terms in q which arise in the inversion. We can obtain a more explicit result by rewriting Eq. (5) as

$$\int d^3x\, \vec{J}(\vec{x})(1+i\vec{q}\cdot\vec{x})e^{i\vec{q}\cdot\vec{x}} = -i\int d^3x[i\vec{q}\cdot\vec{x}\dot{\rho}(\vec{x})+2\vec{q}\times\vec{\mu}(\vec{x})]e^{i\vec{q}\cdot\vec{x}} ,$$

expanding both sides in powers of q, identifying terms with equal powers and, finally, regrouping the terms contributing to $\vec{J}(\vec{q})$. The result is

$$\vec{J}(\vec{q})=\vec{J}_c(\vec{q})+\vec{J}_M(\vec{q}) \tag{6a}$$

with

$$\vec{J}_c(\vec{q})=\int d^3x\, \dot{\rho}(\vec{x})\vec{x}\alpha(\vec{q}\cdot\vec{x}) \equiv i[H_0,\int d^3x\, \vec{x}\alpha(\vec{q}\cdot\vec{x})\rho(\vec{x})] , \tag{6b}$$

where

$$\alpha(z)=\sum_{n=0}^{\infty}\frac{(iz)^n}{(n+1)!} \tag{6c}$$

and

$$\vec{J}_M(\vec{q})=-i\vec{q}\times\int d^3x\, \vec{\mu}(\vec{x})\beta(\vec{q}\cdot\vec{x}) , \tag{6d}$$

where

$$\beta(z)=\sum_{n=0}^{\infty}\frac{2(iz)^n}{n!(n+2)} . \tag{6e}$$

We have denoted by $\vec{J}_c$ and $\vec{J}_M$ the components of the full current density associated with the charge density and the magnetization density, respectively. This result is virtually identical with that of Sachs and Austern, if several of their expressions are rearranged.[14,15]

A few comments are in order at this point: (a) As noted, the current operator has been separated into parts directly associated with fundamental physical quantities: the charge and the magnetization distributions. These components do not, however, have the structure of a conventional Fourier transform. This is a minor drawback and our notation might be somewhat misleading. (b) Our results are identical with those originally presented in Ref. 16. The original derivation was based on a Taylor series expansion; it was logically more direct but the final expression was less compact. (c) Although all magnetic effects are contained in $\vec{J}_M$, it is not true that all electric effects are isolated in $\vec{J}_c$. We know from previous experience, e.g., the traditional multipole-field expansion, that the electric part of the current cannot be fully expressed in terms of the charge density. This is still the case here although the detailed rearrangement has been modified. What remains of the electric current is now expressed in terms of the magnetization density; no additional physical quantity is needed. (d) The magnetic term is purely transverse, but the charge term is not purely longitudinal. As anticipated, it contains a significant transverse component. (e) Both terms contain effects related to charge-exchange forces. Those in $\vec{J}_c$ are quite trivial, however; in the nonrelativistic limit they appear only through the Hamiltonian in $i[H_0,\rho(\vec{x})]$ and are replaced by energy differences when matrix elements are taken. The real observable effects associated with mesonic currents, etc., are all included in $\vec{\mu}(\vec{x})$. Obviously this is an advantage both conceptually and in practice. (f) After scalar multiplication by $\vec{q}$, Eq. (6b) gives the continuity equation, while in the limit $\vec{q}\to 0$ it reduces to Siegert's result. Equation (6d) gives zero in both cases. The coefficient of the linear term in $\vec{q}$ in Eq. (6d) in the small-q limit is simply the magnetic dipole operator of the system. (g) The inversion leading to Eqs. (6) can produce homogeneous solutions of the form $\vec{q}\times\vec{C}/q^2$ for arbitrary $\vec{C}$. These terms have been ignored since they violate our analyticity requirement and destroy Siegert's theorem. (h) Both $\vec{J}(\vec{q})$ and its individual components satisfy the familiar relations

$$\vec{J}^{\dagger}(\vec{q})=\vec{J}(-\vec{q})=-J^*(\vec{q}) .$$

(i) If the mathematical identities

$$\alpha(\vec{q}\cdot\vec{x})=\sum_{n=0}^{\infty}\frac{(i\vec{q}\cdot\vec{x})^n}{(n+1)!}=\int_0^1 d\lambda\, e^{i\lambda\vec{q}\cdot\vec{x}} \tag{7a}$$

and

$$\beta(\vec{q}\cdot\vec{x})=\sum_{n=0}^{\infty}\frac{2(i\vec{q}\cdot\vec{x})^n}{(n+2)n!}=2\int_0^1 d\lambda\,\lambda e^{i\lambda\vec{q}\cdot\vec{x}} \quad (7b)$$

are inserted in Eqs. (6b) and (6c), Eq. (6a) becomes essentially identical to the expansion first introduced by Sachs and Austern[14,15] in their discussion of the interaction of nuclei with the electromagnetic field. The connection with other conventional expressions for the current is established in Sec. III.

Many applications require forms of the current in configuration space. An example of this is the DWBA analysis of inelastic transitions.[8] In order to facilitate these applications we present here the appropriate forms for $\vec{J}_c(\vec{x})$ and $\vec{J}_M(\vec{x})$. Using Eqs. (7) we find

$$\vec{J}_c(\vec{x})\equiv\int d^3q\,e^{-i\vec{q}\cdot\vec{x}}\vec{J}_c(\vec{q})=i[H_0,\vec{d}(\vec{x})]\,, \quad (8a)$$

where

$$\vec{d}(\vec{x})=\vec{x}\int_0^1 d\lambda\,\rho(\vec{x}/\lambda)/\lambda^4 \quad (8b)$$

and

$$\vec{J}_M(\vec{x})\equiv\int d^3q\,e^{-i\vec{q}\cdot\vec{x}}\vec{J}(\vec{q})=\vec{\nabla}\times\vec{m}(\vec{x})\,, \quad (8c)$$

where

$$\vec{m}(\vec{x})=2\int_0^1 d\lambda\,\vec{\mu}(\vec{x}/\lambda)/\lambda^2\,. \quad (8d)$$

