New class of antisymmetrized optical potentials

Sadhan K. Adhikari
Departamento de Física, Universidade Federal de Pernambuco, 50.000 Recife, PE, Brasil

R. Kozack
Physics Department, Brown University, Providence, Rhode Island 02912

F. S. Levin*
Departamento de Física, Universidade Federal de Pernambuco, 50.000 Recife, PE, Brasil
and Physics Department, Brown University, Providence, Rhode Island 02912
(Received 11 April 1983)

This paper is mainly concerned with the construction and various properties of a new class of antisymmetric, two-fragment, elastic scattering optical potentials. The construction is based directly on the wave function formulation of N-particle collision theories, which are antisymmetrized herein using the method of Adhikari and Glöckle. Those theories which are label transforming are considered first. For such theories, the resulting optical potentials are asymmetric but nevertheless real for energies below the inelastic threshold E_{inel}, and are asymmetric and absorptive for $E \geq E_{\text{inel}}$. This unusual feature of asymmetry is a consequence of working directly with N-particle collision theories, almost all of which are expressed in non-Hermitian matrix form, and arguments are presented as to why asymmetry is not a practical problem. Not only are the new class of optical potentials asymmetric, it is also found that exchange effects generally enter them in an extremely simple fashion. These latter two features distinguish the members of the new class from the optical potential developed recently by Goldflam and Kowalski. Examples of label-transforming theories producing such potentials are the extended Faddeev theory of Levin, the precursor form of the Bencze, Redish, Sloan theory, and the wave function component theory of L'Huillier, Redish, and Tandy. Their existence establishes that formalisms other than those based on the Alt, Grassberger, Sandhas transition operator lead to symmetrized optical potentials free of elastic unitarity cuts, albeit potentials which are not Hermitian analytic. In addition to the above theories, a new, symmetrized form of the equations of the channel permuting array theory is developed.

I. INTRODUCTION

The principal goal of this paper is the construction and description of a new class of two-fragment, elastic-scattering optical potentials (or complex potential wells)[1] for a scattering system composed of N identical fermions.[2] After a period of theoretical research some years ago,[3] little additional work on this problem was done until the recent investigations of Goldflam, Kowalski, and Picklesimer cited in Ref. 3. Like these latter investigations, the work of this paper is based on modern collision theory,[4] but in contrast to them, we base our analysis on a wave function and not a transition operator formalism. This approach leads to a class of potentials in which exchange effects enter in a relatively simple fashion and which possesses the unusual but not impractical property, as discussed in Sec. V, of being real (i.e., free of elastic unitarity cuts) and asymmetric for E below the inelastic threshold E_{inel}, and absorptive and asymmetric for $E \geq E_{\text{inel}}$. These are the main features distinguishing the present class of optical potentials from those of Ref. 3. These features and the existence of a class of optical potentials possessing them is a result of our working directly with the integral equations of N-particle collision theories, and we examine this class in detail after deriving our main theoretical results.

It is not simply these features that distinguish our work on optical potentials from that of Goldflam, Kowalski, and Picklesimer (GKP). There is also the crucial fact that in these two approaches the phrase "optical potential" (i.e., "unaveraged complex potential well") has different meanings. In this paper and others connected with it, we depart from what is no doubt the more traditional meaning and denote as an optical potential *any* one-body effective potential which, when used in the relevant, one-body, relative-motion Schrödinger equation, leads to the exact, two-fragment, elastic scattering amplitude (to within a phase factor). It is within this context that we derive and examine the new class of antisymmetrized optical potentials. GKP on the other hand, have confined their use of the phrase optical potential to its more traditional meaning of an effective potential which is real and symmetric for $E < E_{\text{inel}}$ and is Hermitian analytic for $E \geq E_{\text{inel}}$.[5] Because of this "strict constructionist" approach, they formulated their Hermitian analytic optical potential by means of a symmetric framework. Nevertheless, the quantities studied in Sec. II of Ref. 6 form a natural path to an asymmetric formulation such as ours: in that sense, the work of Ref. 6 anticipates some of our (independently derived) results (although no work formulating an antisymmetric, asymmetric optical potential has been published by GKP). It should be clear therefore, that the op-

 ©1984 The American Physical Society

tical potentials of the transition-operator approach of GKP and of the wave function approach used herein are very different objects.

In order to derive an antisymmetrized, two-fragment optical potential using a wave function approach, it is necessary to develop a scheme for antisymmetrizing (or at least partially antisymmetrizing) the quantities that occur in this formulation of many-particle scattering theory. The relevant quantities are known as wave function components;[4,7] the development of a procedure for antisymmetrizing them is a second goal of this paper. The method followed is essentially that of Adhikari and Glöckle.[8]

II. BACKGROUND

A. Notation

We consider a scattering system composed of N particles labeled $1,\ldots,N$, initially assumed to be distinguishable. Corresponding to the various asymptotic arrangements into bound fragments are partitions of the particles into distinct clusters. Partitions will be denoted $a(i)$, $b(j)$, etc., and can contain m clusters, $2\leq m\leq N$. The parenthetic labels i, j, etc., will distinguish the various orderings of the particle labels, e.g., $b(0)=(1)(2)(34)$, and $b(1)=(1)(3)(24)$ in the $N=4$ case. For any given b, the index j will take on the N_b+1 values of $0\leq j\leq N_b$. The parenthetic label "0" will always refer to a specific serial ordering, as in the $b(0)$ cited above. The set $\{b(0)\}$ will be referred to as the set of *canonical labels*.[10] Two-cluster partitions will be denoted by lower case Greek letters, e.g., $\alpha(i)$ and $\beta(j)$. When it is necessary to specify the number of clusters m in a given partition, we shall append it as a subscript, e.g., $b_m(j)$. Two partitions $b(j)$ and $b(k)$ are related by the operator $P_{b(j)b(k)}$,

$$b(j)=P_{b(j)b(k)}b(k)\ . \tag{2.1}$$

We are interested in collisions initiated in a two-cluster channel, e.g., $\alpha(i)$. The corresponding Schrödinger wave function will be denoted $\Psi(\alpha(i)n)$; it obeys

$$(E-H)\Psi(\alpha(i)n)=0\ , \tag{2.2}$$

where H is the Hamiltonian for the system and n denotes the internal states of the colliding fragments. Corresponding to partition $b(j)$ is the decomposition $H=H_{b(j)}+V^{b(j)}$, where $V^{b(j)}$ is the intercluster interaction in partition $b(j)$ and $H_{b(j)}$ is the Hamiltonian governing the behavior of the system with the $V^{b(j)}=0$. The eigenstates of $H_{b(j)}$ will be denoted $\Phi_{b(j)n}$ and are a product of bound states $\phi_{b(j)n}$ for the m clusters forming $b(j)$ times relative motion plane wave states. The subscript n will denote the bound state quantum numbers, just as in $\Psi(\alpha(i)n)$ of (2.2). The only $\Phi_{b(j)n}$ that we shall be interested in have the same total energy E as appears in (2.2); this energy dependence of $\Phi_{b(j)n}$ will be suppressed.

From the $H_{b(j)}$ we can form the resolvents $G_{b(j)}(z)=(z-H_{b(j)})^{-1}$, whose $z\to E+i0$ limits define the usual outgoing wave Green's functions $G_{b(j)}^{(+)}(E)$. Transition operators will generally be denoted $T_{b(j)\alpha(i)}$; they describe transitions from states in partition $\alpha(i)$ to those in partition $b(j)$, i.e., processes of the form $\alpha(i)n\to b(j)m$. The wave function components of an N-particle scattering theory[7] will be denoted $\psi_{b(j)}(\alpha(i)n)$. We shall add superscripts when needed to distinguish analogous quantities occurring in different theories.

B. Antisymmetrization procedure

When the N particles forming the scattering system are identical fermions, which we assume, the solution to (2.2) must be antisymmetrized. The procedure we follow to achieve this is a slight generalization of that of Ref. 8. (It also holds for identical bosons by setting the fermion parity phase factors and internal symmetrizers equal to unity.) We illustrate the procedure in this subsection by applying it to $\Psi(\alpha(i)n)$ of (2.2).

We first define the b-channel antisymmetrizer $A_{b(0)}$, generalizing the analogous antisymmetrizer of Ref. 8:

$$A_{b(0)}\equiv(N_b+1)^{-1/2}\sum_{i=0}^{N_b}(-1)^{\sigma_{b(i)}}R_{b(i)}P_{b(i)b(0)}\ , \tag{2.3}$$

where, for the assumed case of identical fermions, $(-1)^{\sigma_{b(i)}}$ is the parity of the permutation and $R_{b(i)}$ antisymmetrizes the internal states of $b(i)$. As in Goldberger and Watson,[14] we employ a normalization such that $A_{b(0)}$ produces a normalized state. Then, if $\Psi(\alpha(i)n)$ solves (2.2) in accord with the usual boundary conditions, the properly antisymmetrized solution of (2.2) is $\Psi^A(\alpha n)$ given by

$$\begin{aligned}\Psi^A(\alpha n)&=A_{\alpha(0)}\Psi(\alpha(0)n)\\&=\hat{N}_\alpha^{-1/2}\sum_{i=0}^{N_\alpha}(-1)^{\sigma_{\alpha(i)}}R_{\alpha(i)}\Psi(\alpha(i)n)\ ,\end{aligned} \tag{2.4}$$

where

$$\Psi(\alpha(i)n)=P_{\alpha(i)\alpha(0)}\Psi(\alpha(0)n)$$

and where $\hat{N}_\alpha=N_\alpha+1$, $\hat{N}_b=N_b+1$, etc. Henceforth, we shall assume, unless otherwise stated, that the i, j, etc., sums in this paper run over all allowed values, including zero. With this understanding, the limits on these sums and those, e.g., on b or β will generally be suppressed in all remaining equations where they appear.

Integral equations are the standard means of combining (2.2) with the relevant boundary conditions. If particle distinguishability is assumed, then a unique solution to (2.2) obeys the $N_2=2^{N-1}-1$ Lippmann-Schwinger (LS) equations[9]

$$\Psi(\alpha(i)n)=\Phi_{\alpha(i)n}\delta_{\beta(j)\alpha(i)}+G_{\beta(j)}^{(+)}(E)V^{\beta(j)}\Psi(\alpha(i)n)\ , \tag{2.5}$$

where $\beta(j)$ runs over all two-cluster partitions. We now let the particles be identical fermions. Application to (2.5) of the procedure embodied in (2.4) then yields $\Psi^A(\alpha n)$ obeying

$$\begin{aligned}\Psi^A(\alpha n)=&\sum_i\delta_{\beta(j)\alpha(i)}(-1)^{\sigma_{\alpha(i)}}R_{\alpha(i)}\Phi_{\alpha(i)n}/\hat{N}_\alpha^{1/2}\\&+G_{\beta(j)}^{(+)}(E)V^{\beta(j)}\Psi^A(\alpha n),\ \text{all }\beta\text{ and }j\ .\end{aligned} \tag{2.6}$$

This set of N_2 equations with the terms $\beta \neq \alpha$ omitted was originally derived in Ref. 8. Because questions concerning its validity have been raised, we present in Appendix A an alternative derivation involving Møller operators. In general, we shall assume that the internal states $\phi_{\alpha(i)n}$ are already each antisymmetric, so that in (2.6) and also in similar equations below, we may replace $R_{\alpha(i)}\Phi_{\alpha(i)n}$ by $\Phi_{\alpha(i)n}$.

III. ANTISYMMETRIZATION OF LABEL-TRANSFORMING WAVE FUNCTION EQUATIONS

A. General results

The procedure of Eq. (2.4) can be used to antisymmetrize any type of wave function equation. Because of their relative simplicity, we shall restrict our discussion of connected-kernel, N-particle scattering equations to those in which the relevant labels correspond to partitions rather than chains of partitions.[4]

The general form of the wave function component equations of interest to us is

$$\psi'_{b(j)}(\alpha(i)n)=\Phi_{\alpha(i)n}\delta_{b(j)\alpha(i)} + G^{(+)}_{b(j)}(E)\sum_{a,k} V_{b(j)a(k)}\psi'_{a(k)}(\alpha(i)n)\ , \tag{3.1}$$

where $\psi'_{b(j)}(\alpha(i)n)$ is the (distinguishable-particle) component in partition $b(j)$ generated by a two-body collision in partition $\alpha(i)$. The connectivity property of (3.1) is maintained by the coupling potentials $V_{b(j)a(k)}$, whose form depends on the particular theory. In general, $V_{b(j)a(k)}$ is asymmetric, so that (3.1) is a non-Hermitian set of equations. It is this non-Hermiticity that leads to the asymmetry of the new class of optical potentials of Sec. IV. Some examples of $V_{b(j)a(k)}$ are given below.

We now assume that the N particles are identical fermions and apply the method of Eq. (2.4) to Eq. (3.1). This produces components $\psi'_{b(j)}(\alpha n)=A_{\alpha(0)}\psi'_{b(j)}(\alpha(0)n)$, which obey

$$\psi'_{b(j)}(\alpha n)=\sum_i (-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}\delta_{b(j)\alpha(i)}/\hat{N}_\alpha^{1/2} + G^{(+)}_{b(j)}(E)\sum_{a,k} V_{b(j)a(k)}\psi'_{a(k)}(\alpha n)\ . \tag{3.2}$$

We have again assumed that each of the two bound states in $\phi_{\alpha(i)n}$ is antisymmetric; hence $R_{\alpha(i)}$ does not appear in the first term in the right-hand side (RHS) of (3.2).

Unlike the solutions $\Psi^A(\alpha n)$ of (2.4) or (2.6), the $\psi'_{b(j)}(\alpha n)$ of (3.2) are generally not antisymmetric. The reason for this is the dependence of $\psi'_{b(j)}(\alpha n)$ on its distinguishable-particle label $b(j)$. The exception to this general situation occurs when the exact solution to (3.1) is *independent* of the subscript $b(j)$ and is equal to the Schrödinger solution $\Psi(\alpha(i)n)$, in which case (3.2) reduces to (2.6), which defines the antisymmetric Schrödinger solution $\Psi^A(\alpha n)$. There are a number of theories (i.e., choices of $V_{b(j)a(k)}$) for which this is true,[7,15] examples of which are encountered below.

