Boson expansions based on the random phase approximation representation

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(Received 14 October 1983)

A new boson expansion theory based on the random phase approximation is presented. The boson expansions are derived here directly in the random phase approximation representation with the help of a technique that combines the use of the Usui operator with that of a new bosonization procedure, called the term-by-term bosonization method. The present boson expansion theory is constructed by retaining a single collective quadrupole random phase approximation component, a truncation that allows for a perturbative treatment of the whole problem. Both Hermitian, as well as non-Hermitian boson expansions, valid for even nuclei, are obtained.

I. INTRODUCTION

Some ten years ago, Kishimoto and Tamura[1,2] reported on a formulation of the boson expansion theory (BET), to use for the analysis of nuclear collective motions. (These two papers will be henceforth referred to as KT-1 and KT-2.) The formalism was worked out, to a large extent, as a compact reformulation of earlier work by Sorensen,[3] but with a few important extensions and modifications. Sorensen's BET was derived for a fermion system which was constructed by using the Tamm-Dancoff approximation (TDA). We may thus call it a TDA-based BET. What was done in KT-1 was to reconstruct a similar BET in a more compact form, with an extension from the fourth to the sixth order. In KT-2, where a very practical version of BET was worked out, however, an important new step was added. It was to perform a random phase approximation (RPA) transformation upon the TDA-based BET, thus obtaining what may be called an RPA-based BET.

To perform this transformation was crucially important. (For a discussion of the RPA, and its superiority over the TDA, see e.g., Ref. 4.) As is well known,[4] TDA predicts too weak collectivity, which reflects, e.g., in too wide spacings of the calculated levels. This trouble was in fact experienced by Sorensen,[3] as well as by Lie and Holzwarth,[5] who also constructed a TDA-based BET by using the method of Marumori, Yamamura, and Tokunaga (MYT),[6] rather than the commutator method used in Refs. 1—3. On the other hand, such trouble was not encountered in our RPA-based BET. As shown in KT-2, and in a number of publications that followed,[7] we were able to fit spectra of a variety of collective nuclei, without making any fudging of the calculated spacings. See Ref. 8 for a summary of the numerical results.

The procedure of performing the TDA-to-RPA transformation adopted in KT-2 was, however, rather lengthy and complicated. It also contained in it a few steps, which could not be justified much beyond plausibility arguments. (See Secs. 3—5 and 7 of KT-2 for the details of this procedure.) Thus, although we believe even to date that the method used there was basically correct, we have also felt for a long time that it was desirable to derive an RPA-based BET directly, without going through the TDA-based BET. It is the purpose of this paper to present a method to solve this long standing problem.

Recently, Kishimoto and Tamura (KT-3) (Ref. 9) carried out a major reformulation of BET, with the purpose of putting our previous work[1,2,7,8] on a firmer basis, and also to prepare for their further extension. In doing this, we switched from the commutator method[1,2] to the MYT method,[6] and obtained results that were very general and flexible so that one could bosonize essentially any fermion system, so long as the latter was given in the TD representation (TDR).

More recently, Tamura[10] noticed that one of the key steps that characterized the MYT method could be avoided and replaced by a new procedure, called the term-by-term bosonization (TTB) method; as was shown in Ref. 10, the bosonization procedure, e.g., of KT-3, can be simplified significantly by the use of this method.

The essence of the TTB method is to recognize first that the fermion matrix elements of any operator have a certain tensorial structure, and can (in general) be reduced to a finite sum of irreducible tensors. Each term of this sum can, however, be easily replaced by the matrix element of a boson operator. If these operators are summed up, the result is nothing but the boson expanded form (boson image) of the original fermion operator. In other words, once the irreducible tensor reduction of a fermion matrix element is completed, the ensuing bosonization procedure can be readily carried out with the use of the TTB method.

When a very general TDR space is taken, as it was in KT-3, the above first step, i.e., the reduction of the fermion matrix elements to their irreducible forms, remains rather lengthy, and the TTB method does not help very much in simplifying it. However, if we truncate the TDR space so as to retain only one kind of collective component, matters change drastically. It is well known that, under the above truncation, the boson expansion converges very fast (see, e.g., KT-3), and thus the expansion can be terminated at a lower order. What was noticed in Ref. 10 was that, under the same truncation, the fermion matrix elements, notably the norm matrix elements, themselves can be expanded similarly, i.e., they can be treated perturbatively, thus making the procedure of the irreducible-tensor reduction very easy to carry out. This fact, combined with the use of the TTB method, makes the whole *procedure of the bosonization extremely simple and transparent.*

In Ref. 10 we have demonstrated this for the TDA case. In the present paper we apply the same procedure to solve the problem mentioned above, i.e., to carry out the direct (perturbative) bosonization of a fermion system in the RPA representation in the case when the sole quadrupole collective RPA component is retained.

We may remark at this stage that an earlier work by Almoney and Borse[11] is, in a sense, closely related to what we have done here. However, our work goes significantly beyond theirs, as will be explained after we present our formalism. There were also other authors who worked on constructing theories that went beyond the lowest RPA, i.e., the harmonic RPA. The earliest version of such an attempt was called higher RPA (HRPA).[12] The new versions[13,14] appear to have closer similarity to what we do here, at least in that the boson technique is used, one way or another. Nevertheless, their approach to the problem differs significantly from ours. Somewhat crudely speaking, it appears more appropriate to regard them as an improved version of HRPA, rather than as an RPA-based BET, as it is understood in the present paper.

We define first in Sec. II A the various fermion quantities, including the RPA pair operators, and then give the commutation relations they satisfy. We emphasize there the differences between these commutation relations and the corresponding ones that are satisfied by the TDA operators, pointing out the reasons why the direct construction of the RPA-based BET could be so much harder than that of the TDA-based BET. In Sec. II B, we then introduce the RPA product space, i.e., the space spanned by states constructed by multiplying the powers of the RPA pair operators upon the RPA vacuum, and show how to evaluate perturbatively their overlap integrals. In Sec. II C, the results of Sec. II B are used to construct, again perturbatively, orthonormal RPA states. As seen, the norm matrix cannot be constructed uniquely due to the presence of an unfixed parameter called z. However, it is shown in Secs. II C and III C that different choices of z simply specify different representations, all of which are related by unitary transformations, and all of which are thus physically equivalent. The bosonization, with the use of the TTB method, of the fermion system constructed in Sec. II is then carried out in Secs. III A and III B, respectively, for the Hermitian and non-Hermitian cases. As seen, the boson expansions obtained there are indeterminate due to the presence of the unfixed parameter z, originating from the fermion norm matrix. However, since the fermion representations are equivalent under a unitary transformation, so are the resulting boson expansions. This means that we have complete freedom in the choice of z. Such freedom is then used in Sec. III C in order to simplify the form of the boson expansions obtained in the preceding subsections. We also show there that, in the RPA-based form, Dyson's (non-Hermitian) BET has lost most of its attractive features, which it had in the TDA case, namely its finiteness and exactness. Furthermore, the non-Hermitian boson expansions turn out to be somewhat more complicated than the Hermitian expansions. In Sec. III D we discuss a few additional features pertaining to the RPA-based BET, and finally in Sec. IV we summarize the present paper.

II. FERMION DESCRIPTION

A. Definition of the fermion quantities

We start with a fermion system that is described in terms of the quasiparticles in the BCS theory. Denoting by $d_{jm}^\dagger$ and d_{jm} the quasiparticle creation and annihilation operators, we shall define the pair creation and scattering operators as

$$B^\dagger_{j_1j_2\lambda\mu}=\sum_{m_1m_2}(j_1m_1j_2m_2\,|\,\lambda\mu)d^\dagger_{j_1m_1}d^\dagger_{j_2m_2}\ , \tag{2.1a}$$

$$C^\dagger_{j_1j_2\lambda\mu}=\sum_{m_1m_2}(j_1m_1j_2m_2\,|\,\lambda\mu)d^\dagger_{j_1m_1}d_{j_2\tilde{m}_2}\ , \tag{2.1b}$$

where $(j_1m_1j_2m_2\,|\,\lambda\mu)$ is a Clebsch-Gordan coefficient, and

$$d_{j_2\tilde{m}_2}=(-)^{j_2-m_2}d_{j_2-m_2}\ .$$

The RPA operators may then be defined as

$$B^\dagger_{\alpha\lambda\mu}=\tfrac{1}{2}\sum_{j_1j_2}D_{j_1j_2}(\psi^{(\alpha)}_{j_1j_2\lambda}B^\dagger_{j_1j_2\lambda\mu}-\phi^{(\alpha)}_{j_1j_2\lambda}B_{j_1j_2\lambda\tilde{\mu}})\ ,$$
$$B_{\alpha\lambda\mu}=(B^\dagger_{\alpha\lambda\mu})^\dagger,\ \ D_{j_1j_2}=\sqrt{1+\delta_{j_1j_2}}\ . \tag{2.2}$$

In (2.2) ψ and ϕ are the RPA amplitudes that satisfy the usual orthonormal and completeness relations (see, e.g., Ref. 15).