Current conservation follows from the identity

$$\vec{\nabla}\cdot\vec{d}(\vec{x})=\left[3+\frac{x\partial}{\partial x}\right]\int_0^1 d\lambda\,\rho(\vec{x}/\lambda)/\lambda^4=-\rho(\vec{x})\,, \quad (8e)$$

while consistency with $\vec{x}\times\vec{J}$ leads to

$$\left[1+\frac{x\partial}{\partial x}\right]\vec{m}(\vec{x})=-2\vec{\mu}(\vec{x})\,. \quad (8f)$$

Using these relations the usual form of the electromagnetic interaction $H_{\rm EM}=\int\rho\phi-\int\vec{J}\cdot\vec{A}$, can be cast in the manifestly gauge invariant form

$$H_{\rm EM}=-\int d^3x[\vec{m}(\vec{x})\cdot\vec{B}(\vec{x})+\vec{d}(\vec{x})\cdot\vec{E}(\vec{x})]\,, \quad (8g)$$

where $\vec{E}$, $\vec{B}$, ϕ, and $\vec{A}$ are the external electric and magnetic fields and scalar and vector potentials. This is a remarkably simple and satisfying result but, in view of our conserved current, hardly surprising. We should emphasize also that $\vec{m}(\vec{x})$ and $\vec{d}(\vec{x})$ are not the usual magnetic and electric dipole moment densities. They are extensions which have the property that their volume integrals are identical with the usual dipole moments.

III. MULTIPOLE ANALYSIS

Our remaining task is to perform a multipole expansion of the currents given in Eqs. (6). The results for the electric terms were already given in Ref. 16. What we want to do here is to present some details of a rather straightforward derivation and write, also, the corresponding expansion of the magnetic terms. As is customary, we shall choose the z axis in the direction of $\hat{q}$ and introduce the spherical polarization vectors

$$\hat{\epsilon}_m=\hat{\epsilon}_\pm=\mp(\hat{\epsilon}_x\pm i\epsilon_y)/\sqrt{2}\,.$$

Our starting point will be Eqs. (6b) and (6c) together with the identities (7a) and (7b). For $J\neq 0$, the formulas

$$\hat{\epsilon}_m\cdot\vec{\nabla}_q Y_{JM}(\hat{q})=-\frac{1}{q}\left[\frac{J(J+1)(2J+1)}{8\pi}\right]^{1/2}\delta_{M,m}$$

and

$$\int_0^1\frac{d\lambda}{\lambda}j_J(\lambda z)=\int_0^z\frac{dz'}{z'}j_J(z')\equiv\frac{z^J}{J(2J+1)!!}g_J(z)$$

will be kept in mind. Using the standard decomposition of a plane wave in terms of spherical waves we find

$$\hat{\epsilon}_m\cdot\vec{J}_c(\vec{q})=-i\hat{\epsilon}_m\cdot\vec{\nabla}_q\int_0^1\frac{d\lambda}{\lambda}\int d^3x\,\dot{\rho}(\vec{x})e^{i\lambda\vec{q}\cdot\vec{x}}$$
$$=\sqrt{2\pi}\sum_{J=1}^{\infty}\left[\frac{(J+1)(2J+1)}{J}\right]^{1/2}\frac{(iq)^{J-1}}{(2J+1)!!}\times\int d^3x\,\dot{\rho}(\vec{x})x^J g_J(qx)Y_{Jm}(\hat{x})\,. \quad (9)$$

The low-momentum-transfer expansion of the function $g_J(z)$ is given by

$$g_J(z)\cong 1-\frac{Jz^2}{2(J+2)(2J+3)}+\frac{Jz^4}{8(J+4)(2J+3)(2J+5)}+\cdots\,. \quad (10)$$

We note that if we set $g_J(z)\cong 1$ [i.e., if we only keep the first term in Eq. (10)] then Eq. (9) reduces to a familiar form which can also be obtained from the multipole-field expansion, or any other method, under the same conditions. This is the expression which contains all the multipolarities in their lowest order in retardation. Our result differs, however, from what is conventionally written in terms of multipole fields when retardation effects are included, i.e., when the second or higher terms in Eq. (10) are taken into account. The difference appears because we have introduced a nonconventional decomposition of the electric current into terms determined exclusively by the charge density and a remainder which, in our case, is hidden in the magnetization-density term $\vec{J}_M(\vec{q})$. What we should do now is to analyze this term into multipoles and identify its electric and magnetic components.

We start with Eqs. (6c) and (7b) and expand the plane wave into spherical waves. Choosing, again, the z axis in the direction of $\hat{q}$ we find

$$\vec{J}_M(\vec{q})=-2i\sqrt{4\pi}\vec{q}\int d^3x\,\vec{\mu}(\vec{x})\times\sum_J i^J(2J+1)^{1/2}\int_0^1 d\lambda\,\lambda j_J(\lambda qx)Y_{J0}(\hat{x})\,. \quad (11)$$

We now multiply both sides by the polarization vector $\hat{\epsilon}$, use the identities

$$\hat{\epsilon}_m\times\vec{q}=miq\hat{\epsilon}_m$$

and

$$Y_{J0}(\hat{x})\hat{\epsilon}_m=\left[\frac{J+2}{2(2J+1)}\right]^{1/2}\vec{Y}^m_{J+1,J}(\hat{x})+\left[\frac{J-1}{2(2J+1)}\right]^{1/2}\vec{Y}^m_{J-1,J}(\hat{x})-m\left[\frac{2J+1}{2(2J+1)}\right]^{1/2}\vec{Y}^m_{J,J}(\hat{x}),$$

with

$$\vec{Y}^m_{J,L}(\hat{x})=\sum_\mu (L\;m-\mu\;1\mu\,|\,Jm)Y_{Lm-\mu}(\hat{x})\hat{\epsilon}_\mu ,$$

and introduce the radial functions $h_J(qx)$ by

$$\int_0^1 d\lambda\,\lambda j_J(\lambda z)\equiv\frac{z^J}{(J+2)(2J+1)!!}h_J(z). \tag{12}$$

Use of the Bessel-function identity

$$j_J(z)=-z^{J-1}\frac{d}{dz}z^{-2J}\frac{d}{dz}z^{J+1}j_J(z)$$

allows us to rewrite Eq. (12) in the form

$$h_J(z)=-\frac{J+2}{Jz}\frac{d}{dz}z^{-2J}\frac{d}{dz}[z^{2J+1}g_J(z)], \tag{13}$$

which originally appeared in Ref. 16.

