Momentum-space solution of a bound-state nuclear three-body problem with two charged particles

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Momentum-space wave function equations are derived for the three-body system of one neutral and two charged particles where the separable interaction is spin and charge independent. The three-body wave function is decomposed so that the equations for the "pure" nuclear components contain the two-nucleon t matrix, as usual, but the equation for the additional Coulomb component is formulated in terms of the Coulomb potential rather than introducing the Coulomb t matrix. The relationship of these equations to the scattering-amplitude equation of Veselova, in which the Coulomb t matrix appears explicitly, as applied to the same problem by Kok *et al.*, is given. After partial-wave decomposition, the wave function equations are solved numerically with partial waves through $l=4$ retained. The logarithmic singularity that appears when the Coulomb potential is expressed in momentum space is handled for each l by a subtraction technique originated by Lande. Because of the symmetry of the problem, only even values of l arise throughout and only the $l=0$ and $l=2$ partial-wave contributions are required to predict the binding energy (and wave-function components) to four significant figures. An order α (fine-structure constant) approximation (i.e., decoupling the Coulomb wave function component from itself) is made to check its validity. It is shown that this $O(\alpha)$ approximation is equivalent to replacing the Coulomb t matrix by the Coulomb potential in the Veselova equations. The wave functions without Coulomb effects and with Coulomb effects [exact and $O(\alpha)$] are used to calculate the expectation value of the Coulomb operator to examine perturbation theory, cross check the numerical results, and to clarify the physics. We conclude that the Coulomb interaction can be incorporated easily into momentum-space three-nucleon, bound-state calculations.

I. INTRODUCTION

Nuclear three-body problems in which two of the particles are charged are basic to our understanding of nuclear structure and nuclear reactions. Two examples of current interest are the neutron-two-proton (npp) and neutron-proton-alpha-particle systems (npα with the α treated as an elementary particle). To fully understand the structure of ^{3}He and ^{6}Li, p-deuteron (p-d) scattering, and d-α scattering, the Coulomb interaction must be incorporated. Inclusion of the Coulomb interaction in a nuclear three-body problem is considered difficult even when only two of the particles are charged.[1] The difficulty arises, of course, due to the long-range nature of the Coulomb interaction. Progress is being made in the sense that calculational techniques are being developed and used. For example, modifications of the three-body scattering equations in momentum space have been worked out by Veselova,[2] and by Alt, Sandhas, and Ziegelmann;[3] these have been used to carry out p-d scattering calculations.[3,4] More recently, threshold p-d scattering calculations have been performed in configuration space.[5] In fact, configuration-space calculations appear to be the preferred method for treating bound states when the Coulomb interaction is included.[6] However, besides the work described herein, only one other *exact* (numerically) momentum-space, bound-state, three-body calculation with two charged particles has been performed.[7]

Why is there a paucity of work on the momentum-space, bound-state, three-body problem with two charged particles? We surmise that the main reasons concern the worry of dealing with the logarithmic singularity that arises in the kernels of the equations due to the long-range nature of the Coulomb interaction and of working with the fully off-shell Coulomb t matrix. These are unnecessary concerns. The problem can be formulated without introducing the Coulomb t matrix—but only using the Coulomb potential in momentum space. Furthermore, the logarithmic singularity from the momentum-space Coulomb potential can be handled by the method of Lande as reported by Kwon and Tabakin.[8] The purpose of the present article is to describe work that illustrates the latter points as applied to the problem of three spinless, equal-mass nucleons, two of which are charged, and for which the nuclear interaction of any pair is represented by a one-term, attractive, s-wave, separable potential.

The aim is to simulate the ^{3}He system, using sets of separable potential parameters which approximate an average of the two-nucleon s-wave triplet and singlet interactions.

Our work[9] complements that of Kok, van Haeringen, and their collaborators.[7] Kok *et al.* use the homogeneous version of Veselova's system of integral equations.[2] Veselova's equations with the inhomogeneous terms present and the energy set equal to scattering values yield the amplitudes for charged-particle—correlated pair (one of which is charged) scattering.[10] Kok *et al.* make a thorough investigation of the nature of the kernels (driving terms) in Veselova's equations. The Coulomb t matrix appears linearly in some of the kernels; thus, an exact solution of these equations necessitates dealing with the fully off-shell Coulomb t matrix. For a bound-state problem it is the off-shell Coulomb t matrix at negative energies that appears. Nevertheless, after the bound-state equations have been partial-wave projected, they are equations in a single variable. This is in contrast to our approach where the problem is formulated in terms of the bound-state wave function components and does not involve the Coulomb t matrix, but only the Coulomb potential. However, the partial-wave projected form of our equations leads to two-variable integral equations. Regardless, these equations are easily solved numerically and permit direct construction of the bound-state wave function. These aspects and other comparisons of the two methods are examined thoroughly below.

The structure of our presentation is as follows. Section II of the paper contains our formulation of the problem, while in Sec. III we relate our formulation to that of Kok *et al.* The method of handling the logarithmic singularity in the Coulomb kernel of our equations is presented in Sec. IV. In Sec. V we give the numerical results, followed by a discussion in Sec. VI. Section VII contains our conclusion. Finally, an Appendix lists the formulas we used in calculating the wave function normalization and expectation value of the Coulomb operator.

II. DERIVATION OF EQUATIONS

The goal of our formulation of the momentum-space, bound-state, three-particle equations with Coulomb interaction between one pair is to obtain *both* the energy eigenvalue and the wave function. A momentum-space formulation of the problem is most useful when the nuclear interactions are of separable form, because the problem without the Coulomb interaction leads to coupled integral equations in a single variable. In addition, most of the "realistic" local-potential calculations for ^{3}H have been done in momentum space.[11] Thus, the question arises as to what is a straightforward method for including the Coulomb interaction between a pair of particles in a momentum-space formulation. We suggest a method that is independent of the form of the nuclear interaction. Without loss of generality, we illustrate it for the special case of three spinless, equal-mass nucleons, two of which are charged, and with all pairs interacting through an attractive, s-wave, separable potential. With this simple model we can explore how Coulomb effects modify the momentum-space wave function in addition to shifting the eigenvalue. Also, we can check the validity of possible approximations.