For future convenience let us define here a few abbreviations; $a=\{\alpha\lambda\mu\}$ and $p=\{j_1j_2kq\}$. By using (2.1) and the properties of the RPA amplitudes, we can evaluate in a straightforward manner the commutation relations for the RPA operators. They are given as

$$[B_a,B_b^\dagger]=\delta_{ab}-\sum_p P^{(p-)}_{\tilde{a},b}C_p^\dagger\ , \tag{2.3a}$$

$$[C_p^\dagger,B_a^\dagger]=\sum_g P^{(p+)}_{\tilde{a},g}B_g^\dagger+\sum_g Q^{(p+)}_{\tilde{a},\tilde{g}}B_g\ , \tag{2.3b}$$

$$[B_a,C_p^\dagger]=\sum_g P^{(p+)}_{\tilde{g},a}B_g+\sum_g \tilde{Q}^{(p+)}_{\tilde{a},\tilde{g}}B_g^\dagger\ , \tag{2.3c}$$

$$[B_a^\dagger,B_b^\dagger]=\sum_p \tilde{Q}^{(p-)}_{\tilde{b},\tilde{a}}C_p^\dagger\ . \tag{2.3d}$$

The coefficients that appear in (2.3) are defined as

$$P^{(p\pm)}_{\tilde{a},b}=(\lambda_a\tilde{\mu}_a\lambda_b\mu_b\,|\,kq)\hat{P}^{(j_1j_2k\pm)}_{a,b}\ , \tag{2.4}$$

with

$$\hat{P}^{(j_1j_2k\pm)}_{a,b}=\hat{\lambda}_a\hat{\lambda}_b\sum_j[\psi^{(\alpha)}_{jj_2\lambda_a}\psi^{(\beta)}_{j_1j\lambda_b}\pm(-)^{j_1-j_2+k}P_{j_1j_2}\phi^{(\alpha)}_{jj_2\lambda_a}\phi^{(\beta)}_{j_1j\lambda_b}]W(j_2j_1\lambda_a\lambda_b;kj)D_{jj_1}D_{jj_2} \tag{2.5}$$

and

$$Q^{(p\pm)}_{\tilde{a},\tilde{b}}=(\lambda_a\tilde{\mu}_a\lambda_b\tilde{\mu}_b \mid kq)\hat{Q}^{(j_1j_2k\pm)}_{a,b} , \tag{2.6}$$

with

$$\hat{Q}^{(j_1j_2k\pm)}_{a,b}=\hat{\lambda}_a\hat{\lambda}_b\sum_j [\psi^{(\alpha)}_{jj_2\lambda_a}\phi^{(\beta)}_{j_1j\lambda_b}\pm(-)^{j_1-j_2+k}P_{j_1j_2}\phi^{(\alpha)}_{jj_2\lambda_a}\psi^{(\beta)}_{j_1j\lambda_b}]W(j_2j_1\lambda_a\lambda_b;kj)D_{jj_1}D_{jj_2} . \tag{2.7}$$

Also

$$\tilde{Q}^{(j_1j_2kq,\pm)}_{a,b}\equiv(-)^{j_1-j_2+k}Q^{(j_2j_1k\tilde{q},\pm)}_{a,b}, \text{ and } \hat{\lambda}=\sqrt{2\lambda+1} . \tag{2.8}$$

In (2.5) and (2.7) the operator $P_{j_1j_2}$ exchanges the indices j_1 and j_2 in the expression that follows it.

There are three features that we want to emphasize about the commutation relations (2.3): (i) the $[B,B^\dagger]$ commutator has the same structure as does the one obtained in the Tamm-Dancoff (TD) case; however, the coefficient $P^{(p-)}$ is somewhat more complicated; (ii) unlike in the TD case, the commutators $[C^\dagger,B^\dagger]$ and $[C^\dagger,B]$ consist of two terms, showing that the scattering operator is not diagonal in the RPA phonon numbers; (iii) the RPA creation operators $B^\dagger$ do not commute among themselves. Needless to say, if we set the RPA amplitudes $\phi=0$, then $Q^{(p\pm)}=0$, and (2.3) reduces to the commutation relations for the TD case.

As stressed in Sec. I, we consider, in the rest of the paper, the case in which the RPA system is truncated to retain only a single *collective component of the quadrupole nature*. Under such a truncation, the RPA index a stands only for the magnetic quantum number. Also, the quantities defined in (2.5) and (2.7) become somewhat simpler and can be rewritten as (where $\lambda=2$ and c stands for "collective")

$$\hat{P}^{(j_1j_2k\pm)}_{c,c}=\hat{\lambda}^2\sum_j [\psi_{jj_2}\psi_{j_1j}\pm(-)^k\phi_{jj_2}\phi_{j_1j}] \times W(j_2j_1\lambda\lambda;kj)D_{jj_1}D_{jj_2} , \tag{2.9a}$$

$$\hat{Q}^{(j_2j_2k\pm)}_{c,c}=\hat{\lambda}^2\sum_j \{\psi_{jj_2}\phi_{j_1j}[1\pm(-)^k]\} \times W(j_2j_1\lambda\lambda;kj)D_{jj_1}D_{jj_2} . \tag{2.9b}$$

From (2.9b) we see that the quantity $Q^{(p+)}_{a,b}$ ($Q^{(p-)}_{a,b}$), which appears in (2.6), is nonvanishing only for even (odd) k values, and it is symmetric (antisymmetric) under the interchange of the indices a and b (i.e., of μ_a and μ_b). Note that this property holds only when we have solely one RPA component of even multipolarity, as we do here.

B. The RPA product space and the overlap matrix

One of the major tasks in the formulation of the present paper is the construction of an orthonormal fermion basis, which will be carried out in the following subsection. Quantities necessary for this purpose are presented in this subsection.

As in the TD case,[9,10] we begin by introducing the basis states

$$| a;n \rangle=\frac{1}{\sqrt{n!}}(B^\dagger_a)^n \,| 0\rangle\equiv\frac{1}{\sqrt{n!}}B^\dagger_{a_1}B^\dagger_{a_2}\cdots B^\dagger_{a_n} \,| 0\rangle , \tag{2.10}$$

and call the space they span the RPA *product space*. In (2.10) we have introduced an abbreviation a to mean (when it appears in combination with the number n) that $a=\{a_1a_2,\ldots,a_n\}$. In the following we use this abbreviation consistently. We also use $\bar{a}_1\equiv\{a_2,\ldots,a_n\}$, $\bar{a}_2\equiv\{a_3,\ldots,a_n\}$, and so forth. We shall also use $b,\bar{b}_1,\bar{b}_2,\ldots,$ in the same way. On the other hand, $d,e,f,g,\ldots,$ will be used to denote a single index, rather than a set of them. (When $a_1,a_2,\ldots,$ appear without a bar, they also denote individual indices.)

The central problem for the construction of an orthonormal fermion basis is the evaluation of the overlap matrix with elements given by

$$(O^2_{mn})_{a;b}=\langle\langle m;a \mid b;n \rangle\rangle . \tag{2.11}$$

This matrix is nondiagonal, not only with respect to a and b, but also with respect to m and n. (In the TD case,[10] the corresponding matrix was diagonal in the pair number n.) This is due to the presence of the second terms in the right-hand side (rhs) of (2.3b) and (2.3c). These terms also force m and n to have the same parity.

In evaluating (2.11) explicitly, it is very important to note that $C^\dagger_p$ does not annihilate the RPA vacuum;

$$C^\dagger_p \,| 0\rangle\neq0, \text{ and } \langle 0 \,| C^\dagger_p\neq0 . \tag{2.12}$$

In the following we take this into account. Nevertheless, we set everywhere

$$\langle 0 \,| C^\dagger_p \,| 0\rangle=0 . \tag{2.13}$$

In Sec. III D we discuss the consequences of approximation (2.13), and show that, up to the order at which we choose to work, it is consistent with the perturbative treatment of the other terms in the theory.

By using (2.3) we can readily evaluate the overlap matrix element (2.11). The algebra is straightforward, but quite lengthy, and is given in Appendix A. The final result for $(O^2_{mn})_{a;b}$ can be written as

$$(O^2_{mn})_{a;b} \equiv \langle\langle m,a \mid b,n \rangle\rangle = \left[\Delta_{a,b} - \frac{2}{n(n-1)} \mathscr{S}^{(2)}_{m\{a\}} \mathscr{S}^{(2)}_{n\{b\}} Y_{a_1a_2,b_1b_2} \Delta_{\bar{a}_2;\bar{b}_2} \right] \delta_{n,m}$$
$$- \frac{2}{\sqrt{n(n-1)}} \frac{1}{(n-2)} \mathscr{S}^{(1)}_{m\{a\}} \mathscr{S}^{(3)}_{n\{b\}} W_{a_1;b_1b_2b_3} \Delta_{\bar{a}_1;\bar{b}_3} \delta_{n,m+2}$$
$$- \frac{2}{\sqrt{m(m-1)}} \frac{1}{(m-2)} \mathscr{S}^{(1)}_{n\{b\}} \mathscr{S}^{(3)}_{m\{a\}} W_{b_1;a_1a_2a_3} \Delta_{\bar{a}_3;\bar{b}_1} \delta_{n,m-2} \, . \tag{2.14}$$

In writing (2.14) we have introduced the new quantities Y and W which are defined as

$$Y_{ef;gh} = \tfrac{1}{2} \sum_p P^{(p-)}_{\tilde{e},h} P^{(p+)}_{\tilde{g},f} \, ,$$

and (2.15a)

$$W_{h;efg} = \tfrac{1}{2} \sum_p P^{(p-)}_{\tilde{h},e} Q^{(p+)}_{\tilde{f},\tilde{g}} \, .$$

Note that $Y_{ef;gh}$ and $W_{h;efg}$ have the following symmetry properties:

$$Y_{ef;gh} = Y_{fe;gh} = Y_{ef;hg} \, , \tag{2.15b}$$

$$W_{h;efg} = W_{h;egf} \, . \tag{2.15c}$$

However, due to the noncommutativity of the $B^\dagger$ operators, the permutation of the indices e and f in $W_{h;efg}$ gives rise to a nonvanishing term given as

$$W_{h;efg} - W_{h;feg} = \tfrac{1}{2} \sum_p P^{(p+)}_{\tilde{g},h} \tilde{Q}^{(p-)}_{\tilde{e},\tilde{f}} \, . \tag{2.15d}$$

Equation (2.15c) follows from the fact that $Q^{(p+)}_{e,f} = Q^{(p+)}_{f,e}$. The relations (2.15b) and (2.15d) are proved in Appendix B. The operator $\mathscr{S}^{(k)}_{n\{a\}}$ is a symmetrizer with respect to the set of indices $\{a\}$ (see Appendix B). Because of properties (2.15), it is clear that the ordering with which the indices appear after they are relocated by the symmetrizer $\mathscr{S}^{(k)}_{n\{a\}}$ is irrelevant for the Y term, but is crucial for the W terms.