The decomposition of the current into its electric and magnetic components is now specified by the angular momentum and the parity of the vector spherical harmonics. The result is

$$\hat{\epsilon}_m\vec{J}_M(\vec{q})=\hat{\epsilon}_m\cdot\vec{J}^{\,\mathrm{el}}_M(\vec{q})+\hat{\epsilon}_m\cdot\vec{J}^{\,\mathrm{mag}}_M(\vec{q}), \tag{14a}$$

with

$$\hat{\epsilon}_m\cdot\vec{J}^{\,\mathrm{el}}_M(\vec{q})=-2q\sqrt{2\pi}\int d^3x\,\vec{\mu}(\vec{x})\cdot\sum_J i^J(2J+1)^{1/2}\vec{Y}^m_{J,J}(\hat{x})\int_0^1 d\lambda\,\lambda j_J(\lambda qx) \tag{14b}$$

and

$$\begin{aligned}\hat{\epsilon}_m\cdot\vec{J}^{\,\mathrm{mag}}_M(\vec{q})&=\pm2q\sqrt{2\pi}\int d^3x\,\vec{\mu}(\vec{x})\cdot\sum_J i^J[(J+2)^{1/2}\vec{Y}^m_{J+1,J}(\hat{x})+(J-1)^{1/2}\vec{Y}^m_{J-1,J}(\hat{x})]\int_0^1 d\lambda\,\lambda j_J(\lambda qx)\\&=\pm2q\sqrt{2\pi}\int d^3x\,\vec{\mu}(\vec{x})\cdot\int_0^1 d\lambda\,\lambda\sum_J[i^{J-1}(J+1)^{1/2}j_{J-1}(\lambda qx)\vec{Y}^m_{J,J-1}(\hat{x})+i^{J+1}(J)^{1/2}j_{J+1}(\lambda qx)\vec{Y}^m_{J,J+1}(\hat{x})].\end{aligned} \tag{14c}$$

Using Eq. (12), we may rewrite the electric component as

$$\hat{\epsilon}_m\cdot\vec{J}^{\,\mathrm{el}}_M(\vec{q})=-2\sqrt{2\pi}\sum_J\frac{(2J+1)^{1/2}i^Jq^{J+1}}{(J+2)(2J+1)!!}\int d^3x\,x^J\vec{\mu}(\vec{x})\cdot\vec{Y}^m_{J,J}(\hat{x})h_J(qx) \tag{15}$$

and we may also simplify Eq. (14c) by first noticing that the expression

$$-\left[\frac{J+1}{2J+1}\right]^{1/2}\vec{Y}^m_{J,J+1}(\hat{x})+\left[\frac{J}{2J+1}\right]^{1/2}\vec{Y}^m_{J,J-1}(\hat{x})=\hat{x}Y_{Jm}(\hat{x})$$

is orthogonal to $\vec{\mu}(\vec{x})=\frac{1}{2}\vec{x}\times\vec{J}(\vec{x})$. This allows us to replace $\vec{Y}^m_{J,J+1}(\hat{x})$ by $[J/(J+1)]^{1/2}\vec{Y}^m_{J,J-1}(\hat{x})$ in Eq. (14c), for example. What remains then in the bracket is a combination of spherical Bessel functions which can be simplified by the relation

$$\int_0^1 d\lambda\,\lambda[(J+1)j_{J-1}(\lambda qx)-Jj_{J+1}(\lambda qx)]=\frac{2J+1}{qx}\int_0^1 d\lambda\frac{d}{d\lambda}[\lambda j_J(\lambda qx)]=\frac{2J+1}{qx}j_J(qx).$$

Our final expression for the magnetic current is

$$\hat{\epsilon}_m\cdot\vec{J}^{\,\mathrm{mag}}(\vec{q})=m2\sqrt{2\pi}\sum_J\frac{(2J+1)i^{J-1}}{(J+1)^{1/2}}\int d^3x\,\vec{\mu}(\vec{x})\cdot\vec{Y}^m_{J,J-1}(\hat{x})\frac{1}{x}j_J(qx), \tag{16}$$

where we have suppressed the subscript "M" which is clearly redundant for the magnetic term.

We should note that the result of Eq. (16) is not in a form that appears traditionally in the literature. A more familiar expression, however, can easily be obtained if use is made of the identity

$$[J(2J+1)]^{1/2}\hat{x}\times\vec{Y}^m_{J,J-1}(\hat{x})=\vec{x}\times\vec{\nabla}Y_{Jm}(\hat{x})=i[J(J+1)]^{1/2}\vec{Y}^m_{J,J}(\hat{x}).$$

This allows us to write

$$\hat{\epsilon}_m\cdot\vec{J}^{\,\mathrm{mag}}(\vec{q})=m\sqrt{2\pi}\sum_J i^J(2J+1)^{1/2}\int d^3x\,\vec{J}(\vec{x})\cdot\vec{Y}^m_{J,J}(\hat{x})j_J(qx), \tag{17}$$

which does correspond to what one usually writes for the magnetic transition amplitude. We recall also the definitions of Ref. 6,

$$\hat{\epsilon}_m \cdot \vec{J}(\vec{q}) = -\sqrt{2\pi} \sum_J i^J (2J+1)^{1/2} [T^{\rm el}_{Jm}(\vec{q}) + m T^{\rm mag}_{Jm}(\vec{q})] , \tag{18}$$

which, after comparison with Eqs. (9) and (15) gives

$$T^{\rm el}_{Jm}(\vec{q}) = \frac{q^{J-1}}{(2J+1)!!} \int d^3x \left[i \left[\frac{J+1}{J} \right]^{1/2} \dot{\rho}(\vec{x}) x^J Y_{Jm}(\hat{x}) g_J(qx) + \frac{2q^2}{J+2} \vec{\mu}(\vec{x}) \cdot \vec{Y}^{\,m}_{J,J}(\hat{x}) h_J(qx) \right] . \tag{19}$$