Consider the three-body Hamiltonian

$$H=H_0+V_1+V_2+V_3+V_3^C\ , \tag{1}$$

where H_0 is the free-particle Hamiltonian, V_α represents the nuclear potential between particles $\beta\gamma$ ($\alpha\neq\beta\neq\gamma\neq\alpha$), and V_3^C is the Coulomb potential between charged particles 1 and 2. Schrödinger's equation,

$$H\Psi=E\Psi,\ \ (E<0)\ , \tag{2}$$

can be broken down according to the standard Faddeev decomposition:[12]

$$\Psi=\psi_1+\psi_2+\psi_3^C\ . \tag{3}$$

Thus we get

$$(H_0-E)\psi_\alpha=-V_\alpha\psi,\ \ (\alpha=1 \text{ or } 2) \tag{4}$$

and

$$(H_0-E)\psi_3^C=-(V_3+V_3^C)\Psi\ . \tag{5}$$

We break Eq. (5) into two equations by writing $\psi_3^C=\bar{\psi}_3+\eta$, where $\bar{\psi}_3$ represents a "Coulomb-modified" ψ_3, and η is the "pure" Coulomb component:

$$(H_0-E)\bar{\psi}_3=-V_3\Psi \tag{6}$$

and

$$(H_0-E)\eta=-V_3^C\Psi\ . \tag{7}$$

Equations (4) and (6) are next expressed in terms of the nuclear t matrix, $T_\alpha(E)$, and the free-particle resolvent,

$$G_0(E)=(E-H_0)^{-1}\ .$$

The Coulomb t matrix is *not* introduced in Eq. (7). Hence, our final set of equations is

$$\psi_\alpha=G_0(E)T_\alpha(E)(\psi_\beta+\bar{\psi}_3+\eta)\ , \tag{8}$$

$$\bar{\psi}_3=G_0(E)T_3(E)(\psi_1+\psi_2+\eta)\ , \tag{9}$$

and

$$\eta=G_0(E)V_3^C(\psi_1+\psi_2+\bar{\psi}_3+\eta)\ . \tag{10}$$

Equations (8)—(10) are the coupled equations that we solve in momentum space and are the central focus of this paper.

Let us look at these equations for the example mentioned in the first paragraph of this section. The nuclear t matrix is

$$T_\alpha(E)=|g_\alpha\rangle\tau_\alpha(E)\langle g_\alpha|\ , \tag{11}$$

where

$$\tau_\alpha(E)=-\frac{\lambda}{2\mu}\left[1+\frac{\lambda}{2\mu}\langle g_\alpha|G_0(E)|g_\alpha\rangle\right]^{-1}\ , \tag{12}$$

μ is the two-nucleon reduced mass ($2\mu=M$), λ is the strength of the interaction,

$$g(k)\equiv\langle\vec{k}|g\rangle=(k^2+\beta^2)^{-1}\ ,$$

and β is the inverse range parameter. This leads us to choose

$$|\psi_\alpha\rangle = NG_0(E)\,|g_\alpha\rangle\,|a_\alpha\rangle\ , \tag{13}$$

$$|\bar{\psi}_3\rangle = NG_0(E)\,|g_3\rangle\,|b_3\rangle\ , \tag{14}$$

and

$$|\eta\rangle = NG_0(E)\,|\tilde{\eta}\rangle\ , \tag{15}$$

where N is the wave-function normalization constant ($\langle\Psi|\Psi\rangle=1$) and $E\equiv -K^2/M$. In the momentum representation, $|a_\alpha\rangle$ and $|b_3\rangle$ are the spectator functions. With $\vec{k}$ the relative momentum between particles 1 and 2 and $\vec{p}$, the relative momentum between 3 and the center of mass of particles 1 and 2, we can express all other permutations in terms of $\vec{k}$ and $\vec{p}$. Accordingly, the equations to be solved in momentum space are

$$D(p,K^2)a(p) = 4\pi\lambda\int_0^\infty k^2dk\left[I(p,k;K^2)[a(k)+b(k)] + \sum_{l\ (\text{even})}(2l+1)J_l^\eta(p,k;K^2)\right]\ , \tag{16}$$

$$D(p,K^2)b(p) = 4\pi\lambda\int_0^\infty k^2dk\left[2I(p,k;K^2)a(k) + \frac{g(k)}{k^2+\frac{3}{4}p^2+K^2}\tilde{\eta}_0(k,p)\right]\ , \tag{17}$$

$$\tilde{\eta}_l(k,p) = -\frac{M\alpha}{4\pi}\int d^3k'\,v_l(k,k')\left[2P_l(\hat{k}'\cdot\hat{p})\frac{g(|\tfrac{1}{2}\vec{k}'+\tfrac{3}{4}\vec{p}|)a(|\vec{k}'-\tfrac{1}{2}\vec{p}|)}{k'^2+\frac{3}{4}p^2+K^2} + \delta_{l0}\frac{g(k')b(p)}{k'^2+\frac{3}{4}p^2+K^2} + \frac{\tilde{\eta}_l(k',p)}{k'^2+\frac{3}{4}p^2+K^2}\right]\ , \tag{18}$$

where

$$D(p,K^2) = 1-\lambda\int d^3k\frac{g^2(k)}{k^2+\frac{3}{4}p^2+K^2}\ , \tag{19}$$

$$I(p,k;K^2) = \tfrac{1}{2}\int_{-1}^{1}dx\frac{g(|\vec{k}+\frac{1}{2}\vec{p}|)g(|\frac{1}{2}\vec{k}+\vec{p}|)}{k^2+p^2+\vec{k}\cdot\vec{p}+K^2}\ , \tag{20}$$

$$J_l^\eta(p,k,K^2) = \tfrac{1}{2}\int_{-1}^{1}dx\frac{P_l(y)g(|\vec{k}+\frac{1}{2}\vec{p}|)\bar{\eta}_l(|\frac{1}{2}\vec{k}+\vec{p}|,k)}{k^2+p^2+\vec{k}\cdot\vec{p}+K^2}\ , \tag{21}$$

$$x = \hat{k}\cdot\hat{p}\ , \tag{22a}$$

$$y = (\tfrac{1}{2}k+px)/|\tfrac{1}{2}\vec{k}+\vec{p}|\ , \tag{22b}$$

$$\tilde{\eta}(\vec{k},\vec{p}) = \sum_{l\ (\text{even})}(2l+1)P_l(x)\tilde{\eta}_l(k,p)\ , \tag{23}$$

$$v_l(k,k') = Q_l(\xi)/\pi kk',\quad \left[\xi=\frac{k^2+k'^2}{2kk'}\right]\ . \tag{24}$$

$P_l(x)$ and $Q_l(x)$ are Legendre functions of the first and second kind, respectively, and α in Eq. (18) is the fine structure constant, equal to $(137.04)^{-1}$, so $M\alpha=0.034\,696$ fm^{-1}. Only even l enters in the partial-wave projection because the ground state is taken to have total angular momentum zero with positive parity (i.e., symmetric under exchange of particles 1 and 2). The $v_l(k,k')$ is the partial-wave projection of the momentum-space form of the Coulomb interaction (apart from a factor α) and it contains a logarithmic singularity through $Q_l(\xi)$. Note that if the Coulomb interaction is turned off, i.e., when the fine structure constant $\alpha=0$, then $\tilde{\eta}(\vec{k},\vec{p})\equiv 0$ and $b(p)=a(p)$.