We remarked above that the overlap matrix O^2_{mn} is nonvanishing for any n and m of the same parity. In (2.14), however, we have obtained only two terms with $n=m+2$ and $n=m-2$ in addition to the term $n=m$. This is, of course, because we have been satisfied with obtained terms to $O(W)$. [Since $O(Y) \simeq O(W)$, we are actually obtaining terms to $O(Y)$ and $O(W)$. For simplicity, we shall henceforth speak only of $O(W)$ to mean both $O(Y)$ and $O(W)$.] The next off diagonal terms would correspond to the cases when $n=m+4$ and $n=m-4$, which would be, however, $O(W^2)$ and thus are neglected.

Let us remark that, because of property (2.15d), the matrix $(O^2_{mn})_{a;b}$ is not totally symmetric under the permutation of the indices $\{a\}$ and $\{b\}$. For reasons that will be explained later, such a nonsymmetric matrix is not a convenient choice for constructing a suitable fermion basis. Therefore, we introduce here a *symmetrized* overlap matrix. To do that, we start by replacing the product states (2.10) with *symmetrized* product states defined as

$$| \overline{a,m} \rangle\rangle = \frac{1}{n!} P \, | a,m \rangle\rangle \, , \tag{2.16}$$

where P permutes all the $\{a\}$ indices. By using (2.16) we can now construct the symmetrized overlap matrix, with elements $\langle\langle \overline{m;a} \mid \overline{b;n} \rangle\rangle$. We shall denote this matrix by Z^2, to discriminate it from O^2. Its explicit form is given, to $O(W)$, by

$$(Z^2_{mn})_{a,b} = \langle\langle \overline{m,a} \mid \overline{b,n} \rangle\rangle = \left[\Delta_{a,b} - \frac{2}{n(n-1)} \mathscr{S}^{(2)}_{m\{a\}} \mathscr{S}^{(2)}_{n\{b\}} Y_{a_1a_2,b_1b_2} \Delta_{\bar{a}_2;\bar{b}_2} \right] \delta_{n,m}$$
$$- \frac{2}{\sqrt{n(n-1)}} \frac{1}{(n-2)} \mathscr{S}^{(1)}_{m\{a\}} \mathscr{S}^{(3)}_{n\{b\}} W^{(s)}_{a_1;b_1b_2b_3} \Delta_{\bar{a}_1;\bar{b}_3} \delta_{n,m+2}$$
$$- \frac{2}{\sqrt{m(m-1)}} \frac{1}{(m-2)} \mathscr{S}^{(1)}_{n\{b\}} \mathscr{S}^{(3)}_{m\{a\}} W^{(s)}_{b_1;a_1a_2a_3} \Delta_{\bar{a}_3;\bar{b}_1} \delta_{n,m-2} \, . \tag{2.17}$$

The Z^2 matrix of (2.17) is very close to O^2 of (2.14), the only, but important difference being that the former contains the symmetrized $W^{(s)}$ coefficients, defined as

$$W^{(s)}_{h;efg} = \tfrac{1}{3} (W_{h;efg} + W_{h;feg} + W_{h;gef}) \, . \tag{2.18}$$

In the rest of the paper we shall (generally) omit the word *symmetrized,* because from now on we consider exclusively symmetrized product states.

C. Orthonormal fermion space

The product states defined by (2.16) are not only nonorthonormal, but also overcomplete. Therefore the overlap matrix (Z^2) with elements given in (2.17) is singular, in general. This means that unphysical components are admixed in the product space. There is, however, a very practical way to overcome this difficulty; it is to choose a fermion subspace so that in it the matrix Z^2 becomes nonsingular. When truncation to a single collective component has been made, a suitable physical subspace can be defined by imposing an upper limit, say $N_{\max}$, to the admissible number of RPA phonons that can be superimposed in (2.10). (In practice, $N_{\max}$ can be quite large.) Once this is done, no further discussion is needed concerning the problem of the unphysical components in the theory, both in the fermion, as well as in the boson descriptions, since no danger of admixture of unphysical components exists.

In order to carry out the bosonization, it is convenient to construct first an *orthonormal fermion subspace.* This is the space to be subsequently mapped onto an ideal boson subspace. Since the ideal boson states are totally symmetric under the interchange of any pair of indices, it is quite obvious that we ought to start with fermion states that possess the same property. This was the reason for introducing the symmetrized states (2.16), and the symmetrized overlap matrix (2.17).

By using the states (2.16) and the (Z^2) of (2.17) as the norm matrix, we can now construct a set (which is complete in our truncated space) of *orthonormal* RPA states, $|a;m\rangle$. From the form of the (Z^2) matrix, we see that, to $O(W)$, an orthonormal m phonon state can be written as a superposition of symmetrized product states with $n=m$, $m-2$, and $m+2$ phonons. However, it is not *a priori* evident that we must retain both the product states with $m>n$ and $m<n$ in order to be able to obtain an orthonormal state. Consider, e.g., using Schmidt's orthonormalization procedure. We can certainly construct an orthonormal state $|a;m\rangle$ by taking only the product states with $n\le m$.

Having this ambiguity in mind, let us choose to write $|a;m\rangle$ as

$$|a;m\rangle=\sum_{n,b}^{N_{\max}}(N_{mn}^{-1})_{a;b}\,|\overline{b;n}\rangle\rangle\ , \tag{2.19}$$

where the norm matrix (N^{-1}), generally different from (Z^{-1}), is to be determined from the condition that the relation

$$\langle m;a\mid b;n\rangle=\sum_{k,a'e,b'}\sum(N_{mk}^{-1})_{a;a'}(Z_{kl}^2)_{a';b'}(N_{\text{ln}}^{-1})_{b';b}=\Delta_{a;b}\delta_{mn} \tag{2.20}$$

is satisfied [up to $O(W)$].

We shall now find the concrete form of the norm matrix by evaluating the matrix element (2.20). It will be very reasonable to expect that N^{-1} has the same *tensorial* structure as does Z^{-1}. We, therefore, *assume* that N^{-1} can be written as

$$\begin{aligned}(N_{mk}^{-1})_{a;a'}=&\left[\Delta_{a,a'}+\frac{1}{k(k-1)}\mathscr{S}^{(2)}_{m\{a\}}\mathscr{S}^{(2)}_{k\{a'\}}Y_{a_1a_2,a_1'a_2'}\Delta_{\bar a_2;\bar a_2'}\right]\delta_{k,m}\\&+\frac{y}{\sqrt{k(k-1)}}\frac{1}{(k-2)}\mathscr{S}^{(1)}_{m\{a\}}\mathscr{S}^{(3)}_{k\{a'\}}W^{(s)}_{a_1;a_1'a_2'a_3'}\Delta_{\bar a_1;\bar a_3'}\delta_{k,m+2}\\&+\frac{x}{\sqrt{m(m-1)}}\frac{1}{(m-2)}\mathscr{S}^{(1)}_{k\{a'\}}\mathscr{S}^{(3)}_{m\{a\}}W^{(s)}_{a_1';a_1a_2a_3}\Delta_{\bar a_3;\bar a_1'}\delta_{k,m-2}\ .\end{aligned} \tag{2.21}$$

Note that we have introduced the unknown numerical coefficients x and y (their presence is the only reason N^{-1} differs from Z^{-1}), which are to be fixed by using the condition (2.20). Keep in mind that in (2.21) x is associated with the term with the running index $k=m-2$, while y is associated with the term with $k=m+2$. [In the matrix N itself, the roles of x and y with respect to the running index are *inverted* so that $NN^{-1}=1$, again up to $O(W)$.]

We now insert (2.21) and (2.17) into (2.20), and find, to $O(W)$, that

$$\begin{aligned}\langle m;a\mid b;n\rangle=&(N_{mk}^{-1})_{a;a'}(Z_{kl}^2)_{a';b'}(N_{ln}^{-1})_{b';b}\\=&\Delta_{a,b}\delta_{mn}+(x+y-2)\frac{1}{\sqrt{n(n-1)}}\frac{1}{(n-2)}\mathscr{S}^{(1)}_{m\{a\}}\mathscr{S}^{(3)}_{n\{b\}}W^{(s)}_{a_1;b_1b_2b_3}\Delta_{\bar a_1;\bar b_3}\delta_{n,m+2}\\&+(x+y-2)\frac{1}{\sqrt{m(m-1)}}\frac{1}{m-2}\mathscr{S}^{(1)}_{n\{b\}}\mathscr{S}^{(3)}_{m\{a\}}W^{(s)}_{b_1;a_1a_2a_3}\Delta_{\bar a_3;\bar b_1}\delta_{n,m-2}\ .\end{aligned} \tag{2.22}$$

We thus see that (2.20) is satisfied if we choose x and y such that

$$x+y=2\ . \tag{2.23}$$

On the other hand, this also means that the problem is not completely determined, since we have one equation for two unknowns. Such an ambiguity, however, does not represent a serious problem. In fact, any pair of x and y [which we shall henceforth denote collectively by z; $z\equiv(x,y)$] that satisfies (2.23), defines a set of orthonormal basis states, i.e., a representation, and the sets belonging to different z's can be transformed one into another by unitary transformations. Therefore, the above indeterminateness simply says that we have an infinite number of physically equivalent representations.

In Sec. III, we shall show how to bosonize the fermion system which we have constructed above. We do this first for a generic z. Subsequently, we take advantage of the freedom in the choice of z and achieve a significant simplification in the obtained boson expansions. In order to make the presentation of Sec. III more transparent, we shall henceforth make the z dependence of the orthonormal states explicit by writing them as

$$|a,m;z\rangle = \sum_{n,b}^{N_{\max}} (N_{mn}^{-1})_{a;b}(z)\,|\overline{b,n}\rangle\rangle\ . \tag{2.24}$$

III. BOSONIZATION

A. Hermitian case

As is well known, and as emphasized in our recent publications,[9] the basic guiding principle of bosonization is to find a boson image $(O_F)_B$ for a given fermion operator O_F, such that the equality

$$\langle z;m,a\,|\,O_F\,|\,b,n;z\rangle = (m,a\,|\,(O_F)_B(z)\,|\,b,n)\ , \tag{3.1}$$

is satisfied. Here, the boson states are defined as

$$|a,m) = \frac{1}{\sqrt{m!}}(A_a^\dagger)^m\,|0) \equiv \frac{1}{\sqrt{m!}} A_{a_1}^\dagger A_{a_2}^\dagger \cdots A_{a_m}^\dagger\,|0);\qquad (m \le N_{\max})\ . \tag{3.2}$$

In (3.2) $A_{a_i}^\dagger$ is a *pure* boson creation operator, and $|0)$ is the boson vacuum. The states defined by (3.2) span the *collective boson subspace* which is the boson image of the *collective (physical) fermion space,* spanned by states (2.24).