This corresponds to Eq. (17) of Ref. 16 and

$$\begin{aligned} T^{\rm mag}_{Jm}(\vec{q}) &= \int d^3x\, \vec{J}(\vec{x}) \cdot \vec{Y}^{\,m}_{J,J}(\hat{x}) j_J(qx) \\ &= 2i \left[\frac{2J+1}{J+1} \right]^{1/2} \int d^3x \frac{1}{x} j_J(qx) \vec{\mu}(\vec{x}) \cdot \vec{Y}^{\,m}_{J,J-1}(\hat{x}) . \end{aligned} \tag{20}$$

As we have already pointed out, there is no difference between our expression for $T^{\rm mag}_J$ and the customary one of Ref. 6. The differences introduced by our approach are in the electric operator $T^{\rm el}_J$. Although it is true that the overall operator has not changed, its detailed separation into parts given in terms of the charge and magnetization density are now different. For small values of q, the lowest-order nonvanishing terms for each multipolarity are still in their standard form. The changes introduced here appear when retardation is considered. Since for many purposes (e.g., photon scattering or integrated photon absorption) contributions from a multipolarity J are of the same order of magnitude as retardation terms of angular momentum $J-1$, the precise form of the retardation corrections is certainly not insignificant. In view of this, we should keep in mind that the calculation of matrix elements of the operator $\rho(\vec{x})$ is both easy and unambiguous, whereas the same is certainly not true for the magnetization density.

IV. ALTERNATIVE PROOF

Although we presented a derivation for an alternative form of the current in Sec. II, and the corresponding electric multipole fields in Sec. III, the simplicity of that derivation obscures questions of uniqueness. There exist several alternative forms of the electric multipole fields. Some give Siegert's limit for long wavelengths and others do not. In order to make a convincing case that we have constructed an optimal form, it is necessary to convince the reader that we can uniquely identify those components of the multipole fields which are determined by current conservation. The simplest example (electric dipole) was illustrated in Ref. 16; we present here, for completeness, the general case.

The Fourier transforms of the charge and current operators have the forms:

$$\begin{aligned} \rho(\vec{q}) &= \int d^3x\, e^{i\vec{q}\cdot\vec{x}} \rho(\vec{x}) \\ &= \sum_{N=0}^{\infty} \frac{i^N}{N!} \int d^3x (\vec{q}\cdot\vec{x})^N \rho(\vec{x}) \end{aligned} \tag{21}$$

and

$$\begin{aligned} \vec{J}(\vec{q}) &= \int d^3x\, e^{i\vec{q}\cdot\vec{x}} \vec{J}(\vec{x}) \\ &= \sum_{N=0}^{\infty} \frac{i^N}{N!} \int d^3x (\vec{q}\cdot\vec{x})^N \vec{J}(\vec{x}) + \frac{(\vec{P}_f+\vec{P}_i)}{2m_t} \rho(\vec{q}) , \end{aligned} \tag{22}$$

and are related by

$$\vec{q}\cdot\vec{J}(\vec{q}) = [H_0, \rho(\vec{q})] + \frac{(\vec{P}_f+\vec{P}_i)\cdot\vec{q}}{2m_t} \rho(\vec{q}) . \tag{23}$$

The nuclear *internal* Hamiltonian H_0 does not contain the recoil term appropriate to the total mass m_t, which is given by the second term in Eq. (23). Corresponding to this term is the total nuclear convection current, the second term in Eq. (22). In what follows we will ignore the recoil term completely; it has a special and simple form, is frame dependent, usually does not contribute to one-photon physical processes in conventional gauges and frames, and is best treated separately.

The second form of each of Eqs. (21) and (22) casts the current and charge as a sum of (reducible) Cartesian multipoles. Conventionally, one uses (irreducible) spherical multipoles. We will work with the Cartesian form, after starting with the spherical tensors, and then transform back to spherical multipoles. In the latter representation the electric and magnetic multipoles are given by

$$\begin{aligned} T^{\rm mag}_{JM} &= \int d^3x\, j_J(qx) \vec{Y}^{\,M}_{J,J} \cdot \vec{J}(\vec{x}) \\ &= 2i \left[\frac{2J+1}{J+1} \right]^{1/2} \int d^3x \frac{j_J(qx)}{x} \vec{Y}^{\,M}_{J,J-1} \cdot \vec{\mu}(\vec{x}) \end{aligned} \tag{24}$$

and

$$T^{\rm el}_{JM} = \frac{1}{q} \int d^3x\, j_J(qx) \vec{Y}^{\,M}_{J,J} \cdot \vec{\nabla} \times \vec{J}(\vec{x}) \tag{25}$$

or

$$T^{\rm el'}_{JM} = \frac{-i}{[J(J+1)]^{1/2} q} \int d^3x\, Y_{JM} \left\{ \vec{\nabla}\cdot\vec{J}(\vec{x}) \frac{\partial}{\partial x} [x j_J(qx)] - q^2 \vec{x}\cdot\vec{J}(\vec{x}) j_J(qx) \right\} , \tag{26}$$

where the latter two forms are equivalent if current conservation is not assumed for $\vec{\nabla}\cdot\vec{J}$. Moreover we have

$$J_\lambda(q) = -(2\pi)^{1/2} \sum_{JM} (+i)^J (2J+1)^{1/2} (T^{\rm el}_{J\lambda} + \lambda T^{\rm mag}_{J\lambda}) , \tag{27}$$

when $\hat{q}$ is conventionally taken along the z axis. In that case J_0 is given by current conservation and $\lambda = \pm 1$ are the only required values. Although the direction of $\hat{q}$ assumed above greatly simplifies some of the algebra, for other applications it is not helpful and we will not restrict ourselves to that choice. It is clear that parity and angular momentum constraints determine the separation into electric and magnetic multipoles and we will use these constraints, rather than derive them. Moreover, we will break our proof into two parts: one part determines the separation into Siegert-type terms and the remainder, and the dependence of both on the direction of $\hat{q}$; the second part determines the form of the auxiliary functions g_J and h_J. The first part also illustrates the essence of the proof, without excessive complication.