After modifying Eq. (21) to handle the logarithmic singularity (see Sec. IV below), we solve Eqs. (16)–(18) numerically, both as they stand and in order-α [$O(\alpha)$] approximation. In the $O(\alpha)$ approximation $\tilde{\eta}_l$ is decoupled from itself, i.e., instead of Eq. (10) [or Eq. (18)], we have

$$\eta \cong G_0(E)V_3^C(\psi_1+\psi_2+\bar{\psi}_3)\ . \tag{25}$$

[This is equivalent to dropping the $\tilde{\eta}_l$ term from the right-hand side of Eq. (18)]. Within our framework, this is equivalent to replacing the Coulomb t matrix by V_3 as can be seen by writing Eq. (10) as

$$\eta = G_0(E)T_3^C(E)(\psi_1+\psi_2+\bar{\psi}_3)\ . \tag{10'}$$

Certainly, this is a natural approximation to make within the context of our equations. We shall check its validity through the eigenvalue and wave function.

III. RELATIONSHIP TO EQUATIONS OF KOK *et al.*

As mentioned in the Introduction, Kok *et al.* use Veselova's equations[2] to calculate the exact solution of a three-particle problem with Coulomb interaction in momentum space. They use the same model of three spinless, equal-mass nucleons, two of which are charged, with all pairs interacting through an attractive, s-wave separable potential. For this problem, the symmetrized (between particles 1 and 2) form of their equations is

$$|X_+\rangle = [Z_{\alpha\alpha}(E)+Z_{\alpha\beta}(E)]\tau_\alpha(E)\,|X_+\rangle + 2Z_{\alpha 3}(E)\tilde{\tau}_3(E)\,|X_3\rangle\ , \tag{26}$$

$$|X_3\rangle = Z_{3\alpha}(E)\tau_\alpha(E)\,|X_+\rangle\ , \tag{27}$$

where

$$Z_{\alpha\alpha}(E)=\langle g_\alpha \mid G_0(E)T_3^C(E)G_0(E) \mid g_\alpha\rangle \ , \tag{28}$$

$$Z_{\alpha\beta}(E)=\langle g_\alpha \mid G_0(E)+G_0(E)T_3^C(E)G_0(E) \mid g_\beta\rangle \ , \tag{29}$$

$$Z_{3\alpha}(E)=\langle \tilde{g}_3 \mid G_0(E) \mid g_\alpha\rangle \ , \tag{30}$$

$$\mid \tilde{g}_3\rangle=[1+T_3^C(E)G_0(E)] \mid g_3\rangle \ , \tag{31}$$

and

$$\tilde{\tau}_3(E)=-\frac{\lambda}{2\mu}\left[1+\frac{\lambda}{2\mu}\langle g_3 \mid G_0(E) +G_0(E)T_3^C(E)G_0(E) \mid g_3\rangle\right]^{-1} . \tag{32}$$

In momentum space, these are equations in one variable in which the Coulomb t matrix appears explicitly. How do the equations for the wave function components, Eqs. (8)—(10), relate to the equations of Kok *et al.*? Equivalently, how are the X amplitudes related to the wave function components?

To obtain the relationship, we return to Eqs. (4) and (5), but this time ψ_3^C is *not* written as the sum of $\bar{\psi}_3+\eta$. Consequently, we have ($\alpha=1$ or 2, $\alpha\neq\beta$)

$$(H_0-E-V_\alpha)\psi_\alpha=-V_\alpha(\psi_\beta+\psi_3^C) \tag{33}$$

and

$$(H_0-E-\tilde{V}_3)\psi_3^C=-\tilde{V}_3(\psi_\alpha+\psi_\beta) \ , \tag{34}$$

where $\tilde{V}_3=V_3+V_3^C$. By inverting the operators on the left-hand sides of the last two equations, we can express the equations in terms of t matrices:

$$\psi_\alpha=G_0(E)T_\alpha(E)(\psi_\beta+\psi_3^C) \tag{35}$$

and

$$\psi_3^C=G_0(E)\tilde{T}_3(E)(\psi_\alpha+\psi_\beta) \ , \tag{36}$$

where for the separable problem under consideration $T_\alpha(E)$ is given by Eqs. (11) and (12), and

$$\tilde{T}_3(E)=T_3^C(E)+\mid \tilde{g}_3\rangle\tilde{\tau}_3(E)\langle \tilde{g}_3 \mid \ . \tag{37}$$

In the momentum representation, Eqs. (35) and (36) are in two variables, whereas Eqs. (26) and (27) are in one variable. Clearly, an inner-product projection must be formed with Eqs. (35) and (36) to extract one-variable equations. We project from the left with $\langle g_\beta \mid$ on Eq. (35) and symmetrize to attain

$$\mid Y_{\beta\alpha}^+\rangle=\langle g_\beta \mid G_0(E) \mid g_\alpha\rangle\tau_\alpha(E) \mid Y_{\beta\alpha}^+\rangle +2\langle g_\beta \mid G_0(E) \mid g_\alpha\rangle\tau_\alpha(E)\langle g_\alpha \mid \psi_3^C\rangle \ , \tag{38}$$

where

$$\mid Y_{\beta\alpha}^+\rangle=\langle g_\beta \mid \psi_\alpha\rangle+\langle g_\alpha \mid \psi_\beta\rangle \ . \tag{39}$$

Then we project from the left with $\langle g_\alpha \mid$ on Eq. (34), symmetrize, and use the definitions in Eqs. (28)—(32) to derive

$$2\langle g_\alpha \mid \psi_3^C\rangle=Z_{\alpha\alpha}(E)\tau_\alpha(E)(\mid Y_{\beta\alpha}^+\rangle+2\langle g_\alpha \mid \psi_3^C\rangle)+[Z_{\alpha\beta}(E)-\langle g_\alpha \mid G_0(E) \mid g_\beta\rangle]\tau_\beta(E)(\mid Y_{\beta\alpha}^+\rangle+2\langle g_\alpha \mid \psi_3^C\rangle) +2Z_{\alpha3}(E)\tilde{\tau}_3(E)Z_{3\alpha}(E)\tau_\alpha(E)(\mid Y_{\beta\alpha}^+\rangle+2\langle g_\alpha \mid \psi_3^C\rangle) \ , \tag{40}$$

where use has been made of

$$\langle g_\alpha \mid \psi_3^C\rangle=\langle g_\beta \mid \psi_3^C\rangle \ , \tag{41}$$

$$Z_{\alpha\alpha}(E)=Z_{\beta\beta}(E) \ , \tag{42}$$

$$Z_{3\alpha}(E)=Z_{3\beta}(E) \text{ or } Z_{\alpha3}(E)=Z_{\beta3}(E) \ , \tag{43}$$

and

$$\tau_\alpha(E)=\tau_\beta(E) \ . \tag{44}$$

The terms in Eq. (40) with the factor $-\langle g_\alpha \mid G_0(E) \mid g_\beta\rangle$ are equal to $-\mid Y_{\alpha\beta}^+\rangle=-\mid Y_{\beta\alpha}^+\rangle$ [see Eq. (38)]. Accordingly, Eq. (40) can be written as

$$\mid Y_{\beta\alpha}^+\rangle+2\langle g_\alpha \mid \psi_3^C\rangle=[Z_{\alpha\alpha}(E)+Z_{\alpha\beta}(E)]\tau_\alpha(E)(\mid Y_{\beta\alpha}^+\rangle+2\langle g_\alpha \mid \psi_3^C\rangle) +2Z_{\alpha3}(E)\tilde{\tau}_3(E)[Z_{3\alpha}(E)\tau_\alpha(E)(\mid Y_{\beta\alpha}^+\rangle+2\langle g_\alpha \mid \psi_3^C\rangle)] \ . \tag{45}$$