There are known a variety of ways to carry out the bosonization procedure. Here we shall apply a technique that combines the use of the modified Usui operator, first introduced by MYT, with that of the term-by-term bosonization (TTB) method, recently proposed by Tamura.[10] The Usui operator is useful in making the formulation transparent, while the TTB method is powerful in deriving very quickly the explicit form of the various boson expansions.

Let us define the Usui operator as[5,6,9]

$$U(z) = \sum_{m,a} |a,m)\langle z;m,a|\ . \tag{3.3}$$

Naturally the explicit form of the Usui operator depends on the representation, i.e., on the z we choose for the fermion basis. By making use of (3.3) and of the closure relation for a complete set, it is easy to see that in our truncated fermion and boson subspaces, $UU^\dagger = 1_B$ (boson unit operator), and $U^\dagger U = 1_F(z)$ (fermion unit operator in the z representation).

By making use of (3.1) and (3.3) and of the above properties of the Usui operator, we readily obtain the relation

$$\langle z;m,a\,|\,O_F\,|\,b,n;z\rangle = \langle z;m,a\,|\,U^\dagger(z)U(z)O_F U^\dagger(z)U(z)\,|\,b,n;z\rangle = (m,a\,|\,(O_F)_B(z)\,|\,b,n)\ , \tag{3.4}$$

where $(O_F)_B$, given by

$$(O_F)_B(z) = U(z)O_F U^\dagger(z)\ , \tag{3.5}$$

is the boson image of the fermion operator O_F. In fact, if the fermion operators are bosonized according to (3.5), Eq. (3.1) is guaranteed to be satisfied. Conversely, the boson image $(O_F)_B$ of any fermion operator O_F constructed so that it satisfies Eq. (3.1), can also be represented in the form of (3.5).

To find the explicit form of $(O_F)_B$, we begin by choosing the case with $O_F = B_e^\dagger$, and rewriting the matrix element $\langle m;a\,|\,B_e^\dagger\,|\,b;n\rangle$ as

$$\begin{aligned}
\langle z;m,a\,|\,B_e^\dagger\,|\,b,n;z\rangle &= \sum_{l,b'}\sum_{k,a'} (N_{mk}^{-1})_{a;a'}(z)\langle\langle k;a'\,|\,B_e^\dagger\,|\,b';l\rangle\rangle(N_{ln}^{-1})_{b';b}(z)\\
&= \sum_{l,b'}\sum_{k,a'} (N_{mk}^{-1})_{a;a'}(z)\sqrt{(l+1)}\Bigg[\Delta_{a',b'e}\delta_{k,l+1} - \frac{1}{l(l+1)}\mathscr{S}^{(2)}_{k\{a'\}}\Big[\mathscr{S}^{(2)}_{l\{b'\}}Y_{a'_1a'_2,b'_1b'_2}\Delta_{\bar{a}'_2;\bar{b}'_2e}\\
&\qquad + \mathscr{S}^{(1)}_{l\{b'\}}Y_{a'_1a'_2,eb'_1}\Delta_{\bar{a}'_2;\bar{b}'_12}\Big]\delta_{k,l+1}\\
&\qquad - \frac{2}{\sqrt{l(l+1)}}\frac{1}{(l-1)}\mathscr{S}^{(1)}_{k\{a'\}}\Big[\mathscr{S}^{(3)}_{l\{b'\}}W^{(s)}_{a'_1;b'_1b'_2b'_3}\Delta_{\bar{a}'_1;\bar{b}'_3e}\\
&\qquad + \mathscr{S}^{(2)}_{l\{b'\}}W_{a'_1;eb'_1b'_2}\Delta_{\bar{a}'_1;\bar{b}'_2}\Big]\delta_{l,k+1}\\
&\qquad - \frac{2}{\sqrt{k(k-1)}}\frac{1}{(k-2)}\mathscr{S}^{(1)}_{k\{a'\}}\Big[\mathscr{S}^{(1)}_{l\{b'\}}W_{b'_1;a'_1a'_2a'_3}\Delta_{\bar{a}'_3;\bar{b}'_1e}\\
&\qquad + W^{(s)}_{e;a'_1a'_2a'_3}\Delta_{\bar{a}'_3;b'}\Big]\delta_{l,k-3}\Bigg](N_{ln}^{-1})_{b';b}(z)\cdot
\end{aligned} \tag{3.6}$$

By using the definition of the norm matrix given by (2.21), we can carry out the summations over k, l, a', and b' in (3.6). The algebra is somewhat lengthy, but straightforward, and we feel it unnecessary to give it here explicitly. Notice in (3.6) a peculiar behavior of the index e relative to the indices $\{b'\}$. Every quantity in (3.6) is totally symmetric with respect to any interchange of $\{b'\}$, while only a partial symmetry exists in the interchange between e and $\{b'\}$. Thus great care must be exercised in manipulating the product of N^{-1} and (Z^2).

The final form of the fermion matrix element of $B_e^\dagger$ is given as

$$\langle z;m,a \mid B_e^\dagger \mid b,n;z\rangle = \sqrt{(n+1)}\left[\Delta_{a,be} - \frac{1}{n(n+1)}\mathscr{S}^{(2)}_{m\{a\}}\mathscr{S}^{(1)}_{n\{b\}}\,Y_{a_1a_2,b_1e}\Delta_{\bar{a}_2;\bar{b}_1}\right]\delta_{n,m-1}$$
$$-\frac{1}{\sqrt{n}}\frac{1}{(n-1)}\mathscr{S}^{(1)}_{m\{a\}}\mathscr{S}^{(2)}_{n\{b\}}(2W_{a_1;eb_1b_2} - yW^{(s)}_{a_1;eb_1b_2})\Delta_{\bar{a}_1;\bar{b}_2}\delta_{n,m+1}$$
$$-\frac{2-x}{\sqrt{m(m-1)(m-2)}}\mathscr{S}^{(3)}_{m\{a\}}\,W^{(s)}_{e;a_1a_2a_3}\Delta_{\bar{a}_3;b}\delta_{n,m-3}\;. \tag{3.7}$$

Note that this result is fairly simple, and that one reason for this is that the *unlinked terms,* i.e., terms in which the coefficients Y or W do not carry the index e, have canceled out completely. Note also that the $W^{(s)}$ terms are totally symmetric, although they contain the index e (these terms originate from the products of the W factors in N^{-1} and the Δ factors in Z^2).

The rhs of (3.7) is nothing but the irreducible tensor expansion [valid to $O(W)$] of the left-hand side (lhs) of (3.1). By inspection, we can readily write the boson expansion for $(B_e^\dagger)_B(z)$:

$$(B_e^\dagger)_B(z) = A_e^\dagger - \tfrac{1}{2}Y_{fg;eh}A_f^\dagger A_g^\dagger A_h - \left[W_{f;egh} - \frac{y}{2!}W^{(s)}_{f;egh}\right]A_f^\dagger A_g A_h - \frac{2-x}{3!}W^{(s)}_{e;fgh}A_f^\dagger A_g^\dagger A_h^\dagger\;. \tag{3.8}$$

(Summation over dummy indices is understood.) We may easily confirm the validity of (3.8) by inserting it into $(m;a \mid (B_e^\dagger)_B \mid b;n)$ and finding that it results in the rhs of (3.7). The bosonization procedure used above, i.e., the procedure to obtain (3.8) from (3.7) by inspection, is nothing but an example of the use of the TTB method.[10] The spirit of the TTB method was explained in some detail in the Introduction. Combining that explanation with its actual use here, the reader will now have a very concrete understanding of this technique. The reader will also see clearly the greater simplicity and transparency of this method over all the other bosonization techniques mentioned in the Introduction.

The boson image $(B_e)_B$ of the annihilation operator B_e is, by construction, just the Hermitian conjugate of (3.8). As for the scattering operator $C_p^\dagger$, its fermion matrix element is written, again to $O(W)$, as

$$\langle z;a;m \mid C_p^\dagger \mid b;n;z\rangle = \mathscr{S}^{(1)}_{m\{a\}}\mathscr{S}^{(1)}_{n\{b\}}\frac{1}{n}P^{(p+)}_{\bar{b}_1,a}\Delta_{\bar{a}_1;\bar{b}_1}\delta_{nm} + \frac{1}{\sqrt{n(n-1)}}\mathscr{S}^{(2)}_{n\{b\}}Q^{(p+)}_{\bar{b}_1,\bar{b}_2}\Delta_{a;\bar{b}_2}\delta_{n,m+2}$$
$$+\frac{1}{\sqrt{m(m-1)}}\mathscr{S}^{(2)}_{m\{a\}}\tilde{Q}^{(p+)}_{\bar{a}_1,\bar{a}}\Delta_{\bar{a}_2;b}\delta_{n,m-2}\;. \tag{3.9}$$

Applying the TTB method, we then find that

$$(C_p^\dagger)_B = P^{(p+)}_{f,e}A_e^\dagger A_f + \tfrac{1}{2}Q^{(p+)}_{\bar{e},\bar{f}}A_eA_f + \tfrac{1}{2}\tilde{Q}^{(p+)}_{\bar{e},\bar{f}}A_e^\dagger A_f^\dagger\;. \tag{3.10}$$

The expansion (3.10) is basically the same as that of Refs. 11 and 13. As we see, neither (3.9) nor (3.10) depends on z. This is because, up to $O(W)$, we were able to set $N^{-1}=\Delta$ in the fermion matrix element (3.9).