We define the reducible Cartesian tensors with indices, $\alpha_1, \alpha_2, \ldots, \alpha_N$,

$$O_N^{\alpha_1\alpha_2\cdots\alpha_N} = \int d^3x\, x^{\alpha_1}\cdots x^{\alpha_N}\rho(\vec{x}) \tag{28}$$

and

$$P_N^{\alpha_1\alpha_2\cdots\alpha_N} = \int d^3x\, x^{\alpha_1}\cdots x^{\alpha_{N-1}}[\vec{x}\times\vec{J}(\vec{x})]^{\alpha_N} . \tag{29}$$

Note that O is completely symmetric in its indices, while P is symmetric in all but one index. Examining the Nth term in the expansion of Eq. (23) in powers of q we can write

$$\int d^3x\, x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_{N-1}}J^{\alpha_N} + (\text{sym}) = i[H_0, O_N^{\alpha_1\alpha_2\cdots\alpha_N}] , \tag{30}$$

where (sym) indicates that the index "α_N" is permuted with the other $(N-1)$ indices so that there are altogether N terms. Equation (30) is the essence of the proof. We can write the $(N-1)$th term in the expansion of the Fourier transform of J^ξ as

$$q^{\alpha_1}\cdots q^{\alpha_{N-1}}\int x^{\alpha_1}x^{\alpha_2}\cdots x^{\alpha_{N-1}}J^\xi ;$$

it can be rewritten as a term symmetric in the indices and another which has mixed symmetry (no antisymmetric term is possible). The decomposition is unique because of the properties of the permutation group, and the former class of term is determined entirely by current conservation through Eq. (30). Thus we can *uniquely* and *optimally* isolate all terms in $\vec{J}(\vec{x})$ which are determined by current conservation.

We perform the indicated decomposition of the $(N-1)$th term of $J^{\alpha_N}(\vec{q})$,

$$J_{N-1}^{\alpha_N}(\vec{q}) = \frac{i^N}{N!}[H_0, O_N^{\alpha_N}] + \frac{(N-1)}{N!} i^{N-1} q^{\alpha_1}\cdots q^{\alpha_{N-1}} \epsilon^{\alpha_{N-1}\alpha_N\xi} P_{N-1}^{\alpha_1\cdots\alpha_{N-2}\xi} , \tag{31}$$

where ϵ is the antisymmetric symbol and we find it convenient to define

$$O_N^\xi = q^{\alpha_1}\cdots q^{\alpha_{N-1}} O_N^{\alpha_1\cdots\alpha_{N-1}\xi} . \tag{32}$$

The first term in (31), determined by O, contains $E(N)$, $E(N-2), \ldots,$ multipoles, while the remaining term contains $M(N-1)$, $E(N-2)$, $M(N-3), \ldots,$ multipoles. Further decomposition is necessary if we are to proceed. This problem also illustrates an essential feature of our procedure: Reducible Cartesian tensors determine the form of the current conservation constraint and upon further decomposition to project out lower-order multipoles, the same constraint which fixes the long wavelength $E(N)$ also fixes part of the first-order retarded $E(N-2)$, second-order retarded $E(N-4)$, etc. This is why we work in Cartesian rather than spherical multipoles.

The symmetric part of P gives the highest spherical multipole, $M(N-1)$; the remainder has $E(N-2)$, etc., multipoles. This decomposition generates

$$J_{N-1}^\xi(\vec{q}) = \frac{i^N}{N!}[H_0, O_N^\xi] + \frac{(N-2)}{N!} i^{N-1}\vec{q}\times(\vec{q}\times\vec{P}_{N-2})^\xi - \frac{i^{N-1}}{N!}(\vec{q}\times\vec{Q}_{N-1})^\xi , \tag{33}$$

where

$$\vec{P}_N^\xi = 2q^{\alpha_1}\cdots q^{\alpha_{N-1}}[x^{\alpha_1}\cdots x^{\alpha_{N-1}}N^\xi + (\text{sym})] , \tag{34}$$

$$\vec{Q}_N^\xi = 2q^{\alpha_1}\cdots q^{\alpha_{N-1}}[x^{\alpha_1}\cdots x^{\alpha_{N-1}}\mu^\xi + (\text{sym})] , \tag{35}$$

$$\vec{\mu}(\vec{x}) = \tfrac{1}{2}\vec{x}\times\vec{J}(\vec{x}) , \tag{36}$$

and

$$\vec{N}(\vec{x}) = \vec{x}\times\vec{\mu}(\vec{x}) . \tag{37}$$

Clearly, we can project out of O, P, and Q the lower-order multipoles by taking traces; this leaves irreducible tensors of the appropriate rank which we will write with an overbar: $\bar{O}$, etc. We project out of O the $(N-2)$nd multipole:

$$J_{N-1}^\xi(\vec{q}) = \frac{i^N}{N!}\left[H_0, \bar{O}_N^\xi + \frac{N(N-1)q^\xi}{2(2N-1)}\vec{\bar{O}}'_{N-2}\cdot\vec{q} - \frac{(N-1)(N-2)}{2(2N-1)}\vec{q}\times(\vec{q}\times\vec{\bar{O}}'_{N-2})^\xi\right] + \frac{(N-2)}{N!} i^{N-1}\vec{q}\times(\vec{q}\times\vec{\bar{P}}_{N-2})^\xi - \frac{i^{N-1}}{N!}(\vec{q}\times\vec{\bar{Q}}_{N-1})^\xi + \cdots , \tag{38}$$

where

$$O_N'^\xi = q^{\alpha_1}\cdots q^{\alpha_{N-1}}\int d^3x\, x^{\alpha_1}\cdots x^{\alpha_{N-1}}x^\xi x^2\rho(\vec{x}) . \tag{39}$$

The Siegert-type terms in the first line correspond to $E(N)$ and $E(N-2)$ multipoles, while the remaining terms correspond to $E(N-2)$ and $M(N-1)$. We can easily deduce the $\vec{q}$ dependence of the $E(N)$ multipole:

$$\vec{J}(EN) \cong \frac{i^N}{N!}\left[H_0, \bar{\bar{O}}_N - \frac{\vec{q}\,\vec{q}\cdot\bar{\bar{O}}'_N}{2(2N+3)} + \frac{N\vec{q}\times(\vec{q}\times\bar{\bar{O}}'_N)}{2(N+2)(2N+3)}\right] + \frac{i^{N+1}N}{(N+2)!}\vec{q}\times(\vec{q}\times\bar{\bar{\vec{P}}}_N) . \quad (40)$$