The equations of Kok *et al.*, Eqs. (26) and (27), follow directly from Eq. (45) with the definitions

$$\mid X_+\rangle\equiv\mid Y_{\beta\alpha}^+\rangle+2\langle g_\alpha \mid \psi_3^C\rangle \tag{46}$$

$$=\langle g_\beta \mid \psi_\alpha\rangle+\langle g_\alpha \mid \psi_\beta\rangle+2\langle g_\alpha \mid \psi_3^C\rangle \tag{47}$$

and

$$\mid X_3\rangle\equiv Z_{3\alpha}(E)\tau_\alpha(E) \mid X_+\rangle \ . \tag{48}$$

The X amplitudes of Kok *et al.* are thus obtained from our wave-function components by overlapping the components with the form factor, $\langle g_\beta \mid$, of the separable nuclear interaction.

The major approximation examined by Kok *et al.* is the replacement of $T^C(E)$ by V^C. The motivation for doing this derives primarily from the considerable simplification in the numerical computation that results. In fact, before the calculations of Kok *et al.* and the present work, replacing $T^C(E)$ by V^C was the extent of previous attempts.[13] Since Kok *et al.* do not break ψ_3^C into the sum of two terms, it is not obvious that their replacement of $T^C(E)$ by V^C is equivalent to our $O(\alpha)$ approximation.

We now demonstrate that they are equivalent by starting from Eq. (36)—the key equation underlying the formulation of Kok *et al.*

Equation (36) can be written as

$$\Psi_3^C = G_0(E)[T_3^C(E)+\bar{T}_3(E)](\psi_\alpha+\psi_\beta)\ , \tag{36'}$$

where

$$\bar{T}_3(E) = |\tilde{g}_3\rangle\tilde{\tau}_3(E)\langle\tilde{g}_3|\ . \tag{49}$$

We introduce the Born (B) approximation, i.e., replace $T_3^C(E)$ by V^C everywhere in Eq. (36'):

$$\psi_3^C \cong G_0(E)V_3^C(\psi_\alpha+\psi_\beta)+G_0(E)[1+V_3^C G_0(E)]\,|g_3\rangle\tilde{\tau}_3^B(E)\langle g_3|\,[1+G_0(E)V_3^C](\psi_\alpha+\psi_\beta)\ , \tag{50}$$

where

$$\tilde{\tau}_3^B(E) = \tau_3(E)[1-\tau_3(E)\langle g|\,G_0(E)V_3^C G_0(E)\,|g\rangle]^{-1} \tag{51}$$

$$= \tau_3(E)[1+\tau_3(E)\langle g|\,G_0(E)V_3^C G_0(E)\,|g\rangle+\tau_3(E)\langle g|\,G_0(E)V_3^C G_0(E)T_3(E)G_0(E)V_3^C G_0(E)\,|g\rangle+\cdots]\ . \tag{52}$$

We insert Eq. (52) into Eq. (50) to derive

$$\psi_3^C \simeq [G_0T_3(1+G_0V_3^C+G_0V_3^CG_0T_3+G_0V_3^CG_0T_3G_0V_3^C+\cdots)$$
$$+G_0V_3^C(1+G_0T_3+G_0T_3G_0V_3^C+G_0T_3G_0V_3^CG_0T_3+\cdots)](\psi_\alpha+\psi_\beta) \tag{53}$$

$$\equiv\bar{\psi}_3+\eta\ , \tag{54}$$

where

$$\bar{\psi}_3 = G_0T_3(1+G_0V_3^C+G_0V_3^CG_0T_3+\cdots)(\psi_\alpha+\psi_\beta) \tag{55}$$

and

$$\eta \cong G_0V_3^C(1+G_0T_3+G_0T_3G_0V_3^C+\cdots)(\psi_\alpha+\psi_\beta)\ . \tag{56}$$

We rewrite $\bar{\psi}_3$ and η in the following manner:

$$\bar{\psi}_3 = G_0T_3(\psi_\alpha+\psi_\beta)+G_0T_3[G_0V_3^C(1+G_0T_3+G_0T_3G_0V_3^C+\cdots)(\psi_\alpha+\psi_\beta)] \tag{57}$$

and

$$\eta \cong G_0V_3^C(\psi_\alpha+\psi_\beta)+G_0V_3^C[G_0T_3(1+G_0V_3^C+G_0V_3^CG_0T_3+\cdots)(\psi_\alpha+\psi_\beta)]\ . \tag{58}$$

From Eqs. (57) and (58), it is clear that Eqs. (55) and (56) are coupled as follows:

$$\bar{\psi}_3 = G_0T_3(\psi_\alpha+\psi_\beta+\eta) \tag{9'}$$

and

$$\eta \cong G_0V_3^C(\psi_\alpha+\psi_\beta+\bar{\psi}_3)\ . \tag{25'}$$

Therefore, we see that the $O(\alpha)$ approximation of the present work is equivalent to the $T_3^C \to V_3^C$ approximation of Kok *et al.*

IV. HANDLING THE LOGARITHMIC SINGULARITY OF $v_l(k,k')$

The integrand of Eq. (18) contains a logarithmic singularity at $k=k'$ due to $v_l(k,k')$:

$$v_l(k,k') \sim \ln\left[\frac{\xi+1}{\xi-1}\right]\ ,$$

where $\xi=1$ when $k=k'$ [see Eq. (24)]. We treat this singularity using the method introduced by Lande, as reported by Kwon and Tabakin.[8] The underlying point is that logarithmic singularities are integrable.

Let us write Eq. (18) as

$$\tilde{\eta}_l(k,p) = -M\alpha\int_0^\infty k'^2dk'\,v_l(k,k')\phi_l(k',p)\ , \tag{59}$$

where

$$\phi_l(k,p) = \tfrac{1}{2}\int_{-1}^{1}dx\left[2P_l(x)\frac{g(|\tfrac{1}{2}\vec{k}+\tfrac{3}{4}\vec{p}\,|)a(|\vec{k}-\tfrac{1}{2}\vec{p}\,|)}{k^2+\tfrac{3}{4}p^2+K^2}+\delta_{l0}\frac{g(k)b(p)}{k^2+\tfrac{3}{4}p^2+K^2}+\frac{\tilde{\eta}_l(k,p)}{k^2+\tfrac{3}{4}p^2+K^2}\right]\ . \tag{60}$$

The method of Lande first involves rewriting Eq. (59) by adding and subtracting a term as follows:

$$\tilde{\eta}_l(k,p)=-M\alpha\left\{\int_0^\infty dk' v_l(k,k')\left[k'^2\phi_l(k',p)-\frac{k^2\phi_l(k,p)}{P_l(\xi)}\right]+k^2\phi_l(k,p)\int_0^\infty dk'\frac{v_l(k,k')}{P_l(\xi)}\right\}. \tag{61}$$

