Three-particle equations for a model field theory

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An analysis is carried out of an extension of the Lee model which describes the interaction of three fermions, V, N, and W, with a scalar boson θ through the virtual processes $V \rightleftarrows N+\theta$ and $W \rightleftarrows V+\theta$. It is shown that the amplitudes for the physical processes $V+\theta \to V+\theta$ and $V+\theta \to N+2\theta$ can be obtained from the solution of three-particle equations which differ from those of the Amado-Lovelace type as a result of the presence of the absorption channel $V+\theta \to W \to V+\theta$. The techniques used to derive the equations are not peculiar to the model, since they rely mainly on unitarity and analyticity in the subenergy and total energy variables, and hence they can be applied to realistic systems.

[NUCLEAR REACTIONS Modified three-particle equations for $V-\theta$ sector of extended Lee model.]

I. INTRODUCTION

The most general framework for dealing with the interactions of mesons with nuclei is quantum field theory. In principle such a theory contains an infinite number of degrees of freedom, whereas in practice only a finite number are important. Reducing a quantum field theory to manageable proportions is therefore a problem of some interest.

For some model field theories it is possible to carry out an exact reduction of the theory to tractable form for certain processes. Such a theory is the Lee model,[1] which describes the interaction of two fermions, V and N, with a scalar boson θ through the virtual process $V \rightleftarrows N+\theta$. This model is tractable because of the conservation of charge and baryon number, and the lack of antiparticles in the theory. The processes that have been analyzed in some detail in this model are $N-\theta$ scattering[1] and $V-\theta$ scattering.[2] In particular, in Ref. 3 (hereafter referred to as L) it is shown that the amplitudes for the processes $V+\theta \to V+\theta$, $V+\theta \to N+2\theta$, and $N+2\theta \to N+2\theta$ can be obtained from the solution of an Amado-Lovelace[4] type of three-particle equation. The technique used in L to derive this equation depends in an essential way on the restricted nature of the states in each sector of the Lee model, and therefore cannot be used to treat field theories with particles and antiparticles present.

In Ref. 5 (hereafter referred to as E) an extension of the Lee model,[6] which contains an antiparticle $\bar{\theta}$, was analyzed, and an Amado-Lovelace equation for $V-\theta$ scattering was obtained. The technique used in E for deriving this equation is based on a dispersion relation obtained from an exact formal expression for the $V+\theta \to N+2\theta$ production amplitude. The dispersion relation is written in terms of the energy ω of one of the θ particles in the final state. The function dispersed, which is a part of the full production amplitude, has a branch cut for $\omega \geq \mu$, where μ is the θ mass. The discontinuity across the low energy end of the cut ($\mu \leq \omega \leq M_V - M_N + 2\mu$) is related linearly to the function itself. By assuming this discontinuity is valid for all $\omega \geq \mu$, a linear scattering integral equation is obtained. This technique is closely related to an approach used by Amado[7] to derive three-particle equations by imposing subenergy unitarity and analyticity on the isobar expansion for production amplitudes.

It turns out that the technique developed in E is still not general enough to treat actual physical systems. This has to do with the fact that in $V-\theta$ scattering there is no absorption channel present, since conservation of charge prevents a V and a θ from combining to form a V or an N. As we shall see here, the lack of an absorption channel has the consequence that the function dispersed in E vanishes at infinity, and hence no subtraction is necessary. The presence of such a channel introduces an unknown function of the total energy into the dispersion relation.

In order to see how to extend the dispersion relation technique developed in E to allow for the presence of an absorption channel, we shall here analyze an extension of the Lee model considered some time ago by Bronzan and Chen-Cheung.[8] In this extension there is an additional W field introduced so that the basic processes are $V \rightleftarrows N+\theta$ and $W \rightleftarrows V+\theta$. This leads to an absorption channel ($V+\theta \to W \to V+\theta$) in $V-\theta$ scattering. As in the original Lee model, this model is tractable because of conservation of charge and baryon number, and the lack of antiparticles in the theory. In particular, the Hilbert space for $V-\theta$ scattering is spanned by the bare states, $|V\theta\rangle$, $|N2\theta\rangle$, and $|W\rangle$. It is not difficult to extend the derivation in L to include the bare W state, and thereby derive three-particle equations. This will not be done here, since the purpose of the present work is to develop a technique for deriving three-particle equations which is of general validity.