The second term proportional to $\vec{q}$ is obviously determined by $\vec{q}\cdot\vec{J}$ and arises from the expansion of

$$\tilde{j}_N \cong 1 - z^2/2(2N+3) + \cdots ,$$

where the reduced spherical Bessel functions are given by

$$\tilde{j}_N = j_N(z)(2N+1)!!/z^N . \quad (41)$$

Thus

$$\vec{J}_{||}(EN) \doteq \frac{i^N}{qN!}\left[H_0, \int d^3x\, \tilde{j}_N(qx)(\vec{q}\cdot\vec{x})^N \rho(\vec{x})\right] , \quad (42a)$$

$$\vec{J}_{\perp}(EN) \doteq \frac{i^N}{N!}\left[H_0, \int d^3x\, g_N(qx)(\vec{q}\cdot\vec{x})^{N-1}\vec{x}_{\perp}\rho(\vec{x})\right] - \frac{2i^{N+1}}{(N+2)!}\int d^3x\, h_N(qx)[(\vec{q}\cdot\vec{x})^{N-1}\vec{N}_{\perp} + (N-1)\vec{x}_{\perp}\vec{q}\cdot\vec{N}(\vec{q}\cdot\vec{x})^{N-2}] , \quad (42b)$$

and

$$\vec{J}(MN) \sim -\frac{2i^N}{(N+1)!}\int d^3x[(\vec{q}\cdot\vec{x})^{N-1}(\vec{q}\times\vec{\mu}) + (N-1)(\vec{q}\cdot\vec{x})^{N-2}(\vec{q}\times\vec{x})(\vec{q}\cdot\vec{\mu})] , \quad (42c)$$

where $||$ and $\perp$ refer to the direction $\hat{q}$. Only the leading-order term in (42c) was developed, but we have clearly displayed the form of the dependence on $\hat{q}$ and note that it is purely transverse $[\hat{q}\cdot\vec{J}(MN) \equiv 0]$. We have also added functions g_N and h_N in (42b) whose form we have not calculated beyond $g_N(0) = h_N(0) \equiv 1$.

Having established the *form* of the expansion, we resort to a trick which avoids the necessity of projecting out magnetic and lower-order electric multipoles at every step of the reduction process. Using $\vec{L}Y_{JM}/[J(J+1)]^{1/2} = \vec{Y}^M_{JJ}$ we can rewrite Eq. (24) as

$$J^{\rm el}_{J\lambda} = \frac{i^{J+1}\sqrt{2\pi}}{[J(J+1)(2J+1)]^{1/2}(2J-1)!!}\int d^3x\, \tilde{j}_J(qx)(\vec{x}\times\vec{\nabla})\cdot(q^{J-1}Y_{J\lambda})\vec{\nabla}\times\vec{J}(\vec{x}) , \quad (43)$$

where $\mathscr{Y}_{JM} = r^J Y_{JM}$ is the solid harmonic of order J. By writing out the Cartesian forms of $q^{J-1}\mathscr{Y}_{J\lambda}$ and $O_{J\lambda} \sim (\vec{q}\cdot\vec{x})^{J-1}x_\lambda$ we can establish the formal equivalence

$$q^{J-1}\mathscr{Y}_{J\lambda} \doteq \bar{O}_{J\lambda}\left[\frac{2J(2J+1)}{(J+1)(4\pi)}\right]^{1/2}\frac{(2J-1)!!}{J!} . \quad (44)$$

We now expand the Bessel function in Eq. (43) as $\sum_n z^{2n}C_n$, integrate $\vec{\nabla}\times\vec{J}$ by parts, and perform all derivatives of $Y_{J\lambda}$. The integrand then has the form

$$-\sum_n C_n\{(2n+J+1)[\vec{q}\cdot\vec{J}\,\vec{x}_\lambda(J-1) + \vec{J}_\lambda(\vec{q}\cdot\vec{x})](\vec{q}\cdot\vec{x})^{J-2}x^{2n} - \vec{x}\cdot\vec{J}\,\vec{x}_\lambda x^{2n-2}(\vec{q}\cdot\vec{x})^{J-1}(2nJ)\}$$
$$\equiv \sum_n C_n[(2N+J+1)(\vec{a}^{\,J}_n)_\lambda - (2nJ)(\vec{b}^{\,J}_n)_\lambda] , \quad (45)$$

where we have ignored all terms $\sim q_\lambda$ and trace terms which make the tensor irreducible. Equation (45) can be rearranged into the desired form determined from current conservation. Relationship (30) of order $N = 2n + J$ becomes, after contracting $2n$ indices in any order (the tensor is symmetric) and dotting $J-1$ of the remaining indices with vectors, $\vec{q}$,

$$\int 2n(\vec{b}^{\,J}_n)_\lambda + (\vec{a}^{\,J}_n)_\lambda = i\int [H_0, O^\lambda_J]x^{2n} , \quad (46)$$

while the symmetric tensor of rank J

$$P^{\alpha_1\cdots\alpha_J}_{N,J} = x^{2n-2}[x^{\alpha_1}\cdots x^{\alpha_J}\vec{x}\cdot\vec{J} - x^{\alpha_1}\cdots J^{\alpha_J}x^2 + ({\rm sym})] \quad (47a)$$

combined with the definition of the operator N becomes

$$2x^{2n-2}[x^{\alpha_1}\cdots N^{\alpha_J} + ({\rm sym})] , \quad (47b)$$

and leads to

$$\int J(\vec{b}^{\,J}_n)_\lambda - (\vec{a}^{\,J}_n)_\lambda = \int P^\lambda_n x^{2n-2} . \quad (48)$$

Equations (46) and (48) can be solved for $\vec{a}^{\,J}_n$ and $\vec{b}^{\,J}_n$:

$$(\vec{a}^{\,J}_n)_\lambda = [i[H_0, O^\lambda_J]x^{2n} + P^\lambda_n x^{2n-2}]/(2n+J) , \quad (49a)$$

$$(\vec{b}^{\,J}_n)_\lambda = [i[H_0, O^\lambda_J]Jx^{2n} - 2nP^\lambda_n x^{2n-2}]/(2n+J) . \quad (49b)$$