The first integral in Eq. (61) has an integrand that vanishes at $k=k'$, since the quantity in the square brackets goes to zero like $(k-k')$ as $k\to k'$. The second integral in Eq. (61) does possess the logarithmic singularity in its integrand, but we handle this integral analytically. Following Kwon and Tabakin, we write

$$S_l(k)\equiv\int_0^\infty dk'\frac{v_l(k,k')}{P_l(\xi)}\,, \tag{62}$$

$$=\frac{1}{\pi k}\int_0^\infty\frac{dk'}{k'}\frac{Q_l(\xi)}{P_l(\xi)}\,, \tag{63}$$

$$=\frac{1}{2\pi k}\int_0^\infty\frac{dk'}{k'}\ln\left[\frac{\xi+1}{\xi-1}\right]-\frac{I_l(k)}{k}\,, \tag{64}$$

where the integrand of $I_l(k)$ has no singularities along the path of integration:

$$I_l(k)\equiv\frac{1}{\pi}\int_0^\infty\frac{dk'}{k'}\frac{W_{l-1}(\xi)}{P_l(\xi)} \tag{65}$$

and

$$W_{l-1}(\xi)=\sum_{l'=1}^{l}\frac{1}{l'}P_{l'-1}(\xi)P_{l-l'}(\xi)\,, \tag{66}$$

with $W_{-1}=0$. Therefore, we can do the integrals analytically and write

$$S_l(k)=\frac{\pi}{2k}\left[1-\frac{2}{\pi}I_l\right], \tag{67}$$

where

$$I_0=0\,, \tag{68}$$

$$I_2=\sqrt{3/2}\,, \tag{69}$$

and

$$I_4=\frac{\sqrt{7}}{24}\left[\frac{6-\sqrt{10/3}}{(1-\sqrt{3/10})^{1/2}}+\frac{6+\sqrt{10/3}}{(1+\sqrt{3/10})^{1/2}}\right], \tag{70}$$

(we need only $l=0$, 2, and 4 in the present work). Clearly, the problem of the logarithmic singularity has been eliminated.

V. RESULTS

In the solution of Eqs. (16)—(18), we consider two different interaction-parameter sets: a set originally generated by Tabakin[14] and a set used by Kok *et al.*[7] The values of the parameters are given in Table I [note that $Mc^2/(\hbar c)^2=41.5016$ MeV^{-1}fm^{-2}]. The binding energy of the corresponding two-nucleon bound state is given under E_2. At the two-body level, these interactions are not markedly different.

TABLE I. Interaction parameters.

Source	λ (fm^{-3})	β (fm^{-1})	$\lvert E_2\rvert$ (MeV)
Tabakin (Ref. 14)	0.182	1.15	0.413
Kok *et al.* (Ref. 7)	0.1540	1.082	0.4420

First, we solve Eqs. (16)—(18) for the two parameter sets when $\alpha=0$, i.e., without Coulomb interaction. In that case, Eqs. (16)—(18) reduce to a single equation:

$$D(p,K^2)a(p)=8\pi\lambda\int_0^\infty k^2dk\,I(p,k;K^2)a(k)\,. \tag{71}$$

We solve Eqs. (16)—(18) by iteration. The procedure is to estimate a value of K^2; from initial estimates for the $a(k)$, $b(k)$, and $\tilde{\eta}_l(k,p)$ we iterate until the iterates approach a constant ratio, r. If r equals one, then the proper value of K^2 has been obtained. The angular integrals in Eqs. (20), (21), and (60) are done with Gaussian quadrature and all infinite integrations by Gegenbauer quadratures.[15] Besides varying the number of points in the integration grids, we also set tolerances for $r(\Delta r)$ and the sum of the absolute values of the $a(p)$, $b(p)$, and $\tilde{\eta}_l(k,p)$ at the grid points $(\Delta\Sigma)$. The results for $\alpha=0$ are given in Table II.

The results in Table II indicate that the fine structure constant $\alpha=0$ eigenvalue, E_3^0, is given to four significant figures for a grid combination of sixteen Gegenbauer points and ten Gaussian points with tolerances $\Delta r=10^{-6}$ and $\Delta\Sigma=10^{-4}$. For such a grid, the wave-function normalization constant, N_0, is determined to three significant figures. With this wave function, we compute the expectation value of the Coulomb operator:

$$\Delta E_0^C=\left\langle\Psi(\alpha=0)\left|\frac{\alpha}{r_{12}}\right|\Psi(\alpha=0)\right\rangle. \tag{72}$$

ΔE_0^C is determined to within ±5 in the third significant figure (or to better than 0.6%) on a 16—10 grid with 10^{-6}—10^{-4} tolerances.

Next, we solve Eqs. (16)—(18) for the case in which the fine structure constant $\alpha\neq0$. The results in Table III are delineated according to the number of partial waves included in the construction of $\tilde{\eta}(\vec{k},\vec{p})$. To a precision of better than ±5 in the fifth significant figure, the eigenvalue E_3^α is determined by including only the $l=0$ and 2 partial-wave projections of $\tilde{\eta}(\vec{k},\vec{p})$. Likewise, the wave-function normalization is stable to one part in 10^4. In order to determine the contribution to ΔE^C from the second-order perturbation theory, we evaluate the expectation value of the Coulomb operator with the full Coulomb wave function, i.e.,

$$\Delta E_\alpha^C=\left\langle\Psi(\alpha\neq0)\left|\frac{\alpha}{r_{12}}\right|\Psi(\alpha\neq0)\right\rangle, \tag{73}$$

TABLE II. Pure nuclear results ($\alpha=0$).

Model	$\lvert E_3^0 \rvert$ (MeV)	r	Grids Gegenbauer	Grids Gaussian	Tolerances Δr	Tolerances $\Delta\Sigma$	N_0 (fm^{-1})	ΔE_0^c (MeV)
Tabakin	9.236 8	0.999 999 40	10	6	10^{-6}	10^{-4}	0.320 0	0.935 4
	9.237 0	1.000 000 00	16	10	10^{-6}	10^{-4}	0.321 8	0.921 8
Kok *et al.*	8.802 0	1.000 7	10	6	10^{-3}	10^{-3}	0.300 48	0.895 6
	8.809 2	0.999 999 82	10	6	10^{-4}	10^{-3}	0.300 46	0.895 4
	8.809 2	0.999 999 82	10	6	10^{-6}	10^{-3}	0.300 46	0.895 4
	8.810 1	1.000 000 95	10	6	10^{-6}	10^{-4}	0.300 46	0.895 4
	8.802 0	1.000 7	10	10	10^{-3}	10^{-3}	0.300 48	0.895 5
	8.810 2	0.999 999 94	10	10	10^{-6}	10^{-4}	0.300 46	0.895 4
	8.802 00	1.000 8	16	10	10^{-3}	10^{-3}	0.302 29	0.882 4
	8.810 14	1.000 000 95	16	10	10^{-6}	10^{-4}	0.302 28	0.882 2
	8.791 22	1.000 7	16	16	10^{-3}	10^{-3}	0.302 38	0.881 6
	8.809 76	1.000 000 00	16	16	10^{-6}	10^{-4}	0.302 28	0.882 2
	8.809 85	1.000 000 00	24	16	10^{-6}	10^{-4}	0.302 63	0.876 8

where $a(p)$ and $b(p)$ are different due to the Coulomb interaction, and contributions from $\tilde{\eta}(\vec{k},\vec{p})$ are included as well. Clearly, the value of ΔE_α^C is stable with respect to additional partial-wave contributions from $\tilde{\eta}(\vec{k},\vec{p})$. Overall, including only the $l=0$ and 2 partial-wave components of $\tilde{\eta}(\vec{k},\vec{p})$ is sufficient to determine Coulomb effects on E_3, N, and ΔE^C (equations for N and ΔE^C are given in the Appendix).