We shall see that the unknown function introduced into the dispersion relation by the absorption channel can be determined from the unitarity relation for the elastic scattering amplitude, and is related, not too surprisingly, to the dressed propagator for the W particle. The expression for the $V-\theta$ elastic scattering amplitude that emerges is similar in structure to that found by other authors[9] for the π-N elastic scattering amplitude, with the important difference that the expression developed here includes the effects of three-particle unitarity. Thus, the present work has the important consequence of suggesting a way of including three-particle effects in existing models of the π-N amplitude; moreover, it provides a method for calculating the production process $N+\pi\to N+2\pi$.

The Hamiltonian for the extended Lee model[8] is given in Sec. II, and its basic features are summarized. In Sec. III the formulas for the $N-\theta$ T matrix are given without derivation, since it turns out that the analysis of the $V-N\theta$ sector is identical to that of the original Lee model.[1] The $N-\theta$ T matrix plays an important role in the three-particle equations for $V-\theta$ scattering developed in Sec. IV. A brief discussion is given in Sec. V.

II. THE MODEL

The Hamiltonian for the model is given by

$$H=H_0+H_1\ , \tag{1a}$$

$$H_0=M_V^{(0)}V^\dagger V+M_W^{(0)}W^\dagger W+M_N N^\dagger N + \int d^3k\, a^\dagger(\vec{k})a(\vec{k})\omega_k\ , \tag{1b}$$

$$H_1=\int d^3k[a(\vec{k})J(k)+a^\dagger(\vec{k})J^\dagger(k)]\ , \tag{1c}$$

where

$$J(k)=u(k)(g_0V^\dagger N+f_0W^\dagger V)\ . \tag{2}$$

Here, $V^\dagger, W^\dagger, N^\dagger$ and V, W, N are creation and annihilation operators, respectively, for the corresponding particles, and obey the usual anticommutation rules for fermions. The bare masses $M_V^{(0)}$ and $M_W^{(0)}$ are renormalized to M_V and M_W by the interaction, while for the N-particle mass there is no renormalization. The operators $a^\dagger(\vec{k})$ and $a(\vec{k})$ create and annihilate θ mesons with three-momentum $\vec{k}$ and energy $\omega_k=(k^2+\mu^2)^{1/2}$, and satisfy the usual commutation rules for bosons; the boson and fermion operators commute with each other. The interaction H_1 contains a cutoff function $u(k)$ and describes the processes

$$V\rightleftarrows N+\theta\ , \tag{3a}$$

$$W\rightleftarrows V+\theta\ , \tag{3b}$$

with bare coupling constants g_0 and f_0, respectively.

It is straightforward to show that the following operators commute with H:

$$B=N^\dagger N+V^\dagger V+W^\dagger W\ , \tag{4a}$$

$$Q=N^\dagger N-W^\dagger W-\int d^3k\, a^\dagger(\vec{k})a(\vec{k})\ . \tag{4b}$$

Clearly B is a baryon number operator and Q is a charge operator. The particles N, V, W, and θ have been assigned the charges 1, 0, -1, and -1, respectively. The existence of these operators and the absence of antiparticles is the reason the model is tractable.

The bare vacuum characterized by

$$V\,|\,0\rangle=W\,|\,0\rangle=N\,|\,0\rangle=a(\vec{k})\,|\,0\rangle=0 \tag{5}$$

is also the physical vacuum, i.e.,

$$H\,|\,0\rangle=0\ . \tag{6}$$

The single particle states

$$|\,\vec{k}\rangle=a^\dagger(\vec{k})\,|\,0\rangle\ , \tag{7a}$$

$$|\,N\rangle=N^\dagger\,|\,0\rangle\ , \tag{7b}$$

satisfy

$$H\,|\,\vec{k}\rangle=\omega_k\,|\,\vec{k}\rangle\ , \tag{8a}$$

$$H\,|\,N\rangle=M_N\,|\,N\rangle\ , \tag{8b}$$

which shows that the θ and the N are not dressed by the interaction.