Combining Eqs. (43) and (45) produces

$$\vec{J}(EJ)=\frac{i^J}{J!}[H_0,\int d^3x\,\vec{O}^J g_J(z)\rho(\vec{x})]-\frac{i^{J+1}}{(J+2)!}\int d^3x\,\vec{P}_J h_J(z) \qquad (50)$$

with $z=qx$. Thus using Eqs. (42) we obtain

$$g_J(z)=\sum_{N=0}^{\infty}\frac{(-z^2/2)^N(2J+1)!!J}{N!(2J+2N+1)!!(2N+J)}=\frac{J(2J+1)!!}{z^J}I_J(z) \qquad (51)$$

and

$$h_J(z)=\sum_{N=0}^{\infty}\frac{(-z^2/2)^N}{N!}\frac{(2J+1)!!(J+2)}{(2J+2N+1)!!(2N+J+2)}=-\frac{(J+2)(2J+1)!!}{z}\frac{d}{dz}\frac{1}{z^{2J}}\frac{d}{dz}(z^{J+1}I_J)\,, \qquad (52)$$

where

$$I_J(z)=\int_0^z dz'\frac{j_J(z')}{z'}\,. \qquad (53)$$

Mapping Eq. (50) to a spherical basis with $\hat{q}=\hat{z}$ we obtain

$$J_\lambda(EJ)=\frac{i^J q^{J-1}}{(2J+1)!!}\left[2\pi\frac{(2J+1)(J+1)}{J}\right]^{1/2}[H_0,\int d^3x\,x^J Y_{J\lambda}(\hat{x})g_J(qx)\rho(\vec{x})]-\frac{i^J 2q^{J+1}}{(2J+1)!!(J+2)}[2\pi(2J+1)]^{1/2}\int d^3x\,x^J\vec{Y}^{\lambda}_{J,J}\vec{\mu}(\vec{x})h_J(qx) \qquad (54)$$

and

$$T^{\rm el}_{J\lambda}(EJ)=-\frac{q^{J-1}}{(2J+1)!!}[(J+1)/J]^{1/2}[H_0,\int d^3x\,x^J Y_{J\lambda}(\hat{x})g_J(qx)\rho(\vec{x})]+\frac{2q^{J+1}}{(J+2)(2J+1)!!}\int d^3x\,x^J\vec{Y}^{\lambda}_{J,J}\cdot\vec{\mu}(\vec{x})h_J(qx)\,. \qquad (55)$$

The form given in Eq. (55) has a number of important features which will be described in Sec. V. Primary among them is the similarity between the last term and Eq. (24) for the magnetic multipoles. In the new representation for the electric multipoles, the current enters *explicitly* only in the combination $\vec{\mu}(\vec{x})$, precisely as it does in magnetic terms. Special cases of $\vec{J}(\vec{x})$ can be shown to lead to equivalent values of $T^{\rm el}$ in all forms. Choosing the spin magnetization current $\vec{J}=\vec{\nabla}\times\vec{\mu}_s(\vec{x})$, it can be shown that the first term in (55) vanishes and the second term leads to

$$T^{\rm el}_{J\lambda}(EJ;{\rm spin})=q\int d^3x\,j_J(qx)\vec{Y}^{\lambda}_{J,J}\cdot\vec{\mu}_s(\vec{x})\,, \qquad (56)$$

a simple and well-known result which follows immediately from Eq. (25). We recommend the form (56) for the multipole expansion of the spin magnetization current and the isobar part of the pion-exchange current, which also has the solenoidal form given above.

V. DISCUSSION AND CONCLUSIONS

Several features of our new electric multipole formula should be stressed. Although a number of different forms exist in the literature, only those that give Siegert's long wavelength limit are reasonably accurate for nuclear physics. None of the latter isolates those retarded terms which are also specified by current conservation, as we have done. Our method, which utilizes reducible Cartesian multipoles, is unique in the sense that we have used a symmetry to isolate the terms we sought. The reducibility allows retarded contributions of lower multipole order to be related in magnitude to unretarded higher multipoles. No other formula exists with the properties we have developed. We emphasize, however, that if the model of the current which one uses in a calculation is conserved all electric multipole forms are *precisely* equivalent. One is neither better nor worse than the others. In physical situations where the current is not completely known (e.g., nuclear physics) the differentiation of formulae is particularly useful. Moreover, if the charge operator were less well known than the current, our efforts would have been pointless; this is not the case in nuclear physics, however.

We reemphasize that the rearranged current developed in Sec. II and implicit in Sec. IV is manifestly conserved. What role does current conservation play, then, in discussing meson-exchange currents? If one chooses to use Eq. (55) for the electric multipole fields, as one should, the current enters only as $\vec{\mu}(\vec{x})$, which necessarily contains exchange currents. A nonconserved model of the nuclear current provides a quantitative measure of how those former currents might affect $\vec{\mu}(\vec{x})$, and that is all. Questions[17] about which electromagnetic form factors to associate with various processes (e.g., G_E, G_A, and G_π) and detailed considerations about incorporating meson-nucleon form factors into exchange-current operators affect only $\vec{\mu}$ and ρ. All "missing" parts of $\vec{J}$ associated with current conservation have been automatically incorporated by our formalism.

It has been known for a long time that magnetic transitions are not constrained by current conservation. Although it should not be too much of a surprise that our electric multipole form depends on the current through $\vec{\mu}(\vec{x})$, it is nevertheless highly satisfying. The current enters both electric and magnetic multipoles on the same footing. Another benefit of this dependence lies in shell model calculations. Equation (25) builds in the (retarded) convection current in the form $\vec{x}\cdot\vec{J}\sim r\partial/\partial r$, which involves *radial* derivatives of wave functions. The quantity

$\vec{\mu}(\vec{x})$ generates $\vec{x}\times\vec{J}\sim\vec{L}$, which involves only *angular* derivatives. Angular momentum reductions for this operator are trivial and are left to the reader. Radial derivatives are best avoided, however, because they enhance sensitivity to noise, either theoretical or computational. They also enhance sensitivity to the short-range part of the wave function, which is usually less well known and hard to determine. An example of this difficulty is found in calculations of *H-D* molecules,[18] which are used to determine the deuteron's quadrupole moment. Indeed, the ease of accurate calculation using the Siegert prescription for the long wavelength current lies largely in the x^J factor which generates large contributions in the exterior or "tail" part of the wave function. The form of the latter is usually well determined once the energy is known.