In the general case, properly symmetrized components may be obtained from the $\psi'_{b(j)}(\alpha n)$ of (3.2) by forming appropriate linear combinations. The simplest examples to deal with are those for which $V_{b(j)d(k)}$ is label transforming,[10] and we restrict ourselves to that class in the remainder of this section. For future comparison with the transition operator formalism of Ref. 10, it is helpful to display explicitly the implicit presence in Eq. (3.2) of the operator $R_{b(k)}$. We can do this most simply by allowing $R_{b(j)}$ to act on each side of this equation. From arguments similar to those used in Sec. III B below it follows that

$$\psi'_{b(j)}(\alpha n)=R_{b(j)}\psi'_{b(j)}(\alpha n)\ ,$$

and hence, that (3.2) is equivalent to

$$\psi'_{b(j)}(\alpha n)=\sum_i (-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}\delta_{b(j)\alpha(i)}/\hat{N}_\alpha^{1/2} + G^{(+)}_{b(j)}(E)\sum_{a,k} R_{b(j)}V_{b(j)a(k)}\psi'_{a(k)}(\alpha n)\ . \tag{3.3}$$

To obtain (3.3) we have also used the following relations:

$$R_{b(j)}\Phi_{\alpha(i)n}\delta_{b(j)\alpha(i)}=\Phi_{\alpha(i)n}\delta_{b(j)\alpha(i)}$$

and

$$[R_{b(j)},G^{(+)}_{b(j)}(E)]=0\ .$$

The assumption of label transformability now allows us to introduce Eq. (3.5), a result crucial to the further development of our theory. Let P be any element of S_N, the permutation group of order N. When $V_{b(j)a(k)}$ is label transforming, i.e., when

$$PV_{b(j)a(k)}P^{-1}=V_{Pb(j)Pa(k)}\ , \tag{3.4}$$

then as we prove below, $\psi'_{b(j)}(\alpha n)$ obeys

$$\psi'_{b(j)}(\alpha n)=(-1)^{\sigma_{b(j)}}P_{b(j)b(0)}\psi'_{b(0)}(\alpha n),\ \text{all } b\ . \tag{3.5}$$

We next sum both sides of (3.5) on j, which gives

$$\sum_j \psi'_{b(j)}(\alpha n)=\sum_j (-1)^{\sigma_{b(j)}}P_{b(j)b(0)}\psi'_{b(0)}(\alpha n)\ . \tag{3.6}$$

The RHS of (3.6) is almost in the form $A_{b(0)}\psi'_{b(0)}(\alpha n)$, where $A_{b(0)}$ is defined by (2.3). We put it in this form by introducing $\psi_{b(j)}(\alpha n)$:

$$\psi_{b(j)}(\alpha n)=\hat{N}_b^{1/2}\psi'_{b(j)}(\alpha n)\ . \tag{3.7}$$

Then the RHS of (3.6) becomes

$$N_b^{-1/2}\sum_j (-1)^{\sigma_{b(j)}}P_{b(j)b(0)}\psi_{b(0)}(\alpha n) =A_{b(0)}\psi_{b(0)}(\alpha n)\equiv\psi^A_b(\alpha n)\ , \tag{3.8}$$

where we have used the fact that $\psi_{b(j)}(\alpha n)$ like $\psi'_{b(j)}(\alpha n)$ obeys

$$\psi_{b(j)}(\alpha n)=R_{b(j)}\psi_{b(j)}(\alpha n)\ .$$

Thus, (3.3) leads, via (3.5), to *manifestly* antisymmetric components $\psi^A_b(\alpha n)$, defined with a normalization analogous to that of the antisymmetrized plane wave or of the full scattering state $\Psi^A(\alpha n)$. We shall refer to b as a generic partition or channel.

For that special class of theories in which the sum of

the components over the distinguishable-particle partition labels yields the (unsymmetrized) Schrödinger solution,[7,15] i.e., for which

$$\sum_{b,j}\psi'_{b(j)}(\alpha(i)n)=\Psi(\alpha(i)n)\ , \tag{3.9}$$

then (3.6) and (3.7) imply that

$$\sum_{b}\hat{N}_b^{-1/2}\psi^A_b(\alpha n)=\Psi^A(\alpha n)\ . \tag{3.10}$$

For those theories where $\psi'_{b(j)}(\alpha n)$ is independent of $b(j)$ and equal to $\Psi^A(\alpha n)$, (3.5) is an intrinsic property of $\Psi^A(\alpha n)$ and (3.10) simply introduces a renormalization.

Equation (3.5) implies that it is sufficient to determine the canonically labeled components. Setting $j=0$ in (3.3) and using (3.7) we have

$$\begin{aligned}\psi_{b(0)}(\alpha n)=&\sum_i(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}\delta_{b(0)\alpha(i)}\\&+G^{(+)}_{b(0)}(E)\sum_{a,k}(\hat{N}_b/\hat{N}_a)^{1/2}R_{b(0)}\\&\times V_{b(0)a(k)}\psi_{a(k)}(\alpha n)\ .\end{aligned} \tag{3.11}$$

Application of (3.5) to the $\psi_{a(k)}(\alpha n)$ on the RHS of (3.11) then leads to a set of equations involving the canonically labeled components $\psi_{b(0)}(\alpha n)$ alone:

$$\begin{aligned}\psi_{b(0)}(\alpha n)=&\Phi_{\alpha(0)n}\delta_{b(0)\alpha(0)}\\&+G^{(+)}_{b(0)}(E)\sum_a\tilde{V}_{b(0)a(0)}\psi_{a(0)}(\alpha n)\ ,\end{aligned} \tag{3.12}$$

where the exchange-effect potential $\tilde{V}_{b(0)a(0)}$ is defined by

$$\tilde{V}_{b(0)a(0)}=(\hat{N}_b/\hat{N}_a)^{1/2}\sum_k R_{b(0)}V_{b(0)a(k)}(-1)^{\sigma_{a(k)}}P_{a(k)a(0)}\ . \tag{3.13}$$

Equations (3.12) and (3.13) are the basic results of this subsection, supplemented by (3.8) and (3.5). We choose to work with the $\{\psi_{b(0)}(\alpha n)\}$ rather than the $\{\psi'_{b(0)}(\alpha n)\}$ because the former are normalized to $\Phi_{\alpha(0)n}\delta_{b(0)\alpha(0)}$. As a result (see Appendix B), Eq. (3.12) yields amplitudes that are the matrix elements of the properly symmetrized transition operators[10] taken between plane wave states normalized as are the $\{\psi_{b(0)}(\alpha n)\}$ themselves. Note that if we had used the $\{\psi'_{b(0)}(\alpha n)\}$, it would have been necessary to renormalize the resulting amplitudes.

Equation (3.3) is in general a very large set of equations.[16] The assumption of label transformability has effected a reduction to Eq. (3.12), which is $N_e\times N_e$, where N_e is the total number of equivalence classes.[16] From the structure of (3.12) [and also (3.3)] it follows that $\psi_{b(0)}(\alpha n)$ yields asymptotically at least portions of and in some cases the whole transition amplitude for processes of the type $\alpha\to b$.[4,7,15] But (3.5) shows that to within a phase factor, $\psi_{b(j)}$ and $\psi_{b(0)}$ yield the *same* asymptotic quantities. This will prove to be a crucial ingredient in our formulation of the optical potential.

It is useful for further developments to express (3.2) and (3.12) in matrix form. Equation (3.2) will thus read

$$\vec{\psi}'(\alpha n)=\vec{\Phi}^A+\underline{g}\underline{V}\vec{\psi}'(\alpha n)\ , \tag{3.14}$$

where

$$(\vec{\psi}'(\alpha n))_{b(j)}=\psi'_{b(j)}(\alpha n)\ ,$$

$$(\vec{\Phi}^A)_{b(j)}=\sum_{i=0}^{N_a}(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}\delta_{\alpha(i)b(j)}/\hat{N}_a^{1/2}\ ,$$

$$(\underline{g})_{b(j)a(k)}=\delta_{b(j)a(k)}G^{(+)}_{b(j)}(E)\ ,$$

and

$$(\underline{V})_{b(j)a(k)}=V_{b(j)a(k)}\ .$$

Similarly, (3.12) becomes

$$\vec{\psi}_0(\alpha n)=\vec{\Phi}_0+\underline{g}_0\underline{\tilde{V}}_0\vec{\psi}_0(\alpha n)\ , \tag{3.15}$$

with

$$(\vec{\psi}_0(\alpha n))_b=\psi_{b(0)}(\alpha n)\ ,$$

$$(\vec{\Phi}_0)_b=\Phi_{\alpha(0)n}\delta_{b\alpha}\ ,$$

$$(\underline{g}_0)_{ba}=G^{(+)}_{b(0)}(E)\delta_{ba}\ ,$$

and

$$(\underline{\tilde{V}}_0)_{ba}=\tilde{V}_{b(0)a(0)}$$

is defined by (3.13).

In addition, we define the two matrices of Green's functions, $\underline{\mathscr{G}}$ and $\underline{\mathscr{G}}_0$:

$$\underline{\mathscr{G}}=(\underline{g}-\underline{V})^{-1} \tag{3.16}$$

and

$$\underline{\mathscr{G}}_0=(\underline{g}_0-\underline{\tilde{V}}_0)^{-1}\ . \tag{3.17}$$

In terms of these quantities, the solved forms of (3.14) and (3.15) are

$$\vec{\psi}'(\alpha n)=\vec{\Phi}^A+\underline{\mathscr{G}}\,\underline{V}\vec{\Phi}^A \tag{3.18}$$

and

$$\vec{\psi}_0(\alpha n)=\vec{\Phi}_0+\underline{\mathscr{G}}_0\underline{\tilde{V}}_0\vec{\Phi}_0(\alpha n)\ . \tag{3.19}$$

B. Proof of Eq. (3.5).

In this subsection, we shall prove Eq. (3.5). We employ the relations

$$P_{\alpha(i)\alpha(0)}\Phi_{\alpha(0)n}=\Phi_{\alpha(i)n}$$

and

$$P_{b(j)b(0)}\Phi_{b(0)m}=\Phi_{b(j)m}$$

to help construct the unnormalized, N-particle, antisymmetrized plane wave states

$$\begin{aligned}\hat{N}_\alpha^{1/2}\Phi^A(\alpha n)=&\sum_i(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}\\=&\sum_i(-1)^{\sigma_{\alpha(i)}}P_{\alpha(i)\alpha(0)}\Phi_{\alpha(0)n}\end{aligned} \tag{3.20}$$

and

$$\hat{N}_b^{1/2}\Phi^A(bm)=\sum_j(-1)^{\sigma_{b(j)}}P_{b(j)b(0)}\Phi_{b(0)m}\ . \tag{3.21}$$

The states $\Phi_{b(j)m}$ are products of an antisymmetrized bound state for each of the clusters forming $b(j)$ times relative plane wave states.

To establish (3.5), we use the solved form (3.18), which in terms of partition labels reads

$$\hat{N}_\alpha^{1/2}\psi'_{b(j)}(\alpha n)=\sum_i\delta_{b(j)\alpha(i)}(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}+\sum_{d,k,i}\mathcal{G}_{b(j)d(k)}V_{d(k)\alpha(i)}(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}\ . \tag{3.22}$$

For simplicity, we consider $P=P_{b(0)b(j)}$. Then from (3.22) we have

$$\hat{N}_\alpha^{1/2}P\psi'_{b(j)}(\alpha n)=\sum_i P\delta_{b(j)\alpha(i)}P^{-1}P\hat{\Phi}_{\alpha(i)n}+\sum_{d,k,i}P\mathcal{G}_{b(j)d(k)}P^{-1}PV_{d(k)\alpha(i)}P^{-1}P\hat{\Phi}_{\alpha(i)n}\ , \tag{3.23}$$

where

$$\hat{\Phi}_{\alpha(i)n}=(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}\ .$$

By assumption, $\underline{V}$ is label transforming, which means that $\underline{\mathcal{G}}$ is also. Hence (3.23) becomes

$$\hat{N}_\alpha^{1/2}P\psi'_{b(j)}(\alpha n)=\sum_i\delta_{b(0)P\alpha(i)}P\hat{\Phi}_{\alpha(i)n}+\sum_{d,k,i}\mathcal{G}_{Pb(j)Pd(k)}V_{Pd(k)P\alpha(i)}P\hat{\Phi}_{\alpha(i)n}\ . \tag{3.24}$$

Now, $Pb(j)=b(0)$ while the fact that P is a member of the permutation group of order N means that $Pd(k)=d(k')$. Because of this latter relation, the sum on all k of $Pd(k)$ in (3.24) can be replaced by the sum on all k' of $d(k')$, exactly as in Ref. 10. Since k' is a summation variable we may replace it by k. These facts transform (3.24) into

$$\hat{N}_\alpha^{1/2}P\psi'_{b(j)}(\alpha n)=\sum_i\delta_{b(0)P\alpha(i)}P\hat{\Phi}_{\alpha(i)n}+\sum_{d,k,i}\mathcal{G}_{b(0)d(k)}V_{d(k)P\alpha(i)}P\hat{\Phi}_{\alpha(i)n}\ . \tag{3.25}$$

Evaluation of $P\psi_{b(j)}(\alpha n)$ is thus reduced to the determination of $P\hat{\Phi}_{\alpha(i)n}$. We shall prove that

$$P\hat{\Phi}_{\alpha(i)n}=(-1)^{\sigma_{b(j)}}\hat{\Phi}_{P\alpha(i)n}\ ; \tag{3.26}$$

this result, when used in (3.25) establishes (3.5).