III. THE $V-N\theta$ SECTOR

The $V-N\theta$ sector is characterized by baryon number 1 and charge 0. The states of interest are the physical V-particle state $|\,V\rangle_+$ and the $N-\theta$ scattering states. Exact expressions for these in terms of the bare states can be found just as in the Lee model, since the W particle has charge -1, and therefore does not mix in. We will not need the expression for the V and $N-\theta$ state vectors, but we will need the results for the $N-\theta$ T matrix. According to Eqs. (32), (49), and (30) of L, this is given by

$$T(p,q;z)=\frac{gu(p)qu(q)}{h(z)}\ , \tag{9}$$

$$h(z)=(z-\Delta)\left[1+(z-\Delta)g^2\int\frac{d^3q\,u^2(q)}{(\omega_q-\Delta)^2(\omega_q-z)}\right]\ , \tag{10}$$

$$g=Z_V^{1/2}g_0\ , \tag{11}$$

$$Z_V^{1/2}=\langle V\,|\,V\rangle_+\ , \tag{12}$$

$$\Delta=M_V-M_N\ . \tag{13}$$

Here g is a renormalized coupling constant and $Z_V^{1/2}$ is the wave-function renormalization constant for the physical V-particle state.

We see that $h(z)$ has a simple zero at $z=\Delta$ and a right-hand cut beginning at $z=\mu$. The discontinuity of $T(p,q;z)$ across this cut leads to the unitarity relation for $N-\theta$ scattering. This relation implies that the on-shell T matrix has the form

$$T(q,q;\omega_q\pm i\epsilon)=-\frac{e^{\pm i\delta(\omega_q)}\sin\delta(\omega_q)}{4\pi^2 q\omega_q},\quad \omega_q\geq\mu\ . \tag{14}$$

IV. THE THREE PARTICLE EQUATIONS

We now turn our attention to $V-\theta$ scattering. From the conservation of charge and baryon number it follows that the possible reactions are

$$V+\theta\to V+\theta \tag{15a}$$

$$\to N+\theta+\theta\ . \tag{15b}$$

The eigenstate of H with energy

$$E=M_V+\omega_k\ , \tag{16}$$

which describes these reactions, we denote by $|\vec{k}V\rangle_+$, where the plus sign indicates an in state. According to Eqs. (26) and (27) of E, the amplitude for the production process (15b) can be obtained by adding to

$$\begin{aligned}{}_+\langle N\,|\,a(\vec{p})J^\dagger(q)\,|\,\vec{k}V\rangle_+ &= gu(q)\delta^3(\vec{p}-\vec{k})\\ &\quad+gu(p)gu(q)F(E+i\epsilon-M_N-\omega_p,E)\ ,\end{aligned} \tag{17}$$

the same expression with the meson momenta $\vec{p}$ and $\vec{q}$ interchanged. Here

$$F(z,E)={}_+\langle N\,|\,j^\dagger\frac{1}{M_N+z-H}j^\dagger\,|\,\vec{k}V\rangle_+\ , \tag{18}$$

with

$$j=\frac{J(p)}{gu(p)}\ . \tag{19}$$

At this point it is possible to either use bare states for intermediate states in (18) and proceed along the lines of L to obtain three-particle equations, or to use physical states and follow E. The second method is preferable in that it does not depend very much on the peculiarities of the model, and thereby illustrates a procedure of general value. Following E we find

$$\lim_{z\to\Delta}(z-\Delta)F(z,E)={}_+\langle V\,|\,j^\dagger\,|\,\vec{k}V\rangle_+\ , \tag{20}$$

which according to Eq. (12) of L is essentially the $V-\theta$ elastic scattering amplitude, and

$$F(\omega_q+i\epsilon,E)-e^{2i\delta(\omega_q)}F(\omega_q-i\epsilon,E)=-2\pi i\frac{g^2u^2(q)}{h(\omega_q+i\epsilon)}\left[\frac{\delta(\omega_q-\omega_k)}{gu(q)}+4\pi q\omega_qF(E+i\epsilon-M_N-\omega_q,E)\right],\quad \omega_q\geq\mu\ . \tag{21}$$

We define

$$G(z,E)=h(z)F(z,E)\ , \tag{22}$$

and find with the help of (9), (10), and (14) that

$$G(\Delta,E)={}_+\langle V\,|\,j^\dagger\,|\,\vec{k}V\rangle_+ \tag{23}$$

and

$$G(\omega_q+i\epsilon,E)-G(\omega_q-i\epsilon,E)=-2\pi ig^2u^2(q)\left[\frac{\delta(\omega_q-\omega_k)}{gu(q)}+4\pi q\omega_q\frac{G(E+i\epsilon-M_N-\omega_q,E)}{h(E+i\epsilon-M_N-\omega_q)}\right],\quad \omega_q\geq\mu\ . \tag{24}$$

Using Cauchy's theorem we can write

$$G(z,E)=G(\infty,E)+\frac{1}{2\pi i}\int_\mu^\infty\frac{d\omega}{\omega-z}[G(\omega+i\epsilon,E)-G(\omega-i\epsilon,E)]\ . \tag{25}$$

The essential difference between the development here and in E is that here $G(\infty,E)\neq0$. At this point this may appear to be of minor importance; however, we shall see that the nonvanishing of $G(\infty,E)$ is owing to the presence of the θ absorption channel, i.e., $V+\theta\to W\to V+\theta$.