Finally, we solve Eqs. (16)—(18) in $O(\alpha)$ approximation [see Eq. (25)]. The results are in Table IV. For either the Tabakin or Kok *et al.* models, the eigenvalue $E_3^{O(\alpha)}$ is larger than E_3^α by ~ 0.024 MeV. This small shift indicates the validity of the $O(\alpha)$ approximation for a system like ^{3}He. The error in the energy value compared to the exact result is $<0.4\%$. The very small differences in N and ΔE^C between the exact and $O(\alpha)$ calculations indicate that the wave function does not change appreciably.

What actually happens to the wave function in the three different cases? Basically, except for the change in $K^2=M\,|E_3|$ in the energy denominator, a decrease of $\sim 10\%$, the changes in the spectator functions are minimal as can be seen in Table V. When the fine structure constant $\alpha\neq 0$, the symmetry between $a(p)$ and $b(p)$ is broken. In going to $O(\alpha)$ approximation, $a(p)$ remains essentially unchanged, while $b(p)$ changes slightly. The changes in $b(p)$ are similar to those for $\tilde{\eta}_0(k,p)$ given in Table VI. One might expect a larger change in N from N_0 to N_α since E_3 shifts by 10%, but this is compensated for by the presence of the $\tilde{\eta}(\vec{k},\vec{p})$ component in the wave function. $\tilde{\eta}(\vec{k},\vec{p})$ is a negative function and as such leads to destructive interference terms in the evaluation of N_α, since the Faddeev components are nonorthogonal.

TABLE III. Exact results with Coulomb present.

Model (partial waves included)	$\lvert E_3^\alpha \rvert$ (MeV)	r	Grids Gegenbauer	Grids Gaussian	Tolerances Δr	Tolerances $\Delta\Sigma$	N_α (fm^{-1})	ΔE_α^C (MeV)
Tabakin								
($l=0$)	8.329 0	0.999 999 5	16	10	10^{-6}	10^{-4}	0.332 16	0.901 77
($l=0,2$)	8.325 3	0.999 999 2	16	10	10^{-6}	10^{-4}	0.331 74	0.901 95
($l=0,2,4$)	8.325 0	1.000 000 0	16	10	10^{-6}	10^{-4}	0.331 68	0.901 97
Kok *et al.*								
($l=0$)	7.929 1	1.000 000 9	10	6	10^{-6}	10^{-4}	0.310 16	0.875 67
	7.942 0	1.000 000 9	16	10	10^{-6}	10^{-4}	0.312 05	0.862 78
	7.941 1	1.000 000 0	16	16	10^{-6}	10^{-4}	0.312 06	0.862 70
	7.947 3	0.999 999 6	24	16	10^{-6}	10^{-4}	0.312 39	0.857 58
($l=0,2$)	7.937 90	1.000 000 9	16	10	10^{-6}	10^{-4}	0.311 65	0.862 93
($l=0,2,4$)	7.937 70	1.000 000 0	16	10	10^{-6}	10^{-4}	0.311 60	0.862 94

TABLE IV. Order α results.

Model (partial waves included)	$\lvert E_3^{O(\alpha)} \rvert$ (MeV)	r	Grids Gegenbauer	Grids Gaussian	Tolerances Δr	Tolerances $\Delta\Sigma$	$N_{O(\alpha)}$ (fm^{-1})	$\Delta E_{O(\alpha)}^C$ (MeV)
Tabakin								
($l=0$)	8.3060	0.999 999 8	16	10	10^{-6}	10^{-4}	0.332 42	0.901 36
($l=0,2$)	8.3020	0.999 999 6	16	10	10^{-6}	10^{-4}	0.331 99	0.901 538
($l=0,2,4$)	8.3015	0.999 999 5	16	10	10^{-6}	10^{-4}	0.331 94	0.901 536
Kok *et al.*								
($l=0$)	7.919 1	1.000 000 9	16	10	10^{-6}	10^{-4}	0.312 31	0.862 36
($l=0,2$)	7.916 8	0.999 999 2	16	10	10^{-6}	10^{-4}	0.311 90	0.862 53
($l=0,2,4$)	7.915 1	1.000 000 0	16	10	10^{-6}	10^{-4}	0.311 85	0.862 54

VI. DISCUSSION

Before discussing the physics associated with the results, we compare our calculations with those of Kok *et al.*[7] As can be seen from Table VII the agreement is excellent within the accuracy of either calculation. This serves as a check of both approaches because they are markedly different in their formulation and method. Furthermore, in addition to the analytical proof of equivalence given above, it verifies numerically the equivalence of the $O(\alpha)$ approximation of the present work and the approximation of replacing T^C by V^C in the work of Kok *et al.*

All the major results for the Tabakin and Kok *et al.* models are given in Table VIII (16, 10 grids; 10^{-6}, 10^{-4} tolerances). Scanning the table, it is clear that Coulomb effects for the two models are similar. For example, the expectation value of the Coulomb operator from the wave function without Coulomb effects, ΔE_0^C, is $\sim 10\%$ of the non-Coulomb eigenvalue, E_3^0. The exact eigenvalues, E_3^α, are ~ 23 keV larger in absolute magnitude than the $O(\alpha)$ eigenvalues, $E_3^{O(\alpha)}$. Calculation of the Coulomb-operator expectation value yields essentially the same result whether we use the exact or $O(\alpha)$ wave function. Moreover, from perturbation theory (neglecting third order effects),[16] we can estimate the second-order contribution to the Coulomb energy, $E^{(2)}$, from

$$\Delta E_\alpha^C = \Delta E_0^C + 2E^{(2)} .$$

Within the accuracy of the expectation-value calculations (± 5 keV), we obtain $E^{(2)} = -(10\pm 5)$ keV for both models and a good estimate of the Coulomb energy through second order follows: $\Delta\tilde{E}_e^C$ is approximately equal to ΔE_e^C.