To verify (3.26) we expand $\Phi^A(\alpha n)$ via the $\Phi^A(bm)$:

$$\Phi^A(\alpha n)=\sum\!\!\!\!\!\!\int d\mu(m)\Phi^A(bm)\langle\Phi^A(bm)\,|\,\Phi^A(\alpha n)\rangle\ .$$

The antisymmetry property guarantees that P has the same effect on $\Phi^A(\alpha n)$ and on $\Phi^A(bm)$, viz.,

$$\begin{aligned}&P\Phi^A(bm)=f_p\Phi^A(bm)\\ \text{and}\qquad&\\ &P\Phi^A(\alpha n)=f_p\Phi^A(\alpha n)\ ,\end{aligned} \tag{3.27}$$

where f_p is a phase factor independent of the state labels m and n.

The factor f_p can be easily determined by using the expansion (3.21):

$$\hat{N}_b^{1/2}P\Phi^A(bm)=\sum_{k=0}^{N_b}P\hat{\Phi}_{b(k)m}=\sum_{k'=0}^{N_b}f_p\hat{\Phi}_{b(k')m}\ . \tag{3.28}$$

Since the same f_p multiplies each term in the k' sum, and the two sums in (3.28) are identical, we may evaluate f_p by choosing $k=j$ in the k sum and $k'=0$ in the k' sum. That is, the same factor must multiply $\Phi_{b(0)m}$ in each of the two expressions for $P\Phi^A(bm)$. We readily find that $f_p=(-1)^{\sigma_{b(j)}}$. This latter result, plus the relations (3.27) and (3.28) then yield (3.26), which establishes (3.5). A similar analysis holds for (3.3) as well.

C. Projected equations

It is often useful to work with only a few of the channel components $\psi_{b(0)}(\alpha n)$ rather than with the entire set obeying the $N_e\times N_e$ equations (3.15); we may even wish to be more restrictive and work with only certain relative motion states $\langle\phi_{b(0)m}\,|\,\psi_{b(0)}(\alpha n)\rangle$. Both choices can be singled out by means of projection operators, leading to equations for projected components $\vec{\psi}_0^{\,P}(\alpha n)$ obeying equations with effective interactions, exactly as in the one-channel problem.[1]

Let $\underline{I}_0$ be the $N_e\times N_e$ unit matrix, and define $\underline{\mathscr{P}}_0$ to be a diagonal matrix having nonzero entries only on the first n_0 diagonal elements; these may each be unity or a projector of the form $\sum_m|\phi_m\rangle\langle\phi_m|$, where the partition index on $|\phi_m\rangle$ is suppressed. The complement to $\underline{\mathscr{P}}_0$ is $\underline{\mathscr{Q}}_0=\underline{I}_0-\underline{\mathscr{P}}_0$. We also define $\vec{\psi}_0^{\,P}(\alpha n)=\underline{\mathscr{P}}_0\vec{\psi}_0(\alpha n)$. Then inserting $\underline{I}_0=\underline{\mathscr{P}}_0+\underline{\mathscr{Q}}_0$ into (3.15) and eliminating the $\underline{\mathscr{Q}}_0\vec{\psi}_0(\alpha n)$ portion in the usual manner,[1] we find that $\vec{\psi}_0^{\,P}(\alpha n)$ obeys

$$\vec{\psi}_0^{\,P}(\alpha n)=\underline{\mathscr{P}}_0\vec{\Phi}_0+\underline{g}_0\underline{\mathscr{P}}_0\underline{\tilde{U}}_0\underline{\mathscr{P}}_0\vec{\psi}_0^{\,P}(\alpha n)\ , \tag{3.29}$$

where

$$\underline{\tilde{U}}_0=\underline{\tilde{V}}_0+\underline{\tilde{V}}_0\underline{\mathscr{Q}}_0\underline{\Gamma}_0\underline{\tilde{V}}_0\ , \tag{3.30}$$

with

$$\underline{\Gamma}_0=(\underline{g}_0{}^{-1}-\underline{\mathscr{Q}}_0\underline{\tilde{V}}_0\underline{\mathscr{Q}}_0)^{-1}\ , \tag{3.31}$$

and where we have assumed that $[\underline{\mathscr{P}}_0,\underline{g}_0]=0$. The effective interaction $\underline{\tilde{U}}_0$ is the formal solution to

$$\underline{\tilde{U}}_0=\underline{\tilde{V}}_0+\underline{\tilde{V}}_0\underline{\mathscr{Q}}_0\underline{g}_0\underline{\tilde{U}}_0\ . \tag{3.32}$$

Equation (3.29) is the desired result; it will be employed in the next section in our derivation of an antisymmetric optical potential. Our interest is in cases for which $\underline{\mathscr{P}}_0$ projects only onto two-body bound states for which the assumed connectivity property of $\underline{V}$ ensures the connectivity of (3.29) and (3.32). Furthermore, it is sufficient to consider only the (connected) canonically labeled components since the amplitudes obtained from $\psi_{b(0)}(\alpha n)$ and $\psi_{b(j)}(\alpha n)$ are phase equivalent by Eq. (3.5).

IV. ANTISYMMETRIZED OPTICAL POTENTIAL

A. General properties

Equation (3.29) provides a natural basis for constructing an optical model in which the effects of particle identity are already included, and we undertake such a construction in this section. [Equation (3.29) also provides a formalism for treating nuclear reactions, a topic that will be discussed elsewhere.[17]]

The problem of deriving an elastic scattering optical potential or at least a one-body complex potential well[1] which includes the effects of the Pauli principle is an old one. Citations to, as well as comments on, much of the relevant literature can be found, e.g., in Ref. 3. The early formal work stressed the notion of projection operators onto antisymmetric subspaces, while more recent investigations have emphasized group theory and the need to work with connected kernel, multiparticle collision theories, which are essential for determining not only unique results but, for example, the effects of breakup.

In the most recent work on this problem, the following features of the antisymmetrized optical potential formalism are deemed crucial:[3]

(i) it should yield a two-body description of two-fragment elastic scattering;

(ii) it should yield an optical potential which is real below the lowest inelastic threshold (assuming no exoergic reaction channels);

(iii) it should be based on a connected-kernel description which is consistent with (i) and (ii) and which allows for approximations also consistent with (i) and (ii).

An obvious consequence of (iii) is that the detailed structure of the optical potential will depend on which many-particle collision theory one uses; i.e., the optical potential will not be unique.

One way of ensuring (ii) is to require that the optical potential has no elastic unitarity cuts.[18] In recent work on the antisymmetrized optical potential, various transition operators have been investigated and it has been concluded by Kowalski and co-workers that the appropriate one to use is that denoted the Alt-Grassberger-Sandhas (AGS) off-shell extension[12] since it leads to a Hermitian-analytic optical potential free of elastic scattering cuts. For example, Kowalski[3] points out that the prior transition operator[19] (off-shell extension) produces an optical potential that may have elastic scattering singularities.

Our approach to the antisymmetric optical potential problem is based not on transition operators but on the wave function equations developed in Sec. III. We first consider those $\underline{\tilde{V}}_0$ that are label transforming. This allows us to derive a class of asymmetric optical potentials which satisfy the three criteria noted above when the two assumptions on thresholds and nonabsorptiveness stated below hold.

The two assumptions we invoke are first, that there are no reaction channels whose thresholds lie below E_{inel} and second, that the matrix elements $V_{b(0)a(0)}$ are nonabsorptive, e.g., that they are sums of Hermitian interactions and do not involve subcluster transition operators or Green's functions. Examples fulfilling these assumptions are given in Sec. IV B. When either of these assumptions fails to hold, an optical potential can still be defined, but it may be absorptive at energies below the lowest inelastic threshold. We note two examples of the failure of the second assumption in Sec. V, and discuss a non-label-transforming one in detail in Appendix D.

We begin by restricting $\underline{\mathscr{P}}_0$ as follows:

$$(\underline{\mathscr{P}}_0)_{b(0)a(0)}=P_{\alpha(0)0}\delta_{ba}\delta_{b\alpha}\ , \tag{4.1}$$

where $P_{\alpha(0)0}$ is the projector onto the ground state in channel α,

$$P_{\alpha(0)0}=|\phi_{\alpha(0)0}\rangle\langle\phi_{\alpha(0)0}|\ , \tag{4.2}$$

i.e., we set $n=0$ in $\phi_{\alpha(0)n}$. For simplicity we shall drop the second index 0, and write $P_{\alpha(0)}$ and $\phi_{\alpha(0)}$. Since only the $\alpha(0)$ element in $\underline{\mathscr{P}}_0$ is nonzero, only the same element in $\underline{\tilde{U}}_0$ will contribute to (3.29). This element will be denoted $\tilde{U}_\alpha$; from Eq. (3.29) a formal expression for it is

$$\tilde{U}_\alpha=\tilde{V}_{\alpha(0)\alpha(0)}+\sum_{a,c}\tilde{V}_{\alpha(0)a(0)}\mathscr{Q}_{a(0)}\Gamma_{a(0)c(0)}\tilde{V}_{c(0)\alpha(0)}\ , \tag{4.3}$$

where the nonzero elements of $\mathscr{Q}_0$ are

$$\mathscr{Q}_{\alpha(0)\alpha(0)}\equiv\mathscr{Q}_{\alpha(0)}=1-P_{\alpha(0)}$$

and

$$\mathscr{Q}_{b(0)b(0)}\equiv\mathscr{Q}_{b(0)}=1,\ b\neq\alpha\ .$$

Next, consider $\vec{\psi}_0$. In order that the matrix element $\langle\phi_{\alpha(0)}|\tilde{U}_\alpha|\phi_{\alpha(0)}\rangle$ can be identified with the optical potential, $\psi_{\alpha(0)}(\alpha 0)$ must obey the following boundary condition:

$$(2\pi)^{3/2}\langle\vec{r}_\alpha,\phi_{\alpha(0)}|\psi_{\alpha(0)}(\alpha 0)\rangle\underset{r_\alpha\to\infty}{\sim}e^{i\vec{k}_{\alpha 0}\cdot\vec{r}_\alpha}+f^A_{\alpha 0,\alpha 0}e^{ik_{\alpha 0}r_\alpha}/r_\alpha\ , \tag{4.4}$$

where $\vec{r}_\alpha$ is the intercluster separation in partition α, $\vec{k}_{\alpha 0}$ is the incident momentum, and $f^A_{\alpha(0),\alpha(0)}$ is the exact, antisymmetrized elastic scattering amplitude. This will be the case if $\psi_{b(0)}(\alpha 0)$ is a "true component"[20] and therefore obeys Eq. (3.10). Equation (4.4) is also the correct boundary condition if $\psi_{\alpha(0)}(\alpha 0)$ is independent of the subscript $\alpha(0)$ and equal to $\Psi^A(\alpha 0)$, the exact, antisymmetrized Schrödinger wave function. These statements follow from

the results of Appendix B. There may be other choices of wave function components for which (4.4) holds, but these are sufficient for our purposes and are appropriate to the examples given below.

The final step in our construction is to define the one-body elastic scattering state $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$:

$$|\chi^{(+)}_{\vec{k}_\alpha}\rangle \equiv \langle \phi_{\alpha(0)} | \psi_{\alpha(0)}(\alpha 0)\rangle \ . \tag{4.5}$$

From (3.29), $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ obeys

$$|\chi^{(+)}_{\vec{k}_\alpha}\rangle = |\vec{k}_\alpha\rangle + (E-\epsilon_{\alpha 0}-K_{\alpha(0)}+i0)^{-1}\mathscr{V}^A_\alpha |\chi^{(+)}_{\vec{k}_\alpha}\rangle \ , \tag{4.6}$$

where $K_{\alpha(0)}$ is the kinetic energy operator for the relative motion of the two clusters forming channel α and

$$\mathscr{V}^A_\alpha = \langle \phi_{\alpha(0)} | \tilde{U}_\alpha | \phi_{\alpha(0)}\rangle \tag{4.7}$$

is a one-body operator.

Under the assumption that $\langle \vec{r} | \chi^{(+)}_{\vec{k}_\alpha}\rangle$ satisfies (4.4), $\mathscr{V}^A_\alpha$ is the antisymmetrized complex potential well or (non-energy-averaged) optical potential describing elastic scattering of the two fragments forming channel α and $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ is the optical model state vector. From (4.3), $\mathscr{V}^A_\alpha$ is clearly seen to be nonlocal and energy dependent. When the previously stated assumptions on threshold behavior and nonabsorptiveness are satisfied, $\tilde{U}_\alpha$ and $\mathscr{V}^A_\alpha$ are also seen [via (4.4) and $\mathscr{Q}_0$ in (4.3)] to be real below the inelastic threshold and hence to be free of elastic scattering singularities. (This nontrivial and very important conclusion is discussed in further detail in Sec. V.) Since $\mathscr{V}^A_\alpha$ is a one-body operator and $\tilde{U}_\alpha$ is obtained by solving a connected-kernel integral equation, the three crucial features noted at the beginning of this section are satisfied by our construction. However, $\mathscr{V}^A_\alpha$ is not Hermitian analytic, a point discussed in Sec. V C.

Equation (4.6) is a one-body Lippmann-Schwinger (LS) equation. We may use it to define a one-body transition operator T^A_α obeying a similar LS equation:

$$T^A_\alpha = \mathscr{V}^A_\alpha + \mathscr{V}^A_\alpha (E-\epsilon_{\alpha 0}-K_{\alpha(0)}+i0)^{-1} T^A_\alpha \ . \tag{4.8}$$

The exact, antisymmetrized, elastic scattering transition amplitude $A^A_{\alpha 0,\alpha 0}$ is obviously given by the on-shell matrix element of T^A_α:

$$A^A_{\alpha 0,\alpha 0} = \langle \vec{k}'_\alpha | T^A_\alpha | \vec{k}_\alpha \rangle \ , \tag{4.9}$$

where $|\vec{k}'_\alpha| = |\vec{k}_\alpha|$.