If we let

$$X(p,k;E+i\epsilon)=gu(p)G(E+i\epsilon-M_N-\omega_p,E)\ , \tag{26}$$

we find

$$\begin{aligned}X(p,k;E+i\epsilon)&=gu(p)G(\infty,E)+B(p,k;E+i\epsilon)\\ &\quad+\int d^3q\,B(p,q;E+i\epsilon)\\ &\quad\times\frac{X(q,k;E+i\epsilon)}{h(E+i\epsilon-M_N-\omega_q)}\ ,\end{aligned} \tag{27}$$

where

$$B(p,q;z)=\frac{gu(p)gu(q)}{z-M_N-\omega_p-\omega_q}\ . \tag{28}$$

From (26), (16), (23), and (19) it follows that

$$X(k,k;E+i\epsilon)={}_+\langle V\,|\,J^\dagger(k)\,|\,\vec{k}V\rangle_+\ , \tag{29}$$

which is the elastic $V-\theta$ scattering amplitude. If $G(\infty,E)\to0$, then (27) becomes a standard Amado-Lovelace three-particle equation. We now proceed to see what modifications of these equations are brought about by the presence of this additional term.

From Eq. (10) and Ref. 1 we find

$$h(z)\underset{|z|\to\infty}{\longrightarrow}zZ_V\ , \tag{30}$$

and using (22), (18), (19), (2), and (11) we obtain

$$gG(\infty,E)=f\langle W\,|\,\vec{k}V\rangle_+\ , \tag{31}$$

where $|\,W\rangle$ is a bare W-particle state and

$$f = Z_V^{1/2} f_0 \,. \tag{32}$$

Clearly the nonvanishing of $G(\infty,E)$ is owing to the presence of process (3b). If we contract

$$(E-H)\,|\,\vec{k}V\rangle_+ = 0 \tag{33}$$

with $\langle W\,|$ and use (1) and (2), we find

$$\langle W\,|\,\vec{k}V\rangle_+ = \frac{(f_0/g_0)}{E-M_W^{(0)}} \int d^3q_+ \langle N\,|\,a(\vec{q})J^\dagger(q)\,|\,\vec{k}V\rangle_+ \,. \tag{34}$$

If we insert (17) into (34) and use (11), (32), (22), (26), and (31) we obtain

$$g u(p) G(\infty,E) = \frac{fu(p)fu(k)}{E-M_W^{(0)}} + \int d^3q \frac{fu(p)fu(q)}{E-M_W^{(0)}} \frac{X(q,k;E+i\epsilon)}{h(E+i\epsilon-M_N-\omega_q)} \,, \tag{35}$$

which when combined with (27) shows that the modification of the standard three-particle equations owing to (3b) amounts to the addition of an energy dependent separable potential to $B(p,q;z)$.

A somewhat undesirable feature of this potential is that it contains the bare mass $M_W^{(0)}$ and a coupling constant f which is not completely renormalized. Also, the derivation that led to (35) depends very much on the peculiarities of the model under consideration. For these reasons we consider an alternative procedure for obtaining $G(\infty,E)$.

We introduce two auxiliary functions as solutions of the equations

$$Y(p,q;z) = B(p,q;z) + \int d^3x\, B(p,x;z) \frac{Y(x,q;z)}{h(z-M_N-\omega_x)} \,, \tag{36}$$

$$R(p;z) = u(p) + \int d^3q\, B(p,q;z) \frac{R(q;z)}{h(z-M_N-\omega_q)} \,. \tag{37}$$

It is now easily verified that (27) can be rewritten as

$$X(p,k;E+i\epsilon) = Y(p,k;E+i\epsilon) + R(p;E+i\epsilon) f \langle W\,|\,\vec{k}V\rangle_+ \,, \tag{38}$$

where we have used (31).