This concludes the bosonization of an even fermion system since all the operators can be expressed in terms of $B^\dagger$, B, and $C^\dagger$. We may also notice that it is now a straightforward matter to verify that the expansions of (3.8) and (3.10) do satisfy (perturbatively), within the truncated space, all the original fermion commutation relations given in (2.3). [Note that Eq. (2.15d) is needed to prove (2.3d).]

It is gratifying to find that the forms of $(B_e^\dagger)_B(z)$ and $(C_p^\dagger)_B$, obtained in (3.8) and (3.10), are fairly simple, considering the complexity of their derivation. Still, the z dependence of $(B_e^\dagger)_B$ may be considered embarrassing, because it means, e.g., that the Hamiltonian is indeterminate, to the extent that z is indeterminate. We shall postpone the discussion of this problem, however, to Sec. III C, and address ourselves next to the derivation of the Dyson-type BET.

B. Bosonization: Non-Hermitian case

In this subsection we construct the RPA-based boson expansion which is non-Hermitian, i.e., is of the Dyson form.

As in Refs. 16 and 17, we shall begin by introducing two Usui operators;

$$U_1(z) = \sum_{n,a}\sum_{l,b}\mid a,n)\langle z;l,b \mid (Z_{\ln})_{b;a}\;, \tag{3.11a}$$

$$U_2(z) = \sum_{n,a}\sum_{l,b}\mid a,n)\langle z;l,b \mid (Z^{-1}_{\ln})_{b;a}\;. \tag{3.11b}$$

In our truncated fermion and boson spaces the above Usui operators satisfy the following conditions:

$$U_1(z)U_2^\dagger(z)=\sum_{n,a} |\, a,n)(n,a \,| = 1_B \text{ (boson unit operator)} , \quad (3.12a)$$

$$U_2^\dagger(z)U_1(z)=\sum_{n,a} |\, a,n;z\rangle\langle z;n,a \,| = 1_F(z) \text{ (fermion unit operator)} . \quad (3.12b)$$

Using (3.11) and (3.12b), it is straightforward to derive the z-dependent Dyson boson image of any fermion operator O_F. We rewrite the fermion matrix element as

$$\langle z;n,b \,|\, O_F \,|\, a,m;z\rangle = \langle z;m,a \,|\, U_2^\dagger(z)U_1(z)O_F U_2^\dagger(z)U_1(z) \,|\, b,n;z\rangle = \sum_{k,a'}\sum_{l,b'}(Z_{mk}^{-1})_{a;a'}(k;a' \,|\, (O_F)_D(z) \,|\, b';l)(Z_{ln})_{b';b} , \quad (3.13)$$

with

$$(O_F)_D(z)=U_1(z)O_F U_2^\dagger(z) \quad (3.14)$$

representing the Dyson boson image of the fermion operator O_F.

In order to obtain $(O_F)_D$ explicitly, by using the TTB method, we find it convenient to invert (3.13) as

$$(m;a \,|\, (O_F)_D(z) \,|\, b;n)=\sum_{ka'}\sum_{pa''}\sum_{lb'}\sum_{qb''}(Z_{mp})_{a;a''}(N_{pk}^{-1})_{a'';a'}(z)\langle\!\langle \overline{z;k,a'} \,|\, O_F \,|\, \overline{b',l;z}\rangle\!\rangle(N_{lq}^{-1})_{b';b''}(z)(Z_{qn}^{-1})_{b'';b} . \quad (3.15)$$

If the rhs of (3.15) is rewritten as a sum of irreducible tensors, the TTB method immediately gives rise to the explicit form of $(O_F)_D$.

Note that Z and N appear now in (3.15) in the combinations of ZN^{-1} and $N^{-1}Z^{-1}$. We may work out the combination ZN^{-1} first, which results in the following expression:

$$\sum_{pa''}(Z_{mp})_{a;a''}(N_{pk}^{-1})_{a'';a'}(z)=\Delta_{a;a'}\delta_{mk}+\frac{x-1}{\sqrt{k(k-1)}}\frac{1}{k-2}\mathscr{S}^{(3)}_{k\{a'\}}\mathscr{S}^{(1)}_{m\{a\}}W^{(s)}_{a_1;a_1'a_2'a_3'}\Delta_{\bar{a}_1;\bar{a}_3'}\delta_{k,m-2} + \frac{y-1}{\sqrt{m(m-1)}}\frac{1}{m-2}\mathscr{S}^{(1)}_{k\{a'\}}\mathscr{S}^{(3)}_{m\{a\}}W^{(s)}_{a_1';a_1a_2a_3}\Delta_{\bar{a}_1';\bar{a}_3}\delta_{k,m+2} . \quad (3.16)$$

The combination $N^{-1}Z^{-1}$ can be evaluated in a similarly straightforward manner. By using these results, it is easy to prove that Eq. (3.15) is satisfied for $O_F=1$ [and thus $(O_F)_D=1$] provided, once again, that $x+y=2$. Note that this simple *test case* was used for the choice of Z over other possible normlike matrices in the definition of the two Usui operators in (3.11). Any other choice fails the above test.

We now set $O_F=B_e$ in the rhs of (3.15), and evaluate the fermion matrix element to obtain

$$(m;a \,|\, (B_e)_D(z) \,|\, b;n)=\sqrt{(m+1)}\Delta_{ae;b}\delta_{n,m+1} - \frac{1}{\sqrt{m}}\frac{1}{(m-1)}\mathscr{S}^{(2)}_{m\{a\}}\mathscr{S}^{(1)}_{n\{b\}}[2W_{b_1;ea_1a_2}-(y+1)W^{(s)}_{b_1;ea_1a_e}]\Delta_{\bar{a}_2;\bar{b}_1}\delta_{n,m-1} - \frac{2-(x+1)}{\sqrt{n(n-1)(n-2)}}\mathscr{S}^{(3)}_{n\{b\}}W^{(s)}_{e;b_1b_2b_3}\Delta_{\bar{b}_3;a}\delta_{n,m+3} . \quad (3.17)$$

The validity of the choice of (3.11), using Z and Z^{-1}, reveals itself also in the fact that no unlinked terms are present in (3.17). With any other choice we found that the unlinked terms could not be totally eliminated in the resulting expression corresponding to (3.17). [This is, of course, intimately connected with the fact mentioned below Eq. (3.16).]

We can now apply the TTB method to (3.17) and find that

$$(B_e)_D(z)=A_e-\left[W_{h;efg}-\frac{y+1}{2!}W^{(s)}_{h;efg}\right]A_f^\dagger A_g^\dagger A_h-\frac{2-(x+1)}{3!}W^{(s)}_{e;fgh}A_fA_gA_h . \quad (3.18)$$

Unlike the case of the Tamm-Dancoff based BET (of the Dyson form), in which we had[17]

$$(B_e)_D=A_e;\ \text{(TD case)} \quad (3.19)$$

exactly, the result of (3.18) is rather lengthy.

We next set $O_F=B_e^\dagger$ in (3.15), and find that

$$(m;a\,|\,(B_e^\dagger)_D(z)\,|\,b;n)=\sqrt{(n+1)}\left[\Delta_{a,be}-\frac{2}{n(n+1)}\mathscr{S}^{(2)}_{m\{a\}}\mathscr{S}^{(1)}_{n\{b\}}Y_{a_1a_2,b_1e}\Delta_{\bar{a}_3;\bar{b}_2}\right]\delta_{n,m-1}$$
$$-\frac{1}{\sqrt{n}}\frac{1}{(n-1)}\mathscr{S}^{(1)}_{m\{a\}}\mathscr{S}^{(2)}_{n\{b\}}[2W_{a_1;eb_1b_2}-(y-1)W^{(s)}_{a_1;eb_1b_2}]\Delta_{\bar{a}_1;\bar{b}_2}\delta_{n,m+1}$$
$$-\frac{2-(x-1)}{\sqrt{m(m-1)(m-2)}}\mathscr{S}^{(3)}_{m\{a\}}W^{(s)}_{e;a_1a_2a_3}\Delta_{\bar{a}_3;b}\delta_{n,m-3}\;. \qquad (3.20)$$

Applying the TTB method again, we obtain

$$(B_e^\dagger)_D(z)=A_e^\dagger-Y_{fg;eh}A_f^\dagger A_g^\dagger A_h-\left[W_{f;egh}-\frac{y-1}{2!}W^{(s)}_{f;egh}\right]A_f^\dagger A_g A_h-\frac{2-(x-1)}{3!}W^{(s)}_{e;fgh}A_f^\dagger A_g^\dagger A_h^\dagger\;. \qquad (3.21)$$

To complete the bosonization we must still consider the scattering operator. The algebra is the same as in the Hermitian case, since up to the order we work, we can set $N=Z=\Delta$ in the matrix element of $C^\dagger$. The resulting expansion obviously coincides with that of (3.10) for the Hermitian case:

$$(C_p^\dagger)_D=P^{(p+)}_{\bar{f},e}A_e^\dagger A_f+\tfrac{1}{2}Q^{(p+)}_{\bar{e},\bar{f}}A_eA_f+\tfrac{1}{2}\tilde{Q}^{(p+)}_{\bar{e},\bar{f}}A_e^\dagger A_f^\dagger\;. \qquad (3.22)$$

As in the Hermitian case, it is now straightforward to verify that the expansions of (3.18), (3.21), and (3.22) satisfy all the fermion commutation relations (2.3). As we see, the expansions obtained in the non-Hermitian case are more complicated than we have expected. We shall comment further on this at the end of Sec. III C.

C. Equivalence of all the z representations and the ensuing simplifications of the boson expansions

As we saw in Secs. III A and III B, the boson images of the $B_e^\dagger$ and B_e operators are z dependent. On the other hand, we have remarked at the end of Sec. II that the fermion theories with different choices of z are all equivalent. Since our BET has been constructed in such a way that the fermion calculations are copied faithfully by the boson form of the theory, the above z dependence of the bosonized operators should be only apparent; i.e., any choice of z (so long as the condition $x+y=2$ is fulfilled) ought to result in the same numerical results for any physical quantity. Although the above argument should be sufficiently convincing, we feel it, nevertheless, desirable to formalize it, and this is what we intend to do now.