What is the source of the difference between ΔE_e^C and $\Delta\tilde{E}^C$ (or E_3^α and $E_3^{O(\alpha)}$) that leads to $\Delta\tilde{E}^C$ being ~ 23 keV *larger* than ΔE_e^C? Schematically, the equations for the wave function components have the form [see Eqs. (8)—(10)]

TABLE V. Spectator functions without and with Coulomb. (Parameters of Kok *et al.*, Ref. 7.)

p (fm^{-1})	$\alpha=0$ $a(p)=b(p)$	Exact ($\alpha\neq 0, l=0,2$) $a(p)$	$b(p)$
0.1959×10^{-1}	1.000 000 00	1.000 000 00	0.964 077 77
0.5380×10^{-1}	0.990 217 27	0.989 493 79	0.954 439 16
0.1059×10^{0}	0.958 971 56	0.956 045 27	0.923 681 44
0.1794	0.887 544 45	0.880 231 50	0.853 564 50
0.2795	0.759 907 96	0.746 884 88	0.728 889 47
0.4142	0.579 830 11	0.563 062 91	0.554 241 84
0.5957	0.380 980 07	0.365 352 09	0.362 888 75
0.8428	0.209 386 29	0.198 703 47	0.198 862 20
0.1186×10^{1}	0.093 310 24	0.087 880 43	0.088 422 89
0.1679	0.032 343 70	0.030 302 32	0.030 594 21
0.2414	0.008 236 23	0.007 686 78	0.007 777 46
0.3577	0.001 414 85	0.001 316 24	0.001 333 72
0.5574	0.000 142 65	0.000 132 33	0.000 134 29
0.9441	0.000 006 85	0.000 006 34	0.000 006 44
0.1859×10^{2}	0.000 000 11	0.000 000 11	0.000 000 11
0.5104×10^{2}	0.000 000 00	0.000 000 00	0.000 000 00

TABLE VI. $\tilde{\eta}_0(k,p)$, exact, and $O(\alpha)$, ($l=0,2$). (Parameters of Kok *et al.*, Ref. 7.)

p (fm^{-1})	$k=0.1959\times10^{-1}$ (fm^{-1}) Exact	$O(\alpha)$
0.1959×10^{-1}	-0.1107	-0.1156
0.5380×10^{-1}	-0.1093	-0.1141
0.1059×10^{0}	-0.1049	-0.1094
0.1794	-0.949×10^{-1}	-0.9883×10^{-1}
0.2795	-0.7785	-0.8075
0.4142	-0.5525	-0.5700
0.5957	-0.3257	-0.3339
0.8428	-0.1541	-0.1570
0.1186×10^{1}	-0.5632×10^{-2}	-0.5704×10^{-2}
0.1679	-0.1511	-0.1524
0.2414	-0.2802×10^{-3}	-0.2817×10^{-3}
0.3577	-0.3333×10^{-4}	-0.3344×10^{-4}
0.5574	-0.2278×10^{-5}	-0.2282×10^{-5}
0.9441	-0.6988×10^{-7}	0.6992×10^{-7}
0.1859×10^{2}	-0.1001×10^{-8}	-0.1001×10^{-8}
0.5104×10^{2}	-0.6681×10^{-13}	-0.6672×10^{-13}

TABLE VIII. Comparison of models.

Calculated quantity (MeV)	Tabakin	Kok *et al.*
$\lvert E_2\rvert$	0.413	0.442 0
$\lvert E_3^0\rvert$	9.237 0	8.810 1
$\lvert E_3^\alpha\rvert$	8.325 0	7.937 7
$\lvert E_3^{O(\alpha)}\rvert$	8.301 5	7.915 1
ΔE_0^C	0.921 8	0.882 2
ΔE_α^C	0.902 0	0.862 9
$\Delta E_{O(\alpha)}^C$	0.901 5	0.862 5
$\Delta E_e^C=E_3^\alpha-E_3^0$	0.912 0	0.872 4
$\Delta\tilde{E}^C=E_3^{O(\alpha)}-E_3^0$	0.935 5	0.895 0
$\Delta E_{(2)}^C=E_3^\alpha-E_3^{O(\alpha)}$	$-0.023\,5$	$-0.022\,6$
$E^{(2)}=\frac{1}{2}(\Delta E_\alpha^C-\Delta E_0^C)$	$-0.009\,9$	$-0.009\,6$
$\Delta\tilde{E}_e^C=\Delta E_0^C+E^{(2)}$	0.911 9	0.872 6

$$(E_3^\alpha-H_0-V)\,|\,\psi\rangle=V\,|\,\eta\rangle\ , \tag{74}$$

$$(E_3^\alpha-H_0-V^C)\,|\,\eta\rangle=V^C\,|\,\psi\rangle\ , \tag{75}$$

so

$$\Delta E_e^C=\frac{\langle\psi_0\,|\,V\,|\,\eta\rangle}{\langle\psi_0\,|\,\psi\rangle}=\frac{\langle\psi_0\,|\,VG_0(E_3^\alpha)T^C(E_3^\alpha)\,|\,\psi\rangle}{\langle\psi_0\,|\,\psi\rangle}\ , \tag{76}$$

where

$$(E_3^0-H_0-V)\,|\,\psi_0\rangle=0\ . \tag{77}$$

The $O(\alpha)$ equations are of similar form:

$$(E_3^{O(\alpha)}-H_0-V)\,|\,\psi'\rangle=V\,|\,\eta'\rangle\ , \tag{78}$$

$$(E_3^{O(\alpha)}-H_0)\,|\,\eta'\rangle=V^C\,|\,\psi'\rangle\ , \tag{79}$$

thus,

$$\Delta\tilde{E}^C=\frac{\langle\psi_0\,|\,VG_0(E_3^{O(\alpha)})V^C\,|\,\psi'\rangle}{\langle\psi_0\,|\,\psi'\rangle}\ . \tag{80}$$

TABLE VII. Comparison with calculations of Kok *et al.* (Ref. 7).

Quantity	This work	Kok *et al.*
$\lvert E_3^0\rvert$ (MeV)	8.809 8	8.807 6
$\lvert E_3^\alpha\rvert$ (MeV)	7.937 70	7.937 95
$\lvert E_3^{O(\alpha)}\rvert$ (MeV)	7.915 1	7.915 1
$\Delta E_e^C=E_3^\alpha-E_3^0$ (MeV)	0.872 1	0.869 6

We can expand the resolvent $G_0(E_3^{O(\alpha)})$ about E_3^α:

$$G_0(E_3^{O(\alpha)})\cong G_0(E_3^\alpha)+\Delta E_{(2)}^C G_0^2(E_3^\alpha)\ , \tag{81}$$

where $\Delta E_{(2)}^C=E_3^\alpha-E_3^{O(\alpha)}$. Furthermore, the difference between ψ' and ψ is second order in the Coulomb interaction (this fact is manifest in the near equality of the exact and $O(\alpha)$ Coulomb-operator expectation values, normalization constants, and wave-function components—see Tables III, IV, and VI):

$$|\,\psi'\rangle\cong|\,\psi\rangle-G_0TG_0(T^C-V^C)\,|\,\psi\rangle\ , \tag{82}$$

where all arguments are E_3^α. Therefore,

$$\Delta\tilde{E}^C\cong\frac{\langle\psi_0\,|\,VG_0(E_3^\alpha)V^C\,|\,\psi\rangle}{\langle\psi_0\,|\,\psi\rangle} \tag{83}$$

to within second-order corrections [terms in the expression that are proportional to $(V^C)^3$]. Subtracting Eq. (83) from Eq. (76), we obtain the key result

$$\Delta E_{(2)}^C=\Delta E_e^C-\Delta\tilde{E}^C\cong\frac{\langle\psi_0\,|\,VG_0(E_3^\alpha)[T^C(E_3^\alpha)-V^C]\,|\,\psi\rangle}{\langle\psi_0\,|\,\psi\rangle}\ . \tag{84}$$