B. Specific examples

We illustrate the preceding development with three examples; the first is based on the extended Faddeev theory of N-particle scattering,[15] the second on the precursor form of the Bencze-Redish-Sloan (BRS) theory,[21,15] and the third, equivalent to the second, is based on the AGS off-shell transition operator.[12] The first two of these theories are discussed as a wave function formalism in Ref. 15, the third is discussed in Appendix C. The key element in the derivation of these formalisms is the Benoist-Gueutal, L'Huillier, Redish, Tandy (BLRT) partitioning technique[22] for the external channel interaction V^{b_r} in partition b_r, viz.,

$$V^{b_r} = \sum_{m=2}^{N-1} \sum_a C_m V^{b_r}_{a_m} \ , \tag{4.10}$$

where $C_m = [(-1)^m (m-1)!]$; $V^{b_r}_{a_m}$ is the set of interactions external to (between) the r clusters forming partition b_r and internal in (binding) each of the m clusters forming partition a_m; and here we assume pairwise interactions only. The labels j and i referring to the arrangement of particles in b_r and a_m, e.g., $b_r(j)$, are suppressed in (4.10).

1. *The extended Faddeev (EF) theory*

In this case the matrix $\underline{V}$ of (3.14) has elements

$$V^{\rm EF}_{b_r(j)a_m(k)} = C_r V^{a_m(k)}_{b_r(j)} \ , \tag{4.11}$$

which are nonabsorptive and energy independent. Furthermore, as shown in Ref. 15, the structure of (3.1) guarantees that in a two-cluster partition, the unsymmetrized components $\psi'^{\rm EF}_{\beta(j)}(\alpha(i)n)$ satisfy

$$(2\pi)^{3/2}\langle \vec{r}_\gamma \phi_{\gamma(k)m} | \psi'^{\rm EF}_{\beta(j)}(\alpha(i)n)\rangle \underset{r_\gamma \to \infty}{\sim} \delta_{\gamma(k)\beta(j)}(\delta_{\beta(j)\alpha(i)}\delta_{mn} e^{i\vec{k}_{\alpha n}\cdot\vec{r}_\alpha} + f_{\beta(j)m,\alpha(i)n} e^{ik_{\beta m} r_\beta} r_\beta^{-1}) \ , \tag{4.12}$$

where $f_{\beta(j)m,\alpha(i)n}$ is the exact, distinguishable-particle scattering amplitude. Furthermore, these $\psi'^{\rm EF}_{b_r(j)}$ obey[15]

$$\sum_{r,b,j} \psi'^{\rm EF}_{b_r(j)} = \Psi(\alpha(i)n) \ ,$$

where $\Psi(\alpha(i)n)$ is the exact Schrödinger solution, and hence are true components.[20]

In the EF case the connectivity structure of the unsymmetrized equations leads to Eq. (4.12). Since the connectivity property does not change on symmetrization, then the $\psi^{\rm EF}_{b_r(j)}(\alpha n)$ obey a similar boundary condition. Thus, since in the EF theory $\psi^{\rm EF}_{\alpha(0)}(\alpha 0)$ yields the exact, antisymmetrized scattering amplitude, this theory provides a basis for the optical potential. Inspection of (4.3) shows that in this case, the leading term in $\tilde{U}_\alpha$, viz, $\tilde{V}_{\alpha(0)\alpha(0)}$, is

$$\tilde{V}^{\rm EF}_{\alpha(0)\alpha(0)} = \sum_{k\neq 0} R_{\alpha(0)} V^{\alpha(k)}_{\alpha(0)} (-1)^{\sigma_{\alpha(k)}} P_{\alpha(k)\alpha(0)} \ . \tag{4.13}$$

That is, in the lowest order only the pure exchange terms contribute to the optical potential $\mathscr{V}^A_\alpha$. However, it is shown in Ref. 15 by means of an off-shell transformation, that the next order term in $\tilde{U}_\alpha$ actually leads to the direct (nonexchange) Born term $\langle \vec{k}'_\alpha \phi_{\alpha(0)} | V^{\alpha(0)} | \phi_{\alpha(0)} \vec{k}_\alpha \rangle$. In addition, a multiple scattering series for the distinguishable particle elastic amplitude has been developed in Ref. 15. The extension of this to the present case of identical

particle scattering will be addressed in a future publication.[17]

2. The precursor BRS equations

In this case,[15] $\underline{V}^{\rm BRS}=(\underline{V}^{\rm EF})^T$, where T means transpose, so that

$$V^{\rm BRS}_{b_r(j)a_m(k)}=C_m V^{b_r(j)}_{a_m(k)} , \tag{4.14}$$

which remains nonabsorptive and energy independent. The index $b_r(j)$ in (4.14) labels the external and not the internal portion of $\underline{V}^{\rm BRS}$. The consequence is that each $\psi^{\rm BRS}_{b_r(j)}$ is independent of the subscript $b_r(j)$ and equals the full solution:[15]

$$\psi^{\rm BRS}_{b_r(j)}(\alpha(i)n)=\Psi(\alpha(i)n), \quad \text{all } b_r(j) . \tag{4.15}$$

Hence, $\psi^{\rm BRS}_{b_r(j)}(\alpha(i)n)$ satisfies (4.12) without the factor $\delta_{\alpha(k)\beta(j)}$, i.e., the boundary condition is given by the term in parentheses for (4.12), which is the usual one for the Schrödinger solution.

In order to apply these ideas to the identical particle case, we must show that the analog of (4.15) holds for the antisymmetrized components. The method for establishing this relation, viz.,

$$\psi^{\rm BRS}_{b_r(j)}(\alpha n)=\Psi^A(\alpha n) , \tag{4.16}$$

follows precisely the arguments of Ref. 15 for the distinguishable particle case, the only change being the replacement of the unsymmetrized quantities by their antisymmetrized counterparts. Because of the essential repetition of these steps of Ref. 15, we shall here assume that they have been taken, yielding Eq. (4.16) as the result. As a consequence, the form of the boundary condition obeyed by $\psi^A_\beta(\alpha n)$ in the BRS case is the same as for $\psi_{\beta(0)}(\alpha(0)n)$, viz., the term in parentheses in (4.12) with $f_{\beta(j)m,\alpha(0)n}$ replaced by the exact antisymmetrized amplitude $f^A_{\beta(0)m,\alpha(0)n}$. We therefore conclude that the BRS precursor equations also provide a basis for the antisymmetrized optical potential.

The precursor BRS components $\psi^{\rm BRS}_{b_r(j)}(\alpha(i)n)$ are shown in Ref. 15 to correspond to the precursor form of the BRS transition operators. It is shown in Appendix C that these *same* BRS wave function components also correspond to the transition operators obtained by using the BLRT partitioning technique in conjunction with the AGS off-shell extension. Hence the post and the AGS transition operators combined with the BLRT technique yield the same optical potential.

Since the EF and BRS equations are generated by different off-shell transition operators (the prior and post,[19] respectively), then their respective $\mathscr{V}^A_\alpha$'s are not expected to be identical. Only their on-shell transition amplitudes will be the same. Equivalently, the EF and BRS optical potentials will produce one-body wave functions which will agree only asymptotically.

We remark here that the behavior of the lowest order term in $\tilde{U}^{\rm BRS}_\alpha$ is analogous to that of Eq. (4.13) for the EF case:

$$\tilde{V}^{\rm BRS}_{\alpha(0)\alpha(0)}=\sum_{k\neq 0} R_{\alpha(0)} V^{\alpha(0)}_{\alpha(k)}(-1)^{\sigma_{\alpha(k)}} P_{\alpha(k)\alpha(0)} . \tag{4.17}$$

In this case also only the pure exchange terms contribute. The presence of the full, direct Born term and the existence of a multiple scattering expansion for $\mathscr{V}^A_\alpha$ in the BRS case has been discussed by Kowalski;[23] see also the remarks in Ref. 15.

V. DISCUSSION

A. Elastic scattering singularities and antisymmetry

The analysis of the preceding section clearly establishes the not-unexpected result that wave-function formulations of N-particle collision theories can be used as the basis for deriving an optical potential. Our conclusions go further than this, however, since we also have claimed that when the assumptions on thresholds and nonabsorptiveness are satisfied, our method leads to an optical potential satisfying the three properties noted in Sec. IV A. While the first of these is trivially satisfied by our construction, the validity of the second and third property may not be so obvious, and we discuss them in detail, property two in this subsection and property three in Sec. V D.

There are two cases we consider: true components,[20] which we examine first, and, components like those of BRS,[15] which are independent of partition labels and equal to the full Schrödinger wave function. The crucial property we invoke for true components is the boundary condition (4.12), which holds by definition for all true components and not just for the EF case. The meaning of this boundary condition (bc) is that *only* $\psi_{\beta(j)}(\alpha(i)n)$ contributes asymptotically to the $2\to 2$ amplitude $f_{\beta(j)m,\alpha(i)n}$: no component $\psi_{b(j)}(\alpha(i)n)$, with $b\neq\beta$, contributes to this amplitude. This result, which holds for all b, j, and i, depends only on the connectivity property of the equations of the particular formalism. Furthermore, the connectivity property of (3.1) is maintained when (2.4) is used to form (3.2), since it depends only on the kernel term and not the source (plane wave) terms. Maintenance of the connectivity property thus ensures that $\psi_{\beta(j)}(\alpha n)$ obeys the same kind of bc as does $\psi_{\beta(j)}(\alpha(i)n)$, viz., a form identical to that of (4.12). That is, *only* $\psi_{\beta(j)}(\alpha n)$ contributes asymptotically to the $2\to 2$ processes $\alpha n\to\beta(j)m$. By Eq. (3.5), the amplitudes for $\alpha n\to\beta(j)m$ and $\alpha n\to\beta(0)m$ are identical apart from a phase factor. By the analysis of Appendix B, this amplitude (viz., $f^A_{\beta(0)m,\alpha(0)n}$) is precisely the exact, properly antisymmetrized one obtained by taking plane wave matrix elements of the properly antisymmetrized transition operators. That is, of course, why we need only consider the canonically labelled components. In addition, however, this bc states, for example, that *all* the elastic flux in the EF description of the identical particle case is contained asymptotically in $\langle\phi_{\alpha(0)0}\,|\,\psi_{\alpha(0)}(\alpha 0)\rangle=|\chi^{(+)}_{\vec{k}_\alpha}\rangle$: none of it is obtained from any other projections, such as $\langle\phi_{\alpha(0)0}\,|\,\psi_{\beta(0)}(\alpha(0)\rangle$ or $\langle\phi_{\alpha(0)m}\,|\,\psi_{\alpha(0)}(\alpha 0)\rangle$, $m\neq 0$. To ensure that we obtain only $f^A_{\alpha(0)0,\alpha(0)0}$ of (4.4) asymptotically, we have restricted our description to $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$; this is accomplished by means of $\underline{\mathscr{P}}_0$ defined by (4.1) and hence via the presence in $\mathscr{V}^A_\alpha$ and $\tilde{U}_\alpha$ of the projector $\underline{\mathscr{Q}}_0$ defined below Eq. (4.3).

The presence of the projectors $\underline{\mathscr{P}}_0$ and $\underline{\mathscr{Q}}_0$ have in-

teresting consequences. Use of $\underline{\mathscr{Q}}_0$, as noted above, means that $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ yields asymptotically the entire, antisymmetrized elastic scattering amplitude. But, $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ obeys an equation involving the ground state matrix element of $\tilde{U}_\alpha$ and this latter quantity obeys an equation [e.g., (3.29) or (4.3)] in which $\underline{\mathscr{Q}}_0$ occurs. Hence $\underline{\mathscr{Q}}_0$ ensures that no elastic flux is contained in $\tilde{U}_\alpha$. That is, exactly as in Ref. 6, the presence of $\underline{\mathscr{Q}}_0$ ensures that $\mathscr{V}^A_\alpha$ and $\tilde{U}_\alpha$ are free of elastic unitarity cuts. Therefore, if all reaction channel thresholds lie above $E_{\rm inel}$ and $E<E_{\rm inel}$, then $\mathscr{V}^A_\alpha$ must be real since none of the Green's functions appearing in $\tilde{U}_\alpha$ and in $\mathscr{V}^A_\alpha$ can have any poles. In addition, if $E\geq E_{\rm inel}$, then $\mathscr{V}^A_\alpha$ will become absorptive but will still not contain elastic unitarity cuts due to the presence of $\underline{\mathscr{Q}}_0$. Finally, since $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ and $\langle\phi_{\alpha(0)}|\Psi^A(\alpha 0)\rangle$ are equal asymptotically and $\psi_{b(0)}(\alpha 0)$ is here assumed to be a true component, then $f^A_{\alpha(0)0,\alpha(0)0}$ is necessarily unitary when $E<E_{\rm inel}$ even though $\mathscr{V}^A_\alpha$ is asymmetric.

Similar comments apply to the second case, of which the BRS components are an example. In this case, Eq. (3.1) is a connected-kernel set whose components are each equal to $\Psi(\alpha(i)n)$. The antisymmetrization procedure (2.4) now yields equations whose solutions are each $\Psi^A(\alpha n)$, which also satisfy (2.6) and (3.5). In this case,

$$\langle\phi_{\alpha(0)0}|\Psi^A(\alpha n)\rangle=\langle\phi_{\alpha(0)0}|\psi^{\rm BRS}_{b(0)0}(\alpha n)\rangle=|\chi^{(+)}_{\vec{k}_\alpha}\rangle$$

uniquely yields, by definition, the exact, antisymmetric, elastic-scattering amplitude. For energies less than the first inelastic or reaction channel threshold and for no absorptive interactions appearing in H, then by the preceding arguments the associated optical potential $\mathscr{V}^A_\alpha$ is real.