According to Eq. (7) of L,

$$|\,\vec{k}V\rangle_+ = \left[a^\dagger(\vec{k}) + \frac{1}{E+i\epsilon-H} J(k) \right] |\,V\rangle_+ \,, \tag{39}$$

so that

$$\langle W\,|\,\vec{k}V\rangle_+ = \langle W\,|\, \frac{1}{E+i\epsilon-H} J(k)\,|\,V\rangle_+ \,. \tag{40}$$

If we insert a complete set of physical states ($|\,W\rangle_+$, $|\,V\theta\rangle_+$, $|\,N2\theta\rangle_+$) in (40), we find that $\langle W\,|\,\vec{k}V\rangle_+/u(k)$ is analytic in the complex E plane except for a pole at $E=M_W$ and a right-hand cut beginning at $E=M_V+\mu$. This suggests that we try to find $\langle W\,|\,\vec{k}V\rangle_+$ from a dispersion relation. We shall see that the necessary dispersion relation can be obtained from the unitarity equation for the $V-\theta$ elastic scattering amplitude.

It turns out to be convenient to define a completely off-shell extension of $X(p,k;E+i\epsilon)$ by means of the relation

$$X(p,q;z) = Y(p,q;z) + R(p;z) f^2 G_W(z) R(q;z) \,, \tag{41}$$

where $G_W(z)$ is an unknown function which we shall determine from unitarity. From (28) and (36), it follows that $Y(p,q;z)$ is symmetric in p and q, and hence so is $X(p,q;z)$. If we set $q=k$ and $z=E+i\epsilon$ in (41) and compare with (38), we see that we have written $\langle W\,|\,\vec{k}V\rangle_+$ in a factored form. We will see that the dispersion relation for $G_W(z)$ is quite simple, whereas one for $\langle W\,|\,\vec{k}V\rangle_+$ would be fairly complicated.

In order to make the manipulations we are about to carry out as transparent as possible, it is useful to introduce an operator notation. We write

$$\langle \vec{p}\,|\,X(z)\,|\,\vec{q}\rangle = X(p,q;z) \,, \tag{42}$$

and similarly for $Y(z)$ and $B(z)$; also

$$\langle \vec{p}\,|\,t(z)\,|\,\vec{q}\rangle = \frac{\delta^3(\vec{p}-\vec{q})}{h(z-M_N-\omega_q)} \,, \tag{43}$$

$$\langle \vec{p}\,|\,R(z)\rangle = R(p;z) \,, \tag{44}$$

$$\langle \vec{p}\,|\,u\rangle = u(p) \,, \tag{45}$$

$$\int |\,\vec{p}\rangle d^3p \langle \vec{p}\,| = 1 \,. \tag{46}$$

In this notation (36) and (37) become

$$Y(z) = B(z) + B(z)t(z)Y(z) \tag{47a}$$

$$= B(z) + Y(z)t(z)B(z) \tag{47b}$$

and

$$|\,R(z)\rangle = |\,u\rangle + B(z)t(z)\,|\,R(z)\rangle \,, \tag{48a}$$

$$\langle R(z^*)\,| = \langle u\,| + \langle R(z^*)\,|\,t(z)B(z) \,. \tag{48b}$$

It is easy to see that (47b) is equivalent to (47a) by comparing the iterations of the equations, while (48b) follows from (48a) by simply taking the adjoint. As a shorthand notation, we write

$$X(\pm) = X(E\pm i\epsilon) \,, \tag{49}$$

$$DX = X(+) - X(-) \,, \tag{50}$$

and it will be understood that all discontinuities that we are dealing with are for $E \geq M_V+\mu$.

We will first determine the discontinuity of $|\,R(z)\rangle$. From (48a) and (47), we have

$$[1-B(+)t(+)]D\,|\,R\rangle = [DBt]\,|\,R(-)\rangle \tag{51}$$

and

$$[1+Y(z)t(z)][1-B(z)t(z)] \\ =[1-t(z)B(z)][1+t(z)Y(z)]=1 . \quad (52)$$

Using (52) to solve (51), and doing a little bit of algebra with the help of (47b) we obtain

$$D\,|\,R\rangle=[DB]t(-)\,|\,R(-)\rangle+Y(+)\Gamma(E)\,|\,R(-)\rangle , \quad (53)$$

where

$$\Gamma(E)=Dt+t(+)[DB]t(-) . \quad (54)$$

In exactly analogous fashion it can be shown that

$$D\langle R\,| =-\langle R(-)\,|\,t(+)[DB]-\langle R(-)\,|\,\Gamma(E)Y(-) . \quad (55)$$

It follows from (28) that

$$\langle\vec{p}\,|\,DB\,|\,\vec{q}\rangle=0,\ \ \omega_p=\omega_k \text{ or } \omega_q=\omega_k , \quad (56)$$

so the first terms on the right-hand sides of (53) and (55) vanish on shell.