There exist situations without exchange currents where the current is not automatically conserved: approximate many-body calculations. The exchange terms in many-body theories generate effective momentum-dependent forces. Such momentum dependence leads heuristically to the requirement of additional currents if one invokes the "minimal coupling" prescription. Thus, *approximate* many-body calculations should be best handled in most cases using Siegert-type forms of the electric multipoles. Indirect evidence for this conclusion exists in calculations of electric dipole transitions in He-type atoms.[19,20] Finally, we note that exchange currents should be most important in low-order multipoles, as remarked by Arenhövel.[21] The factor of x^J in Eq. (55) suppresses the contribution of the short-range part of such currents, assisting in the dominance of the long-range one-pion-exchange part.

The remaining question is: How important is the numerical difference between our multipole formula and others, such as Eq. (26)? Such calculations are underway by others. In a specific process involving the deuteron,[22] the differentiation of forms appears to be numerically important in retarded terms. A rough estimate of the effect of the rearrangement can be made using a sum rule:[23] the retarded $E1$ part of the σ_{-2} sum rule. While the latter sum rule has contributions from a variety of multipoles, the once retarded $E1$ part has a particularly simple structure, since the spin magnetization current does not contribute. We also neglect explicit exchange currents. We have calculated the sum rule for our dipole formula and that of Eq. (26) and have separated the contribution from the Siegert-type part and the remainder term. Because of our assumptions, the complete result is identical for the two cases, but the distribution of strength is quite different. We find

$$\text{Eq.(26)}: \ \sigma_{-2}^{\text{ret}}(E1) = -\frac{2\pi^2\alpha Z\langle r^2\rangle}{3M}(2-1)\,,$$
$$\text{Eq.(55)}: \ \sigma_{-2}^{\text{ret}}(E1) = -\frac{2\pi^2\alpha Z\langle r^2\rangle}{3M}\left(\tfrac{1}{3}+\tfrac{2}{3}\right)\,. \tag{57}$$

The first form of Eq. (57) results from the cancellation of substantial terms, unlike the second form. When exchange currents are placed in the second part of each form (-1 or $\frac{2}{3}$), we expect a larger effect in the first form. This suggests that *explicit* exchange currents have a smaller effect in the new electric multipole form. Note also that the Siegert-type parts of the two forms differ by a factor of 6. Retardation is much less important in our form of that term.

In spite of the previous exercise we expect that in isovector transitions the spin current will dominate the retarded orbital current because of its large coefficient, the isovector nucleon magnetic moment. It may therefore be difficult to find a clear-cut case to investigate the difference between multipole forms. We feel that the forms we have introduced are superior to the forms currently in use.

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APPENDIX: MULTIPOLE FUNCTIONS

The multipole functions g_n and h_n both depend on the auxiliary function $I_n(z)$:

$$I_n(z) = \int_0^z \frac{dz'}{z'} j_n(z')\,, \quad \text{for } n \geq 1\,. \tag{A1}$$

Using the properties of the spherical Bessel functions, we can write

$$(n+1)I_n - (n-2)I_{n-2} = \delta_{n,2} - (2n-1)\frac{j_{n-1}}{z}\,, \quad \text{for } n \geq 2\,, \tag{A2}$$

$$I_1 = \tfrac{1}{2}[\mathrm{Si}(z) - j_1(z)]\,, \tag{A3}$$

$$I_2 = \tfrac{1}{3} - \frac{j_1}{z}\,, \tag{A4}$$

where $\mathrm{Si}(z)$ is the usual sine integral: $\int_0^z \sin(z')dz'/z'$. The recursion relation (A2) can be solved in terms of I_1 or I_2 for odd and even n:

$$I_{2n+2} = \frac{(2n)!!}{(2n+3)!!}\left[3I_2 - \frac{1}{z}\sum_{l=0}^{n-1} j_{2l+3}\frac{(4l+7)(2l+3)!!}{(2l+2)!!}\right]\,, \tag{A5}$$

and

$$I_{2n+1} = \frac{(2n-1)!!}{(2n+2)!!}\left[2I_1 - \frac{1}{z}\sum_{l=0}^{n-1} j_{2l+2}\frac{(4l+5)(2l+2)!!}{(2l+1)!!}\right]\,. \tag{A6}$$

Given these forms, g_n and h_n can be calculated easily for large arguments. For small and moderate arguments, the rapidly convergent power series should be used. Only the odd multipoles involve the sine integral, which is extremely tractable numerically.[24,25]

The asymptotic forms of I_j, g_J, and h_J are easily developed using

$$I_J \rightarrow \frac{(J-2)!!}{(J+1)!!}\lambda_J\,, \tag{A7}$$

where

$$\lambda_J=\begin{Bmatrix} 1 & J \text{ even} \\ \pi/2 & J \text{ odd} \end{Bmatrix} . \tag{A8}$$

Similarly,

$$g_J \to \frac{(2J+1)!!J!!\lambda_J}{(J+1)(J-1)!!z^J} \tag{A9}$$

and

$$h_J \to -\frac{(J+2)(2J+1)!!}{z^{J+2}} \left[\cos(z-J\pi/2)-\frac{J!!\lambda_J}{(J-1)!!} \right] . \tag{A10}$$

For comparison the corresponding function in Eq. (5) behaves as

$$\cos(z-J\pi/2)/z^J$$

and

$$\sin(z-J\pi/2)/z^{J+1} .$$

For completeness we give expressions for the common electric dipole and quadrupole cases:

$$g_1=1-\frac{z^2}{30}+\frac{z^4}{1400}+\cdots=\frac{3}{2z}[\mathrm{Si}(z)-j_1(z)] , \tag{A11}$$

$$h_1=1-\frac{3z^2}{50}+\frac{3z^4}{1960}+\cdots=\frac{9}{z^3}[\mathrm{Si}(z)-zj_0(z)] , \tag{A12}$$

$$g_2=1-\frac{z^2}{28}+\frac{z^4}{1512}+\cdots=\frac{10}{z^2}[1-3j_1(z)/z] , \tag{A13}$$

$$h_2=1-\frac{z^2}{21}+\frac{z^4}{1000}+\cdots=-\frac{60}{z^4}[zj_1+2(j_0-1)] . \tag{A14}$$

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