The procedure of this paper may seem unusual in that we partially symmetrize [recall that $\psi^A_b(\alpha n)$ is fully antisymmetric, but $\psi_{b(j)}(\alpha n)$ is not], and then project $\psi_{\alpha(0)}(\alpha n)$ onto the ground state $\phi_{\alpha(0)0}$ to obtain $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$, denoted a one-body elastic scattering state. An antisymmetrical N-body elastic wave function can evidently be formed from $\phi_{\alpha(0)0}\chi^{(+)}_{\vec{k}_\alpha}$ by antisymmetrizing; it will also yield the antisymmetrized elastic scattering amplitude. Suppose, however, that we formed the fully antisymmetric component $\psi^A_\alpha(\alpha n)$ first and then projected it onto $\phi_{\alpha(0)0}$ to form an alternate one-body elastic scattering state

$$|u_\alpha\rangle=\langle\phi_{\alpha(0)0}|\psi^A_\alpha(\alpha n)\rangle=|\chi_{\vec{k}_\alpha}\rangle+\text{exchange terms}\,.$$

It is clear that while $|u_\alpha\rangle$ will obey an equation like (4.6) but with a different optical potential, $|u_\alpha\rangle$ and $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ will each yield the *same* amplitude. That we can generate two different one-body elastic wave functions and optical potentials for a given $\tilde{V}_{b(0)d(0)}$ but still obtain the same amplitude should be no surprise: we only require asymptotic equality. It is evident from this discussion that if we wish to work only with antisymmetrized, N-particle states, then the order of projecting and antisymmetrizing is unimportant as far as the amplitude is concerned, although changing the order will produce a different optical potential. However, if an approximation is introduced, then amplitudes as well as optical potentials will differ. We consider approximations in Sec. V D.

We have referred to $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ as a one-body elastic scattering state. In view of the normalization problems discussed, e.g., by Fliessbach,[24] this nomenclature may seen unwarranted, but it is in fact proper. We use it for two reasons. First $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ does yield the properly normalized, fully antisymmetrized, elastic scattering amplitude, generated by the one-body plane wave state $|\vec{k}_\alpha\rangle=\langle\phi_{\alpha(0)0}|\Phi_{\alpha(0)0}\rangle$. Second, and more importantly, suppose one uses suitably defined projection operators to construct from the set (3.15) a coupled two-channel model in which $\phi_{\alpha(0)0}$ and $\phi_{\beta(0)0}$ are linked. Then, as shown in Ref. 17, the elastic scattering state appearing in the exact, antisymmetrized, rearrangement amplitude $A^A_{\beta(0)0,\alpha(0)0}$ is $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$, normalized to $|\vec{k}_\alpha\rangle$ plus outgoing waves. By working with scattering equations which yield the properly normalized amplitude, e.g., the set (3.15), one automatically produces properly normed one-body states.

B. The LRT and CPA components

The only cases we have considered so far are those in which $V_{b(j)a(k)}$ is nonabsorptive and label transforming. Two exceptions to this are the L'Huillier-Redish-Tandy (LRT) components[25] and the components arising in the various channel permuting array (CPA) versions of the channel coupling array (CCA) theory of N-particle scattering.[13] In the former case, $V^{\rm LRT}_{b(j)a(k)}$ is label transforming but apparently absorptive, due to the presence in it of channel Green's functions. In the CPA case, $V^{\rm CPA}_{b(j)a(k)}$ is nonabsorptive but not label transforming due to the occurrence in it of the channel permuting array, a numerical matrix whose entries are independent of $b(j)$ and $a(k)$. There are both kinds of CPA components: true components and those independent of the channel index and equal to the Schrödinger solution, and we consider them both below. The LRT components are true components and we examine them first.

Since the LRT components are label transforming, the analysis of Secs. III and IV is valid for them. Although $V^{\rm LRT}_{b(j)a(k)}$ contains Green's functions, the fact that the $\psi^{\rm LRT}_{b(j)}(\alpha n)$ are true components means that due to a bc like (4.12), $\psi^{\rm LRT}_{\alpha(0)}(\alpha 0)$ must uniquely yield the antisymmetrized elastic scattering amplitude: none of the $\psi^{\rm LRT}_{\beta(0)}(\alpha 0)$, $\beta\neq\alpha$, can contribute to elastic scattering. Hence,

$$\langle\phi_{\alpha(0)0}|\psi^{\rm LRT}_{\alpha(0)}(\alpha 0)\rangle=|\chi^{\rm LRT}_{\vec{k}_\alpha}\rangle$$

must define a one-body scattering state obeying an equation with an optical potential which is nonabsorptive for incident energies below the inelastic threshold when there are no exoergic rearrangement channels. Furthermore, the LRT-formulated optical potential must be free of elastic scattering singularities since *all* the elastic flux is necessarily contained solely in $|\chi^{\rm LRT}_{\vec{k}_\alpha}\rangle$. Thus, despite the presence of the Green's functions in $\underline{\tilde{V}}^{\rm LRT}_0$, the corresponding optical potential enjoys the three properties listed in Sec. IV A, e.g., $\tilde{U}^{\rm LRT}_{\alpha(0)\alpha(0)}$ does not contain $G^{(+)}_{\alpha(0)}(E)$.

The CPA components are more complicated than those of LRT because of their non-label-transformability. Lack of this property means that the construction of Eqs. (2.4) and (3.3) does not lead in any direct way to $\psi^{\rm CPA}_{b(j)}(\alpha n)$ obeying (3.5). Nevertheless, for those $\psi^{\rm CPA}_{b(j)}(\alpha(i)n) = \Psi(\alpha(i)n)$,[26] application of (2.4) necessarily leads to $\psi^{\rm CPA}_{b(j)}(\alpha n)=\Psi^A(\alpha n)$. Hence lack of label transformability only means that for these particular CPA components, the antisymmetry is not made directly manifest through the CPA equations. That antisymmetry nevertheless results suggests that for the CPA true components, (2.4) and (3.3) will produce $\psi^{\rm CPA}_{b(j)}(\alpha n)$ such that $\sum_j \psi^{\rm CPA}_{b(j)}(\alpha n)$ *is* antisymmetric, i.e., that the non-label-transforming equations are only a temporary and not an ultimate obstacle to proving antisymmetry. We prove this in Appendix D for the $N=3$ case using a comparison method, and we conjecture that for arbitrary N, $\sum_j \psi^{\rm CPA}_{b(j)}(\alpha n)$ is also antisymmetric although we are as yet only able to offer plausibility arguments to support this conclusion. If one accepts this conclusion, then, as discussed in Appendix D, the major effect of the non-label transformability in the CPA true component case is that the set of $N_2=2^{N-1}-1$ equations defining either the $\psi^{\rm CPA}_{b(j)}(\alpha n)$ or the relevant transition operators cannot be reduced to a smaller set indexed only by the canonical labels, except by iteration.[27] [The effect of iteration is to produce a set of equations for the $\psi^{\rm CPA}_{b(0)}(\alpha n)$ in which $\underline{\tilde{V}}_0$ contains Green functions.] But, because these are true components, then, *just as in the LRT case,* when the incident energy is less than the first inelastic threshold and no exoergic rearrangement channels are open, the CPA-formulated $\mathscr{V}^A_\alpha$ will be real.

The foregoing comments are rigorous even though they are stated descriptively. They yield the result that our formulation of the antisymmetric optical potential applies to more N-particle collision theories than those cited in Sec. IV B. This is a consequence of working with a wave function formulation and is thus one of the advantages of such a formulation.

C. Comparison with other formulations

It is easy to realize that the present approach to the optical potential can easily encompass the case of distinguishable particles. In the distinguishable particle case, the optical potential of the present approach is very different than the Feshbach optical potential.[1] For example, and in contrast to the Feshbach potential, the (real) folding potential in the present approach is not contained in the lowest-order approximation [see the remarks following Eq. (4.13)]. The present approach is also distinct from that of Kowalski and collaborators[3,6,18] in that, e.g., in the absence of particle identity, their approach does reduce to the Feshbach potential. In the following, we compare the present approach more closely with those of Refs. 1 and 3.

In the earlier approaches to an antisymmetric optical potential,[1] a projection operator Θ was sought such that

$$\Theta\Psi^A(\alpha n)=A_{\alpha(0)}(\phi_{\alpha(0)0}u_\alpha) .$$

The function u_α asymptotically yields the elastic amplitude and is thus an elastic scattering wave function. The equation determining u_α involves a nonlocal operator K^F, made up of terms such as

$$N_{0i}=(\phi_{\alpha(0)0}\,|\,\phi_{\alpha(i)0}) ,$$

where the notation (|) means integration over all common variables. The nonorthogonality overlap N_{0i} is the same quantity as appears in the coupled reaction channel[28] and resonating group method[29] approaches to nuclear reactions. Connected with its appearance may be problems associated with eigenvalues of the kernel K^F that are equal to unity.[28,29] In our method for constructing $\mathscr{V}^A_\alpha$, no such problems occur, since we determine $\psi_{\alpha(0)}(\alpha n)$ first and then obtain $|\chi^{(+)}_{\vec{k}_\alpha}\rangle$ from it by projection. Correspondingly, full antisymmetrization of $\psi_{\alpha(0)}(\alpha n)$ is unnecessary in order to obtain either an elastic scattering wave function or the antisymmetric, elastic scattering amplitude, since both

$$\langle\phi_{\alpha(0)0}\,|\,\psi_{\alpha(0)}(\alpha n)\rangle = |\chi^{(+)}_{\vec{k}_\alpha}\rangle$$

and

$$\langle\phi_{\alpha(0)0}\,|\,A_{\alpha(0)}\psi_{\alpha(0)}(\alpha n)\rangle$$

yield identical elastic scattering amplitudes.

In the later approaches to the antisymmetric optical potential based on transition operators,[3] full antisymmetry is also introduced at the beginning, but the need for a formulation free of elastic scattering singularities has been stressed. In their analysis of this problem, Kowalski and Picklesimer proved[18] that a formulation based on the transition operator known as the AGS off-shell extension[12] led to a Hermitian-analytic optial potential that was free of such singularities. Because they limited themselves to Hermitian analytic potentials, they selected this formulation as the desirable one to use in discussing the antisymmetric optical potential. By basing our analysis on wave function components with known (and desirable) properties, we have shown herein that other transition operators can be used to formulate optical potentials, albeit asymmetric ones. Thus, in addition to the AGS operator, noted in Sec. IV B 2 and discussed in Appendix C, both the prior (EF) and post (BRS precursor) forms lead to an optical potential with the requisite properties. Other wave function component formulations are allowed as well.

The asymmetry of our class of optical potentials is a reflection of the fact that our construction is based directly on connected-kernel, multichannel scattering theories, and generally in such theories, connectivity is achieved by means of a non-Hermitian, operator-matrix formulation.[4,30] While asymmetry for $E<E_{\rm inel}$ (and lack of Hermitian analyticity for $E\geq E_{\rm inel}$) is an unusual feature, it is not in our opinion of any practical significance. First, by construction the present non-Hermitian-analytic potentials yield the exact, symmetrized transition amplitudes. Second, non-Hermiticity in a connected-kernel formalism has never been a *bar* to successful calculations, exact or approximate;[4] the simplest examples of this are all of the analyses based on the three-particle Faddeev equations.[31] Indeed, if we compare the way exchange effects enter the driving terms of $\tilde{U}_\alpha$ of Eq. (4.3) and the analogous term of the construction of Ref. 3, we see that in the former case exchange effects occur linearly while in the latter case

they occur nonlinearly. This may well allow for simpler approximate calculations using the present formalism (see further comments below). As a third point mitigating against alarm concerning non-Hermiticity for $E<E_{\text{inel}}$, we note that in the opposite limit of $E\gg E_{\text{inel}}$, where a "$t\rho$" type of approximation is expected and can be practical, the transition operator based on the EF theory has been shown[17] to have the same form as obtained from an antisymmetrized spectator expansion.[32] The non-Hermiticity of the present class of optical potentials, arising from the asymmetry of the formalism, is different from that introduced by the energy averaging of theoretical phase shifts and strength functions,[1] since the latter leads to absorption but not symmetry. After an energy average is performed in the present case, one will still have an asymmetric (but not absorptive) potential. However, we note that elastic scattering data are typically fitted through use of empirical potentials which are local in coordinate space. That is, local potential phase shifts can fit data, independent of the structure of the underlying optical potential formalism. Approximating $\mathscr{V}_\alpha^A$ by a local potential will evidently give agreement with data. In addition, use of this approximation will render the asymmetry of $\mathscr{V}_\alpha^A$ an invisible feature in elastic scattering analyses. Replacing $\mathscr{V}_\alpha^A$ and $|\chi_{\vec{k}_\alpha}^{(+)}\rangle$ by local equivalents, however, may not be as straightforward in nuclear reaction applications, as we discuss elsewhere.[17]

D. Approximations

As the preceding discussion indicates, the exact results are straightforward to deal with. The situation is somewhat less clear when one makes approximations, a topic we examine next. There are several points we consider; our only limitation will be to those approximations for which the connectivity property of Eqs. (3.15) or (3.32) holds [this is consistent with requirement (iii) of Sec. IV A]. For simplicity we again assume that all reaction channel thresholds lie above E_{inel} and that H contains no absorptive interactions.