$\Delta E_{(2)}^C$ is *negative*, because $VG_0(E_3^\alpha)$ is a positive operator (remember that V is purely attractive), ψ_0 and ψ are nodeless in momentum space, and for $E_3^\alpha<0$ and V^C repulsive,

$$0\le\frac{\langle\vec{p}\,'\,|\,T^C(E_3^\alpha)\,|\,\vec{p}\,\rangle}{\langle\vec{p}\,'\,|\,V^C\,|\,\vec{p}\,\rangle}\le1 \tag{85}$$

for all $\vec{p}$ and $\vec{p}\,'$.[17] Clearly, the magnitude of $\Delta E_{(2)}^C$ is determined by second and higher-order Coulomb effects:

$$T^C - V^C = V^C G_0 T^C \,. \tag{86}$$

Though this is the case, $\Delta E^C_{(2)}$ is quite distinct from the standard second-order perturbation energy, $E^{(2)}$. This is evident from Eq. (84) compared to the standard formula for $E^{(2)}$ and from the more than factor of 2 difference between the quantities $\Delta E^C_{(2)}$ and $E^{(2)}$ as seen in Table VIII.

VII. CONCLUSION

Until now, the problem of including the Coulomb interaction in bound-state, momentum-space, three-body problems, when two of the particles are charged, with the aim of obtaining the wave function and energy eigenvalue, has been avoided for the most part. We conjecture that the reasons for this are twofold: (1) The complication associated with the fully off-shell Coulomb t matrix and (2) the concern of dealing with the logarithmic singularity that arises, due to the long-range nature of the Coulomb interaction, in the kernels of the wave-function-component equations. We have shown that (1) can be circumvented and that (2) can be handled elegantly. We have formulated the problem without introducing the Coulomb t matrix, using only the Coulomb potential in momentum space, and demonstrated how the logarithmic singularity can be tamed by the subtraction method of Lande. Without loss of generality, we applied our method to a simple model problem to illustrate the method and discuss the underlying physics, as well as to explore the validity of an obvious approximation. We conclude that the Coulomb interaction can be incorporated easily into momentum-space, three-nucleon, bound-state calculations.

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APPENDIX

In this appendix, we display the formulas used to normalize the wave function and calculate the Coulomb expectation values, Eqs. (72) and (73). The equation for the inverse square of the normalization constant is

$$\begin{aligned} N^{-2} = \int d^3k\, d^3p \Bigg[& \left\{ g^2(k)[2a^2(p)+b^2(p)] + 2g(k)b(p)\tilde{\eta}_0(k,p) + \sum_l (2l+1)\tilde{\eta}_l^2(k,p) \right\} / (k^2+\tfrac{3}{4}p^2+K^2)^2 \\ & + 2 \left\{ g(\,|\,\vec{k}+\tfrac{1}{2}\vec{p}\,|\,)a(p)g(\,|\,\tfrac{1}{2}\vec{k}+\vec{p}\,|\,)[a(k)+2b(k)] \right. \\ & \left. + 2\sum_l (2l+1)P_l\left[\frac{\frac{1}{2}k+px}{|\,\frac{1}{2}\vec{k}+\vec{p}\,|}\right] g(\,|\,\vec{k}+\tfrac{1}{2}\vec{p}\,|\,)a(p)\tilde{\eta}_l(\,|\,\tfrac{1}{2}\vec{k}+\vec{p}\,|\,,k) \right\} / (k^2+p^2+\vec{k}\cdot\vec{p}+K^2)^2 \Bigg] \end{aligned} \tag{A1}$$

where $x=\hat{k}\cdot\hat{p}$. Once N is available, then ΔE^C can be calculated from

$$\begin{aligned} \Delta E^C = \hbar c\alpha (4\pi N)^2 \Bigg[& \int p^2dp\, k^2dk \left[\frac{\pi k}{2\Delta_p^2}\{g^2(k)b^2(p)+[2\tilde{\eta}_0(k,p)+4\tilde{\mathscr{D}}_0(k,p)]g(k)b(p)\} \right. \\ & \left. + \sum_l (2l+1)\frac{\pi k}{2\Delta_p^2}\left[1-\frac{2}{\pi}I_l\right][\tilde{\eta}_l(k,p)+2\tilde{\mathscr{D}}_l(k,p)]^2 \right] \\ & + \int p^2dp\, k^2dk\, k'^2dk' \left[\frac{1}{\Delta_p\Delta_p'}\{g(k)P_0^p(k,k')g(k')b^2(p) \right. \\ & \left. + [2\tilde{\eta}_0(k,p)+4\tilde{\mathscr{D}}_0(k,p)]P_0^p(k',k)g(k')b(p)\} \right. \\ & \left. + \sum_l (2l+1)\left\{ \frac{1}{\Delta_p}[\tilde{\eta}_l(k,p)+4\tilde{\mathscr{D}}_l(k,p)]v_l(k,k') \right. \right. \\ & \left. \times \left[\frac{\tilde{\eta}_l(k',p)}{\Delta_p'} - \frac{k^2\tilde{\eta}_l(k,p)}{k'^2P_l(\xi)\Delta_p} \right] + \frac{4}{\Delta_p}\tilde{\mathscr{D}}_l(k,p)v_l(k,k') \right. \\ & \left. \left. \times \left[\frac{\tilde{\mathscr{D}}_l(k',p)}{\Delta_p'} - \frac{k^2\tilde{\mathscr{D}}_l(k,p)}{k'^2P_l(\xi)\Delta_p} \right] \right\} \right] . \end{aligned} \tag{A2}$$

where the notation follows the text plus

$$\Delta_p = k^2 + \tfrac{3}{4}p^2 + K^2 , \tag{A3}$$

$$\Delta'_p = k'^2 + \tfrac{3}{4}p^2 + K^2 , \tag{A4}$$

$$\mathscr{D}_l(k,p) = \tfrac{1}{2}\int_{-1}^{1} dx\, P_l(x) g(\,|\,\tfrac{1}{2}\vec{k} + \tfrac{3}{4}\vec{p}\,|\,) a(\,|\,\vec{k} - \tfrac{1}{2}\vec{p}\,|\,) , \tag{A5}$$

and

$$P_0^p(k,k') = v_0(k,k')\left[1 - \frac{k'^2 g(k')\Delta_p}{k^2 g(k)\Delta'_p}\right] . \tag{A6}$$

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