Let us consider two choices of z; $z=z_1$ ($x=x_1$ and $y=y_1$) and $z=z_2$ ($x=x_2$ and $y=y_2$). The fermion states (2.28), for each of the above choices of z, form an orthonormal complete set (in our truncated space), and the two sets can be transformed from one into the other through a unitary transformation. With obvious notation, this transformation is written as

$$|\,a,m;z_1\rangle=\sum_{n,b}S_{ma;nb}(z_1,z_2)\,|\,b,n;z_2\rangle\;, \qquad (3.23)$$

the element of the unitary matrix S being given as

$$S_{ma;nb}(z_1,z_2)=\langle z_2;n,b\,|\,a,m;z_1\rangle\;. \qquad (3.24)$$

The unitary equivalence of the two representations is embodied in the following equality:

$$\langle z_1;m,a\,|\,H_F\,|\,a',m';z_1\rangle=\sum_{n,b}\sum_{n',b'}S_{ma;nb}(z_1,z_2)\langle z_2;n,b\,|\,H_F\,|\,b',n';z_2\rangle S_{n'b';m'a'}(z_2,z_1)\;. \qquad (3.25)$$

For clarity we have considered in (3.25) the case in which $O_F=H_F$, the Hamiltonian, although any O_F could have been used.

In discussing the boson correspondence of (3.25), we shall take, for concreteness, the Hermitian case of Sec. III A. Since the basic equality (3.1) is satisfied for both $z=z_1$ and $z=z_2$, we can replace (3.25) by

$$(m,a\,|\,H_B(z_1)\,|\,a',m')=\sum_{n,b}\sum_{n',b'}S_{ma;nb}(z_1,z_2)(n,b\,|\,H_B(z_2)\,|\,b',n')S_{n'b';m'a'}(z_2,z_1) \qquad (3.26)$$

[by writing H_B for $(H_F)_B$ for simplicity].

We wrote (3.25) and (3.26) very explicitly in parallel, in order to emphasize the similarity and the difference in the two equalities. The similarity is evident. The difference is that, in the fermion matrix element, the z dependence originates from the fermion states, whereas the boson matrix element depends on z through the boson operators. That the fermion and boson matrix elements differ in this way is very natural. We have no freedom to modify the fermion operator in any arbitrary way. Similarly, the (ideal) boson states are unique; there is again no freedom to choose them arbitrarily. Therefore, in the bosonization the z dependence is transferred from the states to the operators.

In spite of the difference we have just described, the way the z independence of the final physical results is guaranteed via the presence of the unitary operator S, is the same in the fermion and boson descriptions. In other words, we can thus be assured that we may choose any z we like in the bosonized operators.

We now intend to take advantage of the equivalence of all the z representations and simplify the expansions we obtained in the preceding subsections. In the Hermitian case, by looking at $(B_e^\dagger)_B$ given in (3.8), we immediately see that the most desirable choice of z is that $x=2$ and $y=0$. The expansion then takes the form

$$(B_e^\dagger)_B = A_e^\dagger - \tfrac{1}{2} Y_{fg;eh} A_f^\dagger A_g^\dagger A_h - W_{f;egh} A_f^\dagger A_g A_h \ , \qquad (3.27)$$

which is rather simple.

The matter is not so clear-cut in the non-Hermitian case, and it appears that the best we can do is to choose $x=y=1$, obtaining

$$(B_e^\dagger)_D = A_e^\dagger - Y_{fg;eh} A_f^\dagger A_g^\dagger A_h - W_{f;egh} A_f^\dagger A_g A_h - \tfrac{1}{3} W^{(s)}_{e;fgh} A_f^\dagger A_g^\dagger A_h^\dagger \ , \qquad (3.28a)$$

$$(B_e)_D = A_e - (W_{f;egh} - W^{(s)}_{f;egh}) A_f^\dagger A_g^\dagger A_h \ . \qquad (3.28b)$$

The expansions in (3.28) are not much simpler than the original ones given in (3.18) and (3.21). Furthermore, they are by no means simpler that the Hermitian expansion of (3.27).

Dyson's BET has been considered very powerful, because it gives rise to finite expansions for the pair operators [one term for $(B_e)_D$ and $(C_p^\dagger)_D$ and two terms for $(B_e^\dagger)_D$], without employing any perturbative argument, offsetting the complications that come from the non-Hermiticity of the theory. In the RPA case the situation is quite different. Let us stress here a few aspects which characterize the non-Hermitian expansion we obtained: (i) the terms Δ Y, and $P^{(p+)}$ in the expansion which contribute to the diagonal matrix elements are exact, whereas the off-diagonal terms W and $Q^{(p+)}$ have been obtained perturbatively; (ii) unlike the TD case, the non-Hermitian expansions are not *finite;* and (iii) in particular the annihilation operator loses the one-term nature of the TD case. Because of aspects (ii) and (iii), the appealing features of the Dyson-type expansion, namely exactness and finiteness, are all but lost, once one switches to the RPA representation.

D. Comments on the approximation of Eq. (2.13)

Throughout the paper, we have used the approximation (2.13), i.e., we have set $\langle 0 | C_p^\dagger | 0 \rangle = 0$ in evaluating the various matrix elements. Actually, we can evaluate (in a perturbative manner) $\langle 0 | C_p^\dagger | 0 \rangle$, and show that the use of (2.13) is consistent with our perturbative treatment of the other quantities.

The RPA vacuum may be written as[18]

$$|0\rangle = N_0 \exp\left[\sum_{j_1 j_2} \sum_{j_1' j_2'} \sum_{\lambda\mu} V^{(\lambda)}_{j_1 j_2, j_1' j_2'} B^\dagger_{j_1 j_2 \lambda\mu} B^\dagger_{j_1' j_2' \lambda\widetilde{\mu}} \right] |\mathrm{BCS}\rangle \ , \qquad (3.29)$$

where N_0 is the normalization constant, and $|\mathrm{BCS}\rangle$ is the BCS vacuum. Furthermore, the expansion coefficients $V^{(\lambda)}_{j_1 j_2, j_1' j_2'}$ are implicitly given through the equation

$$\phi^{(\alpha)}_{j_1 j_2 \lambda} = \sum_{j_1' j_2'} V^{(\lambda)}_{j_1 j_2, j_1' j_2'} \psi^{(\alpha)}_{j_1' j_2' \lambda} \ , \qquad (3.30)$$

which can be solved perturbatively. To the first order, the solution is given as[19]

$$V^{(\lambda)}_{j_1 j_2, j_1' j_2'} = \sum_\alpha \phi^{(\alpha)}_{j_1 j_2 \lambda} \psi^{(\alpha)}_{j_1' j_2' \lambda} \ , \qquad (3.31)$$

showing that $V^{(\lambda)}_{j_1 j_2, j_1' j_2'} \simeq O(Q^{(p)})$.

From (3.29) we thus see that

$$\langle 0 | C_p^\dagger | 0 \rangle \simeq O(V^2) \simeq O(Q^2) \ .$$

For later convenience we shall write $\langle 0 | C_p^\dagger | 0 \rangle$ as

$$\langle 0 | C^\dagger_{j_1 j_2 kq} | 0 \rangle = \bar{n}_{j_1} \delta_{j_1 j_2} \delta_{k0} \delta_{q0} \ ; \bar{n}_{j_1} \simeq O(Q^2) \ . \qquad (3.32)$$

(The quantity $\bar{n}_{j_1}$ is a constant proportional to the average number of quasiparticles in the orbit $j_1 = j_2$.) Throughout the present paper, we have been setting $\bar{n}_{j_1} = 0$. If (3.32) is taken into account, however, a new term, which we shall denote by $C_{ma;nb}$, appears in the overlap matrix element $\langle\langle m;a | b;n \rangle\rangle$. This additional contribution can be written as

$$C_{ma;nb} = -\frac{1}{n} \mathscr{S}^{(1)}_{m\{a\}} \mathscr{S}^{(1)}_{n\{b\}} \Delta_{\bar{a}_1;\bar{b}_1} \sum_j P^{(jj00-)}_{\bar{a}_1,b_1} \bar{n}_j \delta_{mn}$$
$$= -n \hat{\lambda}^{-1} \sum_j \hat{P}^{(jj00-)}_{c,c} \bar{n}_j \Delta_{a;b} \delta_{mn} \ , \qquad (3.33)$$

where $\hat{P}$ was defined in Eq. (2.9). As we see, $O(C) = O(PQ^2)$, i.e., $|C_{ma;bn}| \ll |W|$, and thus $C_{ma;bn}$ can be neglected consistently in our theory.

It is nevertheless interesting to see how the results of Sec. III would have been modified, had we retained $\bar{n}_{j_1} \neq 0$. We find that, e.g., in the Hermitian case (with $x=2, y=0$), the new expansions can be written as

$$(C^\dagger_{j_1 j_2 kp})_B = \bar{n}_{j_1} \delta_{j_1 j_2} \delta_{k0} \delta_{q0} + P^{(p+)}_{\bar{b},a} A_e^\dagger A_f + \tfrac{1}{2} Q^{(p+)}_{\bar{e},\bar{f}} A_e A_f + \tfrac{1}{2} \widetilde{Q}^{(p+)}_{\bar{e},\bar{f}} A_e^\dagger A_f^\dagger \qquad (3.34)$$

and

$$(B_e^\dagger)_B = \left[1 - \frac{1}{2\sqrt{5}} \sum_j \hat{P}^{(jj00-)}_{c,c} \bar{n}_j \right] A_e^\dagger - \tfrac{1}{2} Y_{fg;eh} A_f^\dagger A_g^\dagger A_h - W_{f;egh} A_f^\dagger A_g A_h \ . \qquad (3.35)$$

As we see from (3.34), a constant term $O(Q^2)$ appears in $(C_p^\dagger)_B$ for $k=0$. This does not concern us in the present paper, since we have considered the case with $\lambda=2$ only.