For exact solutions, $f^A_{\alpha(0)0,\alpha(0)0}$ is necessarily unitary, as already noted. Suppose $E<E_{\text{inel}}$ and that for simplicity, $\mathscr{V}_\alpha^A$ is spherically symmetric in a coordinate representation, i.e.,

$$\langle \vec{r}'_\alpha | \mathscr{V}_\alpha^A | \vec{r}_\alpha \rangle = \langle r'_\alpha | \mathscr{V}_\alpha^A | r_\alpha \rangle \ .$$

Then each partial wave radial function

$$\chi_l(k_\alpha r_\alpha)=\langle r_\alpha lm | \chi_{\vec{k}_\alpha}^{(+)} \rangle$$

will be asymptotic to the unitary form

$$\sin(k_\alpha r_\alpha - l\pi/2+\delta_l^{(\alpha)})/k_\alpha r_\alpha \ ,$$

where $\delta_l^{(\alpha)}$ is the (exact) phase shift. Next, let us introduce an approximation; it will yield the approximate quantities $\hat{\chi}_l(k_\alpha r_\alpha)$ and $\hat{\mathscr{V}}_\alpha^A$, and we again assume that $\hat{\mathscr{V}}_\alpha^A$ is spherically symmetric. If the exact components are true,[20] and if the connectivity property is maintained in the approximated equations, then $|\hat{\chi}_{\vec{k}_\alpha}^{(+)}\rangle$ will necessarily yield the entire, antisymmetrized, approximate elastic scattering amplitude (which will be unitary since no channels other than the elastic one are assumed to be open) and the approximate radial function will again have a unitary form asymptotically, i.e.,

$$\hat{\chi}_l(k_\alpha r_\alpha)\sim\sin(k_\alpha r_\alpha - l\pi/2+\hat{\delta}_l^{(\alpha)})/k_\alpha r_\alpha \ .$$

These remarks follow from the fact that the true component property depends only on the connectivity structure of the defining equations.[20] An approximation for which these remarks hold, in both the EF- and BRS-precursor cases, is $G_{b(j)}^{(+)}(E)=0$, $b\neq$two-cluster channel, and $G_{\gamma(j)}^{(+)}(E)\to P_{\gamma(j)}G_{\gamma(j)}^{(+)}(E)$, where $P_{\gamma(j)}$ projects onto internal bound states in the retained two-cluster channels γ. One can also devise coupled channel approximations in the EF theory which include m-cluster breakup partitions for which the unitary behavior of $\hat{\chi}_l(k_\alpha r_\alpha)$ will continue to obtain.[17] For $E\geq E_{\text{inel}}$, flux is lost out of the incident channel and the unitary form will not occur. However, one can still introduce approximations for which only $|\hat{\chi}_{\vec{k}_\alpha}^{(+)}\rangle$ yields the elastic part of the amplitude.

Next, let us consider the fact that because our class contains more than one member, the introduction of the same mathematical approximation will yield different approximate (asymmetric) $\hat{\mathscr{V}}_\alpha^A$, none of which will be equal to the approximate GKP potential. This raises two questions: First, which set of phase shifts should one use? Second, should one not be disturbed that the same physical approximation leads to different phase shifts? The first question is one that arises *whenever* approximations are used in multiparticle collision theories; a specific answer to it is unknown. A general answer is that the choice depends on such factors as ease of calculation, physical insight, reliability of the model, structure of the formalism, etc. Because exchange effects enter the present class of optical potentials so simply, one or another of them could be preferable to the GKP potential. On the other hand, since it is necessary to consider second-order terms in order to obtain the full (direct plus exchange) Born amplitudes in the EF, BRS-presursor, or CPA cases, it might be preferable to use the GKP potential when making approximations. Clearly, one must have information on the system in order to help decide. As for the problem of the same physical approximation producing formalism-dependent results, it should be obvious that this is a spurious objection: a mathematical approximation is not a physical approximation. The latter occurs only when the former is combined with a particular formalism in which it is to be used. Considered in this way, the second question is simply a variant of the first one, commented on already. We also recall that once approximations are introduced into an asymmetric formalism, then the results need not be time-reversal invariant. References discussing this feature of the theory are given by Kowalski and Levin.[4] The comments on approximations, made just above, clearly apply to this case as well.

The aim of our discussion has been to show that the new class of asymmetric, antisymmetrized optical potentials are a valid and interesting means of describing elastic scattering. However, the scope of an antisymmetrized,

multiparticle collision theory goes well beyond a description of elastic scattering. As noted, the EF theory is being used as a basis for describing nuclear reactions.[17] In its unsymmetrized form, it has been shown by Birse and Redish[33] to be a means of converting the CRC ansatz[28] into a convergeable theory; the symmetrized version has been similarly used by Adhikari, Birse, Kozack, and Levin[34] to embed the RGM ansatz.[29] The symmetrized EF theory has also been used[17] to solve the old, previously unsolved problem of the role of Pauli-principle exchange effects in the distorted wave Born approximation,[35] to derive the standard model of (d,d) and (d,np) processes,[36] and to formulate antisymmetrized, coupled-channel models of direct nuclear reactions, including breakup. This widespread set of applications of the EF theory is, we feel, reason enough to warrant the investigation of non-Hermitian, multiparticle collision theories and their associated asymmetric optical potentials.

VI. SUMMARY

We have achieved two major objectives in this paper. First, a general procedure has been developed for antisymmetrizing wave function descriptions of many-particle scattering, e.g., the EF, LRT, precursor BRS, AGS, and CPA formulations, as discussed in the preceding. Second, this antisymmetrized wave function approach has been used to derive a class of antisymmetrized optical potentials which has a number of features different from the optical potentials discussed in the recent literature. We note again that the development in this paper carries over to the case of identical bosons simply by setting all the phase factors $(-1)^{\sigma_{\alpha(i)}}$ equal to unity.

ACKNOWLEDGMENTS

We express our thanks to Gy. Bencze, M. C. Birse, W. Glöckle, Z. C. Kuruoglu, E. F. Redish, and especially K. L. Kowalski for useful conversations and/or correspondence. F.S.L. is happy to acknowlege the support from the Bank of Interamerican Development that helped to make his visit to Universidade Federal de Pernambuco so pleasant. It is a pleasure to acknowledge financial support from the U.S. Department of Energy (DOE), the Conselho Nacional de Pesquisas (CNPq) of Brasil, and the Financiadora de Estudos e Projetos (FINEP) of Brasil.

APPENDIX A: ANTISYMMETRIZED LS EQUATIONS

Since the validity of Eq. (2.6) for $\Psi^A(\alpha n)$ has been questioned, we established it here using the Møller operator approach of Sandhas.[9] For partition $\beta(j)$ the Møller operator is $\Omega_{\beta(j)}$, whose action on $\Psi(\alpha(i)n)$ is

$$\Omega^{\dagger}_{\beta(j)}\Psi(\alpha(i)n)=\delta_{\beta(j)\alpha(i)}\Phi_{\alpha(i)n}, \text{ all } \beta(j) . \tag{A1}$$

Equation (A1) is a set of $N_2=2^{N-1}-1$ equations which uniquely define $\Psi(\alpha(i)n)$. The equivalent definition is

$$\Psi(\alpha(i)n)=\delta_{\beta(j)\alpha(i)}\Phi_{\alpha(i)n}+G^{(+)}_{\beta(j)}(E)V^{\beta(j)}\Psi(\alpha(i)n) , \tag{A2}$$

which follows from[11]

$$\Psi(\alpha(i)n)=\delta_{\beta(j)\alpha(i)}\Phi_{\alpha(i)n}+(1-\Omega^{\dagger}_{\beta(j)})\Psi(\alpha(i)n) . \tag{A3}$$

There are $N_\alpha+1$ sets of equations (A1)—(A3), each set having N_2 members. Multiply both sides of each equation in these sets by $(-1)^{\sigma_{\alpha(i)}}\hat{N}_\alpha^{-1/2}$, under the assumption that the particles are all identical fermions and that the bound states in $\Phi_{\alpha(i)n}$ are each antisymmetric. The resulting $N_\alpha+1$ sets of equations can now be added, yielding, e.g.,

$$\hat{N}_\alpha^{-1/2}\Omega^{\dagger}_{\beta(j)}\sum_{i=0}^{N_\alpha}(-1)^{\sigma_{\alpha(i)}}\Psi(\alpha(i)n)=\sum_{i=0}^{N_\alpha}\delta_{\beta(j)\alpha(i)}(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}/\hat{N}_\alpha^{1/2}, \text{ all } \beta(j) . \tag{A4}$$

The i sum on the left-hand side of (A4) is just

$$\Psi^A(\alpha n)=\sum_i(-1)^{\sigma_{\alpha(i)}}\Psi(\alpha(i)n)/\hat{N}_\alpha^{1/2}\equiv A_{\alpha(0)}\Psi(\alpha(0)n) ,$$

so that (A4) becomes

$$\Omega^{\dagger}_{\beta(j)}\Psi^A(\alpha n)=\sum_i\delta_{\beta(j)\alpha(i)}(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}/\hat{N}_\alpha^{1/2}, \text{ all } \beta(j) . \tag{A5}$$

On following the steps leading from (A1) to (A2),[11] (A5) yields

$$\Psi^A(\alpha n)=\sum_i\delta_{\beta(j)\alpha(i)}(-1)^{\sigma_{\alpha(i)}}\Phi_{\alpha(i)n}/\hat{N}_\alpha^{1/2}+G^{(+)}_{\beta(j)}(E)V^{\beta(j)}\Psi^A(\alpha n), \text{ all } \beta(j) , \tag{A6}$$

which is just Eq. (2.6). The derivation of Sec. II was based on Eq. (A2); we see that the two derivations are equivalent. The key point is that the antisymmetry is independent of the number of equations indexed by $\beta(j)$. It requires only that the incident wave be antisymmetrized.

APPENDIX B: ANTISYMMETRIZED TRANSITION OPERATORS

When b is a two-cluster channel, say β, then either from the asymptotic form of $\langle \vec{r}_{\beta(0)}\phi_{\beta(0)m} \mid \psi_{\beta(0)}(\alpha n)\rangle$ or from

$\langle \Phi_{\beta(0)m} \mid \psi_{\beta(0)}(\alpha n)\rangle$ one obtains an antisymmetrized transition amplitude which is the matrix element of a transition operator we shall denote $T^A_{\beta(0)\alpha(0)}$ (energy dependence is suppressed). That is, we have

$$A^A_{\beta(0)m,\alpha(0)n} = \langle \Phi_{\beta(0)m} \mid T^A_{\beta(0)\alpha(0)} \mid \Phi_{\alpha(0)n}\rangle \ .$$

When b is an m-cluster channel, $m \geq 3$, on-shell matrix elements $\langle \Phi_{b(0)m} \mid T^A_{b(0)\alpha(0)} \mid \Phi_{\alpha(0)n}\rangle$ define either all or portions of the relevant transition amplitudes. From Eq. (3.12), it follows that the $T^A_{b(0)\alpha(0)}$ obey

$$T^A_{b(0)\alpha(0)} = \tilde{V}_{b(0)\alpha(0)} + \sum_d \tilde{V}_{b(0)d(0)} G^{(+)}_{d(0)}(E) T^A_{d(0)\alpha(0)} \ . \tag{B1}$$

Equation (3.12) follows from (B1) through the relation

$$T^A_{d(0)\alpha(0)} \mid \Phi_{\alpha(0)n}\rangle = \sum_b \tilde{V}_{d(0)b(0)} \mid \psi_{b(0)}(\alpha n)\rangle \ . \tag{B2}$$

Bencze and Redish, in their study of symmetrized transition operators for systems of identical particles,[10] have shown that the properly symmetrized amplitudes are given (in our notation) as the on-shell matrix elements $\langle \Phi_{b(0)m} \mid T^{b(0)\alpha(0)} \mid \Phi_{\alpha(0)n}\rangle$ of class operators $T^{b(0)\alpha(0)}$, which obey

$$T^{b(0)\alpha(0)} = I^{b(0)\alpha(0)} + \sum_d K^{b(0)d(0)} T^{d(0)\alpha(0)} \ . \tag{B3}$$

In terms of the $V_{b(j)d(k)}$ and $G^{(+)}_{d(0)}(E)$ appearing in (B1), the quantities in (B3) are

$$I^{b(0)d(0)} = \left[\frac{\hat{N}_d}{\hat{N}_b}\right]^{1/2} \sum_j R_{b(0)} P_{b(0)b(j)} (-1)^{\sigma_{b(j)}} V_{b(j)d(0)} \tag{B4}$$

and

$$K^{b(0)d(0)} = I^{b(0)d(0)} G^{(+)}_{d(0)}(E) \ , \tag{B5}$$

while our $\tilde{V}_{b(0)d(0)}$ is given by

$$\tilde{V}_{b(0)d(0)} = \left[\frac{\hat{N}_b}{\hat{N}_d}\right]^{1/2} \sum_k R_{b(0)} V_{b(0)d(k)} (-1)^{\sigma_{d(k)}} P_{d(k)d(0)} \ . \tag{3.13}$$

We shall now show that $\tilde{V}_{b(0)d(0)}$ and $I^{b(0)d(0)}$ of (B4) are identical. Use of (B5) and the standard assumption that (B1) and (B3) have unique solutions then yields $T^A_{b(0)\alpha(0)} = T^{b(0)\alpha(0)}$, which establishes the claim made below Eq. (3.13) that Eq. (3.12) leads to the properly normalized amplitudes.

We begin by recalling the proposition in Sec. III of Ref. 10, which in our notation is

$$I^{b(0)d(0)} = I^{b(0)d(k)} P_{\alpha(k)d(0)} \ .$$

Applying this $\hat{N}_d$ times to (B4) gives

$$I^{b(0)d(0)} = (\hat{N}_b \hat{N}_d)^{-1/2} \sum_{j,k} R_{b(0)} P_{b(0)b(j)} (-1)^{\sigma_{b(j)}} \times V_{b(j)d(k)} P_{d(k)d(0)} \ . \tag{B6}$$

Let us now isolate one member of the j sum in (B6). Using exactly the arguments employed in going from Eq. (3.24) to Eq. (3.25) now gives for (B6)

$$I^{b(0)d(0)} = (\hat{N}_b \hat{N}_d)^{-1/2} \sum_{j,k} R_{b(0)} V_{b(0)d(k)} (-1)^{\sigma_{d(k)}} P_{d(k)d(0)} \ . \tag{B7}$$

The j sum in (B7) yields $\hat{N}_b$ so that the RHS of (B7) is seen to be identical to that of $\tilde{V}_{b(0)d(0)}$, thus establishing the desired result.