On the other hand, it is worthwhile to note in (3.35) that the leading $A_e^\dagger$ term has been modified. This represents a qualitatively new feature of the BET in the RPA representation, as opposed to that in the TD representation, where no correction to the leading term could possibly emerge by going to higher orders.

IV. SUMMARY AND DISCUSSION

In the present paper, we have described a method for constructing *directly* an RPA-based BET. As explained in the Introduction, this has been a long standing problem, and in order to simplify our task in the first attempt to solve it, we have limited ourselves to the case in which only one kind of collective RPA mode was retained. This truncation of the RPA space permits us to treat the whole problem perturbatively, and we have taken full advantage of it. Note, nevertheless, that the above truncation is not too restrictive from a practical point of view. Many successful fits to data[7,8] we reported earlier were achieved within the same limitation.

Technically, we used the Usui operator[6] together with the TTB method.[10] Their combined use made the formalism very compact and transparent. Still the problem itself was nontrivial (even under the above truncation), and thus we had to go through fairly lengthy algebra. It is then very pleasing to find that the final results obtained for the boson expansions, given in (3.10), (3.27), and (3.28), for the basic fermion operators, are very simple. This means that, e.g., the bosonized Hamiltonian can be obtained in a quite simple form.

As remarked also in the Introduction, our previous works[7,8] were based on an *indirect* construction of the RPA-based BET, which was done in KT-2. It not only required a lengthy formulation, but also resulted in a complicated Hamiltonian. As discussed in KT-3, one of the (realistic) calculations, which we plan to perform in the near future, by going beyond what we have done before,[7,8] is a calculation in which the coupling between the collective and noncollective modes is taken into account. The complicated expressions of KT-2, which we had to deal with, were rather discouraging in attempting such an extension of the formalism. With the much simpler expressions of the present paper, however, this task appears feasible, and we are in fact preparing for such further work.

At this point, we may explain why we gave up in KT-2 the direct construction of the RPA-based BET. As remarked above, we used there the commutator method. (See Ref. 20 for our significantly improved understanding of the commutator method.) This method worked without any trouble in the TDA case. When the same method was used in the RPA case, however, we found that the number of the unknown expansion coefficients exceeded by one the number of equations for them. From what we have found in the present work, this can be certainly understood as the onset of the indeterminateness of the expansion due to the unfixed parameter z. We now understand the meaning and the role of z, and even know how to take advantage of this indeterminateness. At the time we were working on KT-2, however, we were totally unaware of this feature, and thus gave up the whole project.

We shall now discuss very briefly the work of Almoney and Borse.[11] They also used the commutator method, and obtained (almost by inspection) the result for $(C_p^\dagger)_B$ that agrees with what we gave in (3.10). For $(B_e^\dagger)_B$, however, they were satisfied with having the lowest order expansion in (3.27), i.e., they set $(B_e^\dagger)_B = A_e^\dagger$. In the terminology of KT-1, their theory is thus a third order theory; the highest order terms in the Hamiltonian contain at most three boson operators. To have such an odd order theory is known to be dangerous, however, because, at least conceptually, it leads to a collapsed shape of deformation, a trouble that an even order theory can easily avoid.[21] If this point of view is taken seriously, then the theory must be extended at least to the fourth order. This means that the rest of the terms in (3.27) are needed. As we have explained above, to derive these terms in the commutator method could have been very difficult.

Having obtained the Hermitian expansions given in (3.10) and (3.27), we can construct the fourth order BET Hamiltonian, by using the technique developed in Sec. 4 of KT-1. By going one step further in the expansions in (3.10) and (3.27), we can construct the sixth order Hamiltonian. Since we expect to meet the need for such a higher order BET, under certain circumstances, we are also working on this.

In closing the present paper, we shall address ourselves to some rather general aspects of the BET as a whole. It concerns the question on why bosons are introduced in the first place. This question has been asked repeatedly in the past, but we feel that it has become now much easier to answer, thanks to the introduction of the TTB method. As shown above, the TTB method makes the bosonization procedure almost trivial, once the fermion part of the formalism is completed. It also makes the relation between the fermion and boson descriptions very clear. However, the fact that the fermion part of the formulation has been completed means that we have already concrete forms for the fermion matrix elements. It is thus very natural to ask why these results are not then simply used directly for numerical calculations.

There are several reasons that make the use of a boson representation advantageous. Consider, e.g., using the expansion of (3.10) and (3.27) in the $B^\dagger C^\dagger$ and $B^\dagger B$ terms in the original fermion Hamiltonian. One easily sees that the anharmonic terms like $A^\dagger A^\dagger A$ and $A^\dagger A^\dagger AA$ emerge along with the harmonic $A^\dagger A$ term, in the boson Hamiltonian, allowing us to visualize the onset of the (kinematic) anharmonicity very clearly. In the fermion form, on the other hand, we have to trace the corresponding anharmonicity effects in the individual matrix elements, one by one, which makes it rather hard to get an overall view of these effects.

The use of the boson form also makes the calculations simpler. In fact, e.g., once the boson Hamiltonian has been obtained, we can use the ideal boson states as basis states, and readily carry out the calculation of the matrix elements. Note that the boson expansions in (3.10) and (3.27) were obtained in the so-called M representation. However, the transformation of the Hamiltonian in the boson form from the M to the I representation can be carried out in a straightforward manner.[1] This means that we can use the boson basis states in the I representation as well. To perform the corresponding calculation in the fermion form is rather cumbersome, if not impossible, when one includes many-phonon states.

ACKNOWLEDGMENTS

We are indebted to T. Kishimoto, K. J. Weeks, and T. Udagawa for a number of discussions we had with them during the past years pertaining to the problem of the present paper. This work was supported in part by the U.S. Department of Energy and by the Alexander von Humboldt Stiftung.

APPENDIX A

In this appendix we shall present a few intermediate steps of the algebra needed to derive (2.14). By using (2.3), it is straightforward to find that

$$\begin{aligned}\langle\langle m,a\,|\,b,n\rangle\rangle = {}& \mathscr{S}^{(1)}_{n\{b\}}\delta_{a_1,b}\frac{1}{\sqrt{m}}\frac{1}{\sqrt{n}}\langle\langle m-1,\bar a_1\,|\,\bar b_1,n-1\rangle\rangle - \mathscr{S}^{(2)}_{n\{b\}}\sum_{g,p}P^{(p-)}_{\bar a_1,b_1}P^{(p+)}_{\bar b_2,g(b_2)}\frac{1}{\sqrt{m}}\frac{1}{\sqrt{n}}\langle\langle m-1,\bar a_1\,|\,\bar b_2 g,n-1\rangle\rangle\\ &-\mathscr{S}^{(2)}_{n\{b\}}\sum_{g,p}P^{(p-)}_{\bar a_1,b_1}Q^{(p+)}_{\bar b_2,\bar g(b_2)}\frac{1}{\sqrt{m!}}\frac{1}{\sqrt{n!}}\langle 0\,|\,(B_{\bar a_1})^{m-1}\,|\,B_g(B^\dagger_{\bar b_2})^{n-2}\,|\,0\rangle\\ &-\mathscr{S}^{(1)}_{n\{b\}}\sum_{p}P^{(p-)}_{\bar a_1,b_1}\frac{1}{\sqrt{m!}}\frac{1}{\sqrt{n!}}\langle 0\,|\,(B_{\bar a_1})^{m-1}(B^\dagger_{\bar b_1})^{n-1}C^\dagger_p\,|\,0\rangle\,. \end{aligned}\tag{A1}$$

In (A1), $\mathscr{S}^{(k)}_{n\{b\}}$ is an operator that symmetrizes the operand with respect to the set of n indices $\{b\}$. Thus take, e.g., the symmetrizer $\mathscr{S}^{(1)}_{n\{b\}}$ in the first term of the rhs of (A1). It produces a sum of n terms, the ith term being equal to

$$\delta_{a_1,b_i}\langle\langle m-1;\bar a_1\,|\,b_1b_2\cdots b_{i-1}b_{i+1}\cdots b_n;n-1\rangle\rangle\ .$$

It is important that in

$$|\,b_1b_2\cdots b_{i-1}b_{i+1}\cdots b_n;n-1\rangle\rangle$$

the ordering of the operators

$$B^\dagger_{b_1}\cdots B^\dagger_{b_{i-1}}B^\dagger_{b_{i+1}}\cdots B^\dagger_{b_n}$$

is kept fixed; otherwise error occurs. This is because of the noncommutativity of the $B^\dagger$ operators, as seen in (2.3d). [Note that the symmetrizers select all the appropriate combinations of indices, without, however, changing their original ordering. Thus, e.g., if we have the set $\{a_i\}$ with $i=1,2,\ldots,m$ (in that order), the operator $\mathscr{S}^{(3)}_{m\{a\}}$ would select all the combinations $(a_ia_ja_k)$ with $i<j<k$. It will be seen that this fact is crucial for some kinds of terms, e.g., those that appear in Eq. (2.15) of the text.]

In general, $\mathscr{S}^{(k)}_{n\{b\}}$ gives rise to a sum of ${}_nC_k$ terms, where ${}_nC_k$ is a binomial coefficient. We also understand that $\mathscr{S}^{(k)}_n=0$ if $k>n$ and that $\mathscr{S}^{(n)}_n=1$. The notation $P^{(p+)}_{\bar b_2,g(b_2)}$ used in the second term of (A1) signifies that the creation operator $B^\dagger_{b_2}$ in the matrix element that follows it has been replaced by $B^\dagger_g$. The notation $Q^{(p+)}_{b_2,g(b_2)}$ in the third term of (A1) signifies that the creation operator $B^\dagger_{b_2}$ has now been replaced by the annihilation operator B_g. Because of the presence of the symmetrizer, the B_g operator replaces each of the $B^\dagger_{\bar b_1}$ operators one by one, starting with $B^\dagger_{b_2}$. This is why we were unable to write this matrix element in the form of an overlap integral, as we did for the first two terms.

We note that the second and third terms in the rhs of (A1) are, respectively, $O(P^{(p-)}P^{(p+)})$ and $O(P^{(p-)}Q^{(p+)})$ and are thus small (for the collective RPA component). Since we are satisfied with obtaining all the quantities to these orders, we can treat the RPA operators, that still remain in the matrix elements in these two terms, in the zeroth order, i.e., treat them as bosons. As for the last term in (A1), we first note that (for $m=2$)

$$\langle 0\,|\,B_{a_1}B_{a_2}C^\dagger_p\,|\,0\rangle=\tilde Q^{(p+)}_{\bar a_1,\bar a_2}\,,\tag{A2a}$$

where we have used (2.13). Since the last term of (A1) contains a factor $P^{(p-)}$ already, we need, for a general m, to evaluate the remaining part of the matrix element only up to the same order of (A2a). A direct calculation for this part then gives

$$\langle 0\,|\,(B_{\bar a_1})^{m-1}(B^\dagger_{\bar b_1})^{n-1}C^\dagger_p\,|\,0\rangle=\mathscr{S}^{(2)}_{m-1\{\bar a_1\}}\tilde Q^{(p+)}_{\bar a_2,\bar a_3}\langle 0\,|\,(B_{\bar a_3})^{m-3}(B^\dagger_{\bar b_1})^{n-1}\,|\,0\rangle=(m-3)!\,\mathscr{S}^{(2)}_{m-1\{\bar a_1\}}\tilde Q^{(p+)}_{\bar a_2,\bar a_3}\Delta_{\bar a_3,\bar b_1}\delta_{m,n+2}\tag{A2b}$$

valid up to the term $O(Q^{(p+)})$. In Eq. (2.15b) we have introduced the notation Δ to mean a *normalized* product of Kronecker deltas. Thus

$$\Delta_{a_1b_1}=\delta_{a_1b_1} ,$$

$$\Delta_{a_1a_2;b_1b_2}=(\delta_{a_1b_1}\delta_{a_2b_2}+\delta_{a_1b_2}\delta_{a_2b_1})/2! ,$$

and so forth.

With the above remarks in mind, Eq. (A1) can be rewritten as

$$\begin{aligned}\langle\langle m,a\,|\,b,n\rangle\rangle=&\frac{1}{\sqrt{m}}\frac{1}{\sqrt{n}}\mathscr{S}^{(1)}_{n\{b\}}\delta_{a_1,b_1}\langle\langle m-1,\bar{a}_1\,|\,\bar{b}_1;n-1\rangle\rangle-\frac{2}{n(n-1)}\mathscr{S}^{(1)}_{m-1\{\bar{a}_1\}}\mathscr{S}^{(2)}_{n\{b\}}Y_{a_1a_2,b_1b_2}\Delta_{\bar{a}_2;\bar{b}_2}\delta_{n,m}\\&-\frac{2(n-3)!}{\sqrt{n!m!}}\mathscr{S}^{(3)}_{n\{b\}}W_{a_1;b_1b_2b_3}\Delta_{\bar{a}_1;\bar{b}_3}\delta_{n,m+2}-\frac{2(m-3)!}{\sqrt{n!m!}}\mathscr{S}^{(1)}_{n\{b\}}\mathscr{S}^{(2)}_{m-1\{\bar{a}_1\}}W_{b_1;a_1a_2a_3}\Delta_{\bar{a}_3;\bar{b}_1}\delta_{n,m-2}\ .\end{aligned} \tag{A3}$$

In writing (A3) we have introduced the new quantities Y and W which are defined in (2.15a) of the text. The symmetry properties of Y and W are given in (2.15b) and (15c).

Equation (A3) can be considered as a recurrence relation for the norm matrix $\langle\langle m,a\,|\,b,n\rangle\rangle$ in terms of $\langle\langle m-1,\bar{a}_1\,|\,\bar{b}_1,n-1\rangle\rangle$. We may now apply this recurrence relation $n-1$ or $m-1$ times, whichever is smaller, each time obtaining additional contributions to the last three terms on the rhs of (A3). All the various contributions can be combined together and written in a compact form by using the appropriate symmetrizers. Summing up all the contributions, for the overlap matrix we obtain the expression given in (2.14).

APPPENDIX B

Here we shall prove the equality of (2.15d), i.e.,

$$2(W_{f;e'eg}-W_{f;ee'g})=\sum_p \widetilde{Q}\,^{(p-)}_{\tilde{e}',\tilde{e}}P^{(p+)}_{\tilde{g},f}\ , \tag{B1}$$

as well as the symmetry properties of $Y_{ef;gh}$ of Eq. (2.15b). Let us begin with (B1). By taking advantage of the symmetry (2.15c) of W, and using Eqs. (2.4), (2.6), and (2.9), we may rewrite the lhs of (B1) (keeping in mind that $\lambda=2$ and $p=\{j_1j_2kq\}$) as

$$\begin{aligned}2(W_{f;e'eg}-W_{f;ee'g})=&\sum_p P^{(p-)}_{\tilde{f},e}Q^{(p+)}_{\tilde{g},\tilde{e}}-\sum_p P^{(p-)}_{\tilde{f},e}Q^{(p+)}_{\tilde{g},\tilde{e}'}\\=&\sum_p[(\lambda\tilde{\mu}_f\lambda\mu_{e'}\,|\,kq)(\lambda\tilde{\mu}_g\lambda\tilde{\mu}_e\,|\,kq)-(\lambda\tilde{\mu}_f\lambda\mu_e\,|\,kq)(\lambda\tilde{\mu}_g\lambda\tilde{\mu}_{e'}\,|\,kq)\\&+(\lambda\tilde{\mu}_f\lambda\mu_{e'}\,|\,kq)(\lambda\tilde{\mu}_e\lambda\tilde{\mu}_g\,|\,kq)-(\lambda\tilde{\mu}_f\lambda\mu_e\,|\,kq)(\lambda\tilde{\mu}_{e'}\lambda\tilde{\mu}_g\,|\,kq)]\\&\times\hat{\lambda}^4\sum_{jj'}(\psi_{jj_2}\psi_{j_1j}-\phi_{jj_2}\phi_{j_1j})\psi_{j'j_2}\phi_{j_1j'}W(j_2j_1\lambda\lambda;kj)W(j_2j_1\lambda\lambda;kj')D_{jj_2}D_{jj_1}D_{j'j_2}D_{j'j_1}\ .\end{aligned} \tag{B2}$$

Each of the four terms in the square bracket in (B2) is a product of two Clebsch-Gordan (CG) coefficients. By performing the summation over q, each of the four terms can be transformed into a product of a Racah coefficient and two CG coefficients, one of the latter containing $\tilde{\mu}_a$ and μ_b and the other $\tilde{\mu}_e$ and $\tilde{\mu}_{e'}$. If the summation over k is further carried out, we see that the contributions from the third and the fourth terms in the square bracket cancel out, whereas those from the first and second terms can be combined to give

$$\begin{aligned}2(W_{fje'eg}-W_{f;ee'g})=&\sum_p(\lambda\tilde{\mu}_f\lambda\mu_g\,|\,kq)(\lambda\tilde{\mu}_{e'}\lambda\tilde{\mu}_e\,|\,kq)\\&\times\hat{\lambda}\sum_{jj'}^{4}[\psi_{jj_2}\psi_{j_1j}+(-)^k\phi_{jj_2}\phi_{j_1j}]\psi_{j'j_2}\phi_{j_1j'}[1-(-)^k]W(j_2j_1\lambda\lambda;kj)\\&\times W(j_2j_1\lambda\lambda;kj')D_{jj_2}D_{jj_1}D_{j'j_2}D_{j'j_1}\\=&\sum_p Q^{(p-)}_{\tilde{e}',\tilde{e}}P^{(p+)}_{\tilde{f},g}=\sum_p Q^{(p-)}_{\tilde{e}',\tilde{e}}\widetilde{P}\,^{(p+)}_{\tilde{g},f}=\sum_p\widetilde{Q}\,^{(p-)}_{\tilde{e}',\tilde{e}}P^{(p+)}_{\tilde{g},f}\ ,\end{aligned} \tag{B3}$$

which proves (B1).

To prove (2.15b), it is convenient to rewrite $Y_{ef;gh}$ as

$$
\begin{aligned}
Y_{ef;gh} &= \tfrac{1}{2} \sum_{p} P^{(p-)}_{\tilde{e},h} P^{(p+)}_{\tilde{g},f} \\
&= \tfrac{1}{2} \sum_{p} (\lambda\tilde{\mu}_e \lambda\mu_h \mid kq)(\lambda\tilde{\mu}_g \lambda\mu_f \mid kq)\hat{\lambda}^4 \sum_{jj'} (\psi_{jj_2}\psi_{j_1j}\psi_{j'j_2}\psi_{j_1j'} - \phi_{jj_2}\phi_{j_1j}\phi_{j'j_2}\phi_{j_1j'}) \\
&\quad \times W(j_2j_1\lambda\lambda;kj)W(j_2j_1\lambda\lambda;kj')D_{jj_2}D_{jj_1}D_{j'j_2}D_{j'j_1} \, .
\end{aligned}
\tag{B4}
$$

By performing on (B4) the same kind of angular momentum algebra described above for (B2), we can easily see that the symmetry relations (2.15b) do indeed hold.

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[21] In a third order theory, the potential energy term may be written as

$$V(\beta)=c_2\beta^2+c_3\beta^3\cos(3\gamma) \; ;$$

see, e.g., Sec. 6 of KT-2 for its derivation and for the explanation of the notation. One sees that $V(\beta)\rightarrow -\infty$ as $\beta\rightarrow\infty$, if $c_3<0$ and $\gamma=0°$. This is a needlelike collapse of the nuclear shape. Similarly, a flat disklike collapse occurs if $c_3>0$ and $\gamma=60°$. In the fourth order theory, on the other hand, an additional term, $c_4\beta^4$, appears in $V(\beta)$, which prevents the onset of the above collapse, so long as $c_4>0$. Note that our microscopic calculations in Refs. 2, 7, and 8 always predicted $c_4>0$.