APPENDIX C: THE AGS TRANSITION OPERATOR

The AGS transition operator U_{ba}, defined via[12]

$$U_{ba} = \bar{\delta}_{ba} G_a^{-1} + V^b G G_a^{-1} \ , \tag{C1}$$

is related to the post form of the transition operator $U^{(+)}_{ba}$ via $U_{ba} = \bar{\delta}_{ba} G_a^{-1} + U^{(+)}_{ba}$. The BRS-precursor components are obtained by writing an integral equation for $U^{(+)}_{ba}$ by expanding V^b via the BLRT distribution technique[22] and then using a relation like Eq. (B2) to define the components. Since $U_{ba} \mid \Phi_a\rangle = U^{(+)}_{ba} \mid \Phi_a\rangle$, the components obtained from the AGS operator are identical to the BRS-precursor components, as noted in Sec. IV B 2.

APPENDIX D: ANTISYMMETRIZED CPA EQUATIONS

It was noted in Sec. V B that while the construction of (2.4) and (3.3) does not yield CPA components whose sum on j is manifestly antisymmetric, it does lead both to antisymmetric transition operators obeying noniterated, connected-kernel equations and in the $N=3$ case to wave function components whose sum is antisymmetric even though they do not obey (3.5). We discuss these points in the present appendix, beginning with the $N=3$ components.

Our notation for the $N=3$ case is as follows. The state vector for the canonically labeled incident wave in partition $1=(1)(23)$ will be denoted $\mid \Phi_1\rangle = \mid \phi_1(23)\rangle \mid \vec{k}(1)\rangle$, where we have explicitly displayed particle labels. Partitions 2 and 3, defined, respectively, as (2)(13) and (3)(21), may be obtained from (1)(23) by application to them of the two-particle transposition operators P_{12} and P_{13}. The partition labels 1, 2, and 3 will be used in place of, e.g., $\alpha(0)$, $\alpha(1)$, and $\alpha(2)$. The labels 1, 2, and 3 are, of course, the usual pair indices for the three-particle problem.

There are a variety of different CPA wave function components and corresponding transition operators, each obeying a different set of three coupled equations. We shall concentrate here on those (true) CPA wave function components and transition operators denoted $\hat{\psi}_j$ and $\hat{T}_{jk}$ (for details, see Ref. 7). To generate the relevant $N=3$ CPA equations, we use a channel coupling array $\underline{W}^{(1)}$ of the form[7,13]

$$\underline{W}^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} .$$

When the incident wave is Φ_1, we have the following equations[7,37] for the CPA wave function components $\hat{\psi}_j(1)$:

$$\begin{aligned}\hat{\psi}_1(1)&=\Phi_1+G_1^{(+)}V^2\hat{\psi}_2(1)\,,\\ \hat{\psi}_2(1)&=G_2^{(+)}V^3\hat{\psi}_3(1)\,,\\ \hat{\psi}_3(1)&=G_3^{(+)}V^1\hat{\psi}_1^3(1)\,,\end{aligned}\tag{D1}$$

where the E dependence in $G_j^{(+)}$ and the n dependence of Φ_{1n} is suppressed.

The sets of equations for $\hat{\psi}_j(2)$ and $\hat{\psi}_j(3)$, needed to form

$$\hat{\psi}_j=\sum_{i=1}^{3}(-)^{\sigma_i}\hat{\psi}_j(i)\,,$$

σ_1=even, are generated from the incident waves Φ_2 and Φ_3. These sets are the following:

$$\begin{aligned}\hat{\psi}_1(2)&=G_1^{(+)}V^2\hat{\psi}_2(2)\,,\\ \hat{\psi}_2(2)&=\Phi_2+G_2^{(+)}V^3\hat{\psi}_3(2)\,,\\ \hat{\psi}_3(2)&=G_3^{(+)}V^1\hat{\psi}_1(2)\,,\end{aligned}\tag{D2}$$

and

$$\begin{aligned}\hat{\psi}_1(3)&=G_1^{(+)}V^2\hat{\psi}_2(3)\,,\\ \hat{\psi}_2(3)&=G_2^{(+)}V^3\hat{\psi}_3(3)\,,\\ \hat{\psi}_3(3)&=\Phi_3+G_3^{(+)}V^1\hat{\psi}_1(3)\,.\end{aligned}\tag{D3}$$

Equations (D2) and (D3) use the same $\underline{W}^{(1)}$ as (D1). The only difference is in the position of the incident wave. However, because $W_{jk}^{(1)}V^k$ is not label transforming, (D2) and (D3) cannot be obtained by applying interchange operators to the three equations of (D1), as noted in the text.

We now construct the components

$$\hat{\psi}_j=\hat{\psi}_j(1)+(-1)^{\sigma_2}\hat{\psi}_j(2)+(-1)^{\sigma_3}\hat{\psi}_j(3)\,,$$

as in Eq. (3.2). They obey

$$\begin{aligned}\hat{\psi}_1&=\Phi_1+G_1^{(+)}V^2\hat{\psi}_2\,,\\ \hat{\psi}_2&=(-1)^{\sigma_2}\Phi_2+G_2^{(+)}V^3\hat{\psi}_3\,,\\ \hat{\psi}_3&=(-1)^{\sigma_3}\Phi_3+G_3^{(+)}V^1\hat{\psi}_1\,,\end{aligned}\tag{D4}$$

which is a set of equations based on $\underline{W}^{(1)}$ but with an altered inhomogeneity as compared to the equations for the individual $\hat{\psi}_j(i)$. If we now form $\hat{\psi}=\sum_j\hat{\psi}_j$, it is not easily shown that this $\hat{\psi}$ is antisymmetric. This is a consequence of the fact that $W_{jk}^{(1)}V^k$ is not label transforming.

Although (3.5) does not hold in the case of these CPA components, the sum of the components, denoted $\hat{\psi}$ in the preceding paragraph, *is* nevertheless antisymmetric. Indeed, the construction used to derive (D4) is such—exactly as in the case of (3.2) and label transforming $\underline{V}$'s—as to lead one to *expect* $\hat{\psi}$ to be antisymmetric. The fact that P_{12} applied to (D2) or (D3) produces results not in the form of $\underline{W}^{(1)}$ equations does not mean that $P_{12}\hat{\psi}\neq-\hat{\psi}$, but only that this direct test for antisymmetry is inconclusive in the CPA case. Evidently one must use indirect means to establish the antisymmetry of $\hat{\psi}$. We shall do this by showing that the sum $\hat{\psi}$ in the CPA case is equal to the sum of the Faddeev wave function components, a sum which is *manifestly* antisymmetric.

We proceed as follows. The (unsymmetrized) Faddeev components $\psi_j^F(k)$ obey[31] ($j=1,2,3$)

$$\psi_j^F(k)=\Phi_k\delta_{jk}+G_j^{(+)}V_j\sum_m\bar{\delta}_{jm}\psi_m^F(k)\,,\tag{D5}$$

where $k=1$, 2, or 3 indicates the incident channel. On forming the components

$$\psi_j^F=\psi_j^F(1)+(-1)^{\sigma_2}\psi_j^F(2)+(-1)^{\sigma_3}\psi_j^F(3)\,,$$

we find that they satisfy

$$\begin{aligned}\psi_1^F&=\Phi_1+G_1^{(+)}V_1(\psi_2^F+\psi_3^F)\,,\\ \psi_2^F&=(-1)^{\sigma_2}\Phi_2+G_2^{(+)}V_2(\psi_1^F+\psi_3^F)\,,\\ \psi_3^F&=(-1)^{\sigma_3}\Phi_3+G_3^{(+)}V_3(\psi_1^F+\psi_2^F)\,.\end{aligned}\tag{D6}$$

Since the Faddeev equations are label transforming, then the ψ_j^F of (D6) obey (3.5), and hence $\psi^F=\sum_{j=1}^3\psi_j^F$ is antisymmetric (as is easily shown). In fact, from the property[31] $\sum_{j=1}^3\psi_j^F(k)=\Psi(k)$, where $\Psi(k)$ is the Schrödinger solution in the case where the distinguishable particle k is incident on a pair of identical particles, the construction (D6) leads to $\sum_{j=1}^3\psi_j^F=\Psi^A$, where Ψ^A is the antisymmetric Schrödinger solution describing collisions involving three identical particles.

To show that the CPA sum $\hat{\psi}$ is equal to ψ^F, we first note that (D6) can be rewritten in the form

$$\begin{bmatrix}1 & -G_1^{(+)}V_1 & -G_1^{(+)}V_1\\ -G_2^{(+)}V_2 & 1 & -G_2^{(+)}V_2\\ -G_3^{(+)}V_3 & -G_3^{(+)}V_3 & 1\end{bmatrix}\begin{bmatrix}\psi_1^F\\ \psi_2^F\\ \psi_3^F\end{bmatrix}=\begin{bmatrix}\Phi_1\\ (-1)^{\sigma_2}\Phi_2\\ (-1)^{\sigma_3}\Phi_3\end{bmatrix}.\tag{D7}$$

We shall denote the 3×3 matrix in (D7) by the symbol $\underline{M}^F$, the Faddeev multiplier.[38]

Next, we express the ψ_j^F of (D6) through

$$\begin{bmatrix}\hat{\psi}_1\\ \hat{\psi}_2\\ \hat{\psi}_3\end{bmatrix}=\begin{bmatrix}\psi_1^F\\ \psi_2^F\\ \psi_3^F\end{bmatrix}+\begin{bmatrix}0 & G_0^{(+)}V_3 & -G_0^{(+)}V_1\\ -G_0^{(+)}V_2 & 0 & G_0^{(+)}V_1\\ G_0^{(+)}V_2 & -G_0^{(+)}V_3 & 0\end{bmatrix}\begin{bmatrix}\hat{\psi}_1\\ \hat{\psi}_2\\ \hat{\psi}_3\end{bmatrix};\tag{D8}$$

(D8) yields (D4) when acted on by $\underline{M}^F$.

The antisymmetry of $\hat{\psi}=\sum_{j=1}^3\hat{\psi}_j$ follows from Eq. (D8): summing both sides of (D8) leads to $\hat{\psi}=\psi^F=\Psi^A$, since the kernel term in (D8) makes no contribution to the sum. Hence, $\hat{\psi}$ *is* antisymmetric, even though the $\hat{\psi}_j$ do not obey (3.5). We note two additional points in connec-

tion with (D8). First, the presence of $G_0^{(+)}$ in (D8) means that in the limit $r_j \to \infty$, we have $\langle r_j \to \infty, \phi_j \mid \hat{\psi}_j \rangle = \langle r_j \to \infty, \phi_j \mid \psi_j^F \rangle$, i.e., $\hat{\psi}_j$ and ψ_j^F yield the same $2 \to 2$ transition amplitudes. Since the ψ_j^F obey (3.5), this means that at least in the asymptotic two-body sector, the $\hat{\psi}_j$ also obey (3.5). Put alternatively, each $\hat{\psi}_j$, $j=1,2,3$ yields the *same*, antisymmetrized, two-body transition amplitude [to within the phase factor $(-1)^{\sigma_j}$]. Second, and again due to the $G_0^{(+)}$ in (D8), we see that the contributions to the breakup amplitudes from $\hat{\psi}_j$ and from ψ_j^F are not the same, although in view of the relation $\hat{\psi} = \psi^F$, the *total* CPA and Faddeev breakup amplitudes are the same.

Although the latter two results are not unimportant, the main point of the preceding analysis is the antisymmetry of $\sum_j \hat{\psi}_j$. This is the $N=3$ equivalent of the following property for arbitrary N: $\sum_{\beta,j} \psi_{\beta(j)}(\alpha n)$ is antisymmetric, where $\psi_{\beta(j)}(\alpha n)$ means a two-cluster-labeled, N-particle CPA generalization of the components obeying (D4). But the antisymmetry of the latter sum *always* holds in the CPA case, as we show below, and we conjecture that this is true for any non-label-transforming theory. A more important conjecture is that $\sum_j \psi_{\beta(j)}(\alpha n)$ is also antisymmetric, whether the theory is label transforming or not. We shall first consider the CPA sum $\sum_{\beta j} \psi_{\beta(j)}(\alpha n)$.

There are two cases of interest: that of the so-called channel scattering states[7,39] in which $\psi_{\beta(j)}(\alpha(i)n)$ is independent of the index $\beta(j)$ and is equal to the Schrödinger wave function $\Psi(\alpha(i)n)$, and the case of true components, for which

$$\sum_{\beta,j} \psi_{\beta(j)}(\alpha(i)n) = \Psi(\alpha(i)n) \ .$$

In the former case, since

$$\psi_{\beta(j)}(\alpha(i)n) = \Psi(\alpha(i)n) \ ,$$

the antisymmetrization procedure is precisely that of Appendix A and automatically yields the result that $\sum_{\beta,j} \psi_{\beta(j)}(\alpha n)$ is antisymmetrized, as noted in Sec. V B. In the latter case, since the summations are finite, we may interchange the orders, carrying out the β,j sums first which yields $\Psi(\alpha(i)n)$, and then performing the sum on i in the antisymmetrization procedure yielding $\Psi^A(\alpha n)$. Combining these summations gives

$$\sum_{\beta,j} \psi_{\beta(j)}(\alpha n) = \Psi^A(\alpha n) \ ,$$

thus proving the antisymmetry of the double sum for the case of true CPA components. Similar results hold for the other CPA components discussed in Ref. 7. We expect in all cases that $\sum_{b,j} \psi_{b(j)}(\alpha n)$ will be antisymmetric.

Let us now consider our main conjecture, viz., that in *all* theories the sum $\sum_j \psi_{b(j)}(\alpha n)$ is antisymmetric. In view of the preceding remarks, this is true for the case of the $N=3$ and the channel scattering state CPA components. While we do not have a proof for any other non-label-transforming equations, the conjecture seems to us to be almost self-evident, on the ground that the distinction between generic channels b and d should mean that only the partition labels $\{b(j)\}_{j=0}^{N_b}$ are needed to form an antisymmetric combination: none of the partitions $\{d(k)\}_{k=0}^{N_d}$ should be needed to antisymmetrize quantities indexed by the $b(j)$. For example, if

$$b(0) = (1, \ldots, n_b)(n_b+1, \ldots, N)$$

and

$$d(0) = (1, \ldots, n_d)(n_d+1, \ldots, m_d)(m_d+1, \ldots, N) \ ,$$

then none of the $\{d(k)\}_{k=0}^{N_d}$ should be needed to ensure antisymmetry of $\sum_{j=0}^{N_b} \psi_{b(j)}(\alpha n)$, exactly as in the case of the incident wave $\phi_{\alpha(0)n}$. We are thus claiming that distinction of generic channel labels implies separate antisymmetry of each generally labelled component

$$\psi_b(\alpha n) = \sum_j \psi_{b(j)}(\alpha n) \ .$$

This is only reasonable, given first the expected antisymmetry of $\sum_b \psi_b(\alpha n)$, and second that for the label transforming case $\psi_b(\alpha n)$ has been proved to be antisymmetric.

We turn next to a consideration of the CPA transition operators. We discuss the $N=3$ case first. Although the $\hat{\psi}_j$ of (D4) do not obey (3.5), the amplitudes and transition operators associated with them are manifestly antisymmetrized, i.e., are those appropriate to a system of three identical fermions. We shall prove this after first discussing the antisymmetrized transition operators $\hat{T}_j^A$, obtained from the "distinguishable-particle" transition operators $\hat{T}_{jk}$, which we assume act on the partially symmetrized $\mid \Phi_k \rangle$.

The equations obeyed by the $\hat{T}_{jk}$ are (see, e.g., Ref. 7)

$$\hat{T}_{jk} = W_{jk} V^k + \sum_m W_{jm} V^m G_m^{(+)} \hat{T}_{mk} \ . \tag{D9}$$

We choose $\underline{W} = \underline{W}^{(1)}$ and then construct the three sets of equations for $\hat{T}_{jk} \mid \Phi_k \rangle$, $j=1,2,3$, obtained by allowing k to take on the values 1, 2, and 3. Next, we assume that the particles are identical fermions. As noted in Refs. 10 and 27 the properly antisymmetrized transition operator $\hat{T}_j^A$ to be used in this $N=3$ case is

$$T_j^A = \hat{T}_{j1} + (-1)^{\sigma_2} \hat{T}_{j2} P_{12} + (-1)^{\sigma_3} \hat{T}_{j3} P_{13} \ ;$$

it is to be applied to $\mid \Phi_1 \rangle$. These $\hat{T}_j^A$ are readily found to obey

$$\begin{aligned} \hat{T}_1^A &= (-1)^{\sigma_2} V^2 P_{12} + V^2 G_2^{(+)} \hat{T}_2^A \ , \\ \hat{T}_2^A &= (-1)^{\sigma_3} V^3 P_{13} + V^3 G_3^{(+)} \hat{T}_3^A \ , \\ \hat{T}_3^A &= V^1 + V^1 G_1^{(+)} \hat{T}_1^A \ , \end{aligned} \tag{D10}$$

which is a connected-kernel set of equations involving $\underline{W}^{(1)}$.

We see by this construction that imposition of the Pauli principle has not led to a set of equations for the antisymmetrized CPA transition operators any smaller than the set for the distinguishable-particle case. This is in contrast to the case of the label-transforming Faddeev equations, for which the imposition of identical particle symmetry directly effects a reduction to an $N_e \times N_e$ set (in this

$N=3$ case, $N_e=1$). In the general case of N identical particles and N_2 coupled equations, the analogs of (D4) for the $\hat{\psi}_j$ and of (D10) for the antisymmetrized operators $\hat{T}_j^A$ are also $N_2\times N_2$. Reduction from an $N_2\times N_2$ to an $N_e\times N_e$ set will occur in the general CPA case only if the Goldflam-Tobocman procedure of iterating the equations is followed.[27] Put in another way, the construction yielding (D10) is equivalent to first solving the unsymmetrized transition operator equations and then imposing the effects of identical-particle symmetry by forming the appropriate linear combination of the operators or amplitudes.

Now that ($N=3$) the $\hat{T}_j^A$ have been determined, we establish that these $\hat{T}_j^A$ or their appropriate matrix elements are indeed those associated with the components $\hat{\psi}_j$ of (D4). The generic form of the equations (D1)–(D3) for the $\hat{\psi}_j(i)$ is

$$\hat{\psi}_j(i)=\Phi_i\delta_{ji}+G_j^{(+)}\sum_m W_{jm}^{(1)}V^m\hat{\psi}_m(i)\ . \tag{D11}$$

On multiplying both sides of (D11) by $(-)^{\sigma_i}$ and then summing on i, (D11) becomes

$$\hat{\psi}_j=(-1)^{\sigma_j}\Phi_j+G_j^{(+)}\sum_m W_{jm}^{(1)}V^m\hat{\psi}_m\ . \tag{D12}$$

The amplitudes obtained from (D12) are easily seen to be the on-shell matrix elements of the operators $\hat{U}_j$ defined by

$$\hat{U}_j\,|\,\Phi_1\rangle=\sum_m W_{jm}^{(1)}V^m\hat{\psi}_m\ ,$$

so that $\hat{U}_j$ obeys

$$\hat{U}_j=\sum_m W_{jm}^{(1)}(-1)^{\sigma_m}V^mP_{1m}+\sum_m W_{jm}^{(1)}V^mG_m^{(+)}\hat{U}_m\ . \tag{D13}$$

On the other hand, the equations obeyed by the $\hat{T}_j^A$ of (D10) are obtained by choosing $\underline{W}=\underline{W}^{(1)}$ in (D9) and then summing both sides of (D9) on k after multiplying each term by $(-)^{\sigma_k}P_{1k}$. This yields

$$\hat{T}_j^A=\sum_k W_{jk}^{(1)}(-1)^{\sigma_k}V^kP_{1k}+\sum_m W_{jm}^{(1)}V^mG_m^{(+)}\hat{T}_m^A\ , \tag{D14}$$

whose detailed form is (D10). Since the connectivity of the iterated kernel implies that the solutions of (D13) and (D14) are each unique, comparison of these latter two equations shows that $\hat{U}_j=\hat{T}_j^A$, thus establishing our claim that the nonantisymmetric $\hat{\psi}_j$ yield the antisymmetrized $\hat{T}_j^A$.

It is straightforward to show that the preceding analysis holds for the $(N_2-1)!$ sets of CPA equations that one can generate in the case of an arbitrary number of particles.[7,13]

---

*Permanent address: Physics Department, Brown University, Providence, RI 02912.

[1]The classic works on optical potentials are those of H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958); 19, 287 (1962). The distinction between an optical potential and a complex potential well is made here; we shall use the two interchangeably in the present paper.

[2]A preliminary report of our analysis and results can be found in S. K. Adhikari, R. Kozack, and F. S. Levin, Phys. Lett. (in press).

[3]References to most of the early and recent theoretical research on the antisymmetric optical potential can be found in K.L. Kowalski, Phys. Rev. C 25, 700 (1982). See also Refs. 6 and 18 below.

[4]Reviews of and references to much of the literature on connected kernel theories of N-particle collisions can be found in K. L. Kowalski, in *Lecture Notes in Physics* (Springer, New York, 1978), Vol. 87, p. 343, and F. S. Levin, Nucl. Phys. A353, 143 (1980).

[5]Although this traditional meaning is the one intended by GKP, it is not explicitly stated by them, and for a long time we believed they were employing *our* meaning of the phrase "optical potential." This led us to an interesting misunderstanding of the GKP claim (Refs. 3 and 6) that only the (symmetric) AGS transition operator led to an optical potential which was real for $E<E_{\rm inel}$, since our asymmetric formulations also led to optical potentials enjoying this property (Ref. 2). These different uses of the phrase optical potential were finally clarified in a discussion between K. L. Kowalski and one of us (F.S.L.). Professor Kowalski has also pointed out to us his early awareness that one could obtain antisymmetrized optical potentials with the reality-asymmetry properties we have found. As noted in the text, he and his collaborators chose not to pursue this line of investigation. We are grateful to Prof. Kowalski for his comments and especially for stressing to us the importance of the asymmetry property.

[6]K. L. Kowalski, Phys. Rev. C 23, 597 (1981).

[7]Aspects of the wave function approach to N-particle collisions are reviewed by F. S. Levin, Phys. Rev. C 21, 2199 (1980).

[8]S. K. Adhikari and W. Glöckle, Phys. Rev. C 22, 309 (1980).

[9]W. Glöckle, Nucl. Phys. A141, 620 (1970); W. Sandhas, in *Few Body Nuclear Physics,* edited by G. Pisent *et al.* (IAEA, Vienna, 1978), p. 3; G. Cattapan and V. Vanzani, Phys. Rev. C 19, 1168 (1979).

[10]Gy. Bencze and E. F. Redish, J. Math. Phys. (N.Y.) 19, 1909 (1978).

[11]Our use of Møller operators is based on the discussion of W. Sandhas, Ref. 9.

[12]E. O. Alt, P. Grassberger, and W. Sandhas, Nucl. Phys. B2, 167 (1967).

[13]The literature on the channel coupling array (CCA) theory is extensive and we list only a few original papers and a recent article containing additional references: M. Baer and D. J. Kouri, J. Math. Phys. 14, 1637 (1973); D. J. Kouri and F. S. Levin, Phys. Lett. 50B, 421 (1974); W. Tobocman, Phys. Rev. C 9, 2466 (1974); F. S. Levin, Ref. 5.

[14]M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964).

[15]F. S. Levin, Ann. Phys. (N.Y.) 130, 139 (1980).

[16]Gy. Bencze, Phys. Lett. 72B, 1155 (1977).

[17]R. Kozack and F. S. Levin (unpublished).

[18]See, e.g., R. Goldflam and K. L. Kowalski, Phys. Rev. Lett.

44, 1044 (1980); K. L. Kowalski and A. Picklesimer, Phys. Lett. 95B, 1 (1980); Phys. Rev. Lett. 46, 228 (1981); Nucl. Phys. A369, 336 (1981); K. L. Kowalski (Ref. 3), and additional references cited therein.

[19]See, for example, R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966) for a discussion of the various transition operators.

[20]The concept of "true components" was introduced in Ref. 15. They are wave function components obeying both Eq. (3.9) and the boundary condition that the entire two-body amplitude for the process $\alpha(i)n \rightarrow \beta(j)m$ is contained in $\psi_{\beta(j)}(\alpha(i)n)$, i.e., that $\langle r_{\gamma(k)}\phi_{\gamma(k)m} \mid \psi_{\beta(j)}(\alpha(i)n)\rangle$ is zero when $r_{\gamma(k)} \rightarrow \infty$, $\gamma \neq \beta$, or $k \neq j$, or both.

[21]I. H. Sloan, Phys. Rev. C 6, 1945 (1972); Gy. Bencze, Nucl. Phys. A210, 568 (1973); E. F. Redish, *ibid.* A225, 16 (1974).

[22]P. Benoist-Gueutal, M. L'Huillier, E. F. Redish, and P. C. Tandy, Phys. Rev. C 17, 1924 (1978).

[23]K. L. Kowalski, Ann. Phys. (N.Y.) 120, 328 (1979).

[24]T. Fliessbach, Z. Phys. A 272, 39 (1975). See also W. Tobocman, Phys. Rev. C 17, 2205 (1978).

[25]M. L'Huillier, E. F. Redish, and P. C. Tandy, J. Math. Phys. 19, 1276 (1978).

[26]This property was discovered independently by D. J. Kouri and F. S. Levin, Nucl. Phys. A253, 395 (1975); and by W. Sandhas, in *Few Body Dynamics,* edited by A. N. Mitra *et al.* (North-Holland, Amsterdam, 1976), p. 540.

[27]R. Goldflam and W. Tobocman, Phys. Rev. C 17, 1914 (1978). K. L. Kowalski, Ref. 6, has introduced an alternative symmetrization procedure under which the CPA equations become disconnected.

[28]T. Ohmura, B. Imanishi, M. Ichimura, and M. Kawai, Prog. Theor. Phys. 41, 391 (1969); 43, 347 (1970); 44, 1272 (1970); W. R. Coker, T. Udagawa, and H. H. Wolter, Phys. Rev. C 7, 1154 (1973); S. R. Cotanch and C. M. Vincent, *ibid.* 14, 1739 (1976).

[29]J. A. Wheeler, Phys. Rev. 52, 1083 (1937); 52, 1107 (1937); K. Wildermuth and T. Kanellopoulos, Nucl. Phys. 7, 150 (1958); Y. C. Tang, M. LeMere, and D. R. Thompson, Phys. Rep. 47, 167 (1978).

[30]H. Krüger and F. S. Levin, Phys. Lett. 65B, 109 (1976).

[31]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys.—JETP 12, 1014 (1961)]; and *Mathematical Aspects of the Three Body Problem in the Quantum Scattering Theory* (Davey, New York, 1965).

[32]A. Picklesimer and R. M. Thaler, Phys. Rev. C 23, 42 (1981).

[33]M. C. Birse and E. F. Redish, Nucl. Phys. A406, 149 (1983).

[34]S. K. Adhikari, M. C. Birse, R. Kozack, and F. E. Levin (unpublished).

[35]W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford, London, 1961); N. Austern, *Direct Nuclear Reaction Theories* (Wiley, New York, 1970); F. S. Levin and H. Feshbach, *Reaction Dynamics* (Gordon and Breach, New York, 1973).

[36]See, e.g., R. C. Johnson and P. R. Soper, Phys. Rev. C 1, 976 (1970); H. Amakawa and N. Austern, *ibid.* 27, 299 (1983).

[37]Y. Hahn, D. J. Kouri, and F. S. Levin, Phys. Rev. C 10, 1620 (1974).

[38]S. K. Adhikari and W. Glöckle, Phys. Rev. C 19, 616 (1979).

[39]Channel scattering states obey $\psi_{b(j)}(\alpha(i)n) = \Psi(\alpha(i)n)$; see Refs. 7 and 26.