The discontinuity relations for $X(z)$ and $Y(z)$ are

$$DX=X(+)\Gamma(E)X(-),\ \text{ on shell} , \quad (57)$$

$$DY=Y(+)\Gamma(E)Y(-),\ \text{ on shell} , \quad (58)$$

where "on shell" means, e.g., $\langle\vec{p}\,|\,DX\,|\,\vec{q}\rangle$ with $\omega_p=\omega_q=\omega_k$. The derivation of (58) from (47) is similar to the derivation of (53) from (48a), while (57) could be obtained from (27) and (35). In more realistic field theories it might not be possible to obtain the analog of (35); however, one would still have an equation such as (57), since this relation is simply an expression of unitarity for the elastic scattering amplitude.

In our operator notation (41) becomes

$$X(z)=Y(z)+\,|\,R(z)\rangle f^2G_W(z)\langle R(z^*)\,| . \quad (59)$$

If we insert (59) in both sides of (57) and use (58), (53), (56), and (55) to simplify the result, we obtain

$$DG_W=G_W(+)f^2\langle R(-)\,|\,\Gamma(E)\,|\,R(-)\rangle G_W(-) . \quad (60)$$

The matrix element on the right-hand side of (60) can be written in a more convenient form. If we use (53), (54), (48b), and (52), we find

$$Dt\,|\,R\rangle=[1+t(+)Y(+)]\Gamma(E)\,|\,R(-)\rangle \quad (61)$$

and

$$\langle R(z^*)\,| =\langle u\,|\,[1+t(z)Y(z)] , \quad (62)$$

which combine to give

$$\langle R(-)\,|\,\Gamma(E)\,|\,R(-)\rangle=D\langle u\,|\,t\,|\,R\rangle . \quad (63)$$

In order to solve (60) for $G_W(z)$, we need to know its behavior for large $|\,z\,|$. From (37) we have

$$R(p;z)/u(p) \underset{|z|\to\infty}{\longrightarrow} 1 . \quad (64)$$

Upon comparing (41) and (38), and using (40), we find

$$G_W(E+i\epsilon) \underset{E\to\infty}{\longrightarrow} \frac{\langle W\,|\,J(k)\,|\,V\rangle_+}{fu(k)E} , \quad (65)$$

which can be simplified to

$$G_W(E+i\epsilon) \underset{E\to\infty}{\longrightarrow} E^{-1} , \quad (66)$$

with the help of (2), (12), and (32). Assuming (66) is true for all z and applying Cauchy's theorem to $G_W^{-1}(z)$, we find, using (60) and (63), that

$$\begin{aligned} G_W^{-1}(z)&=z-M_W^{(0)}-\frac{f^2}{2\pi i}\int_{M_V+\mu}^{\infty}\frac{dE}{E-z}D\langle u\,|\,t\,|\,R\rangle \\ &=z-M_W^{(0)}-f^2\langle u\,|\,t(z)\,|\,R(z)\rangle \\ &=z-M_W^{(0)}-f^2\int\frac{d^3p\,u(p)R(p;z)}{h(z-M_N-\omega_p)} . \end{aligned} \quad (67)$$

Here $M_W^{(0)}$ simply plays the role of an arbitrary constant; however, it can be shown to be the bare W-particle mass by turning off the interaction, so we write it as such.

By comparing with the work of Bronzan and Chen-Cheung,[8] it can be shown that $G_W(z)$ is the Fourier transform of the time-dependent W propagator, more precisely,

$$G_W(z)=\langle W\,|\,\frac{1}{z-H}\,|\,W\rangle . \quad (68)$$

The physical W mass M_W can be obtained by solving $G_W^{-1}(z)=0$, and the resulting equation can be used to eliminate $M_W^{(0)}$ from (67). From (41) and (67), it is possible to determine the residue of the W pole in the $V-\theta$ elastic scattering amplitude, and thereby express f in terms of a completely renormalized coupling constant. This has been carried out by Bronzan.[8] It should be noted that his Eq. (18) and Eq. (41) of this work are essentially the same.

V. DISCUSSION

It is worthwhile at this point to briefly summarize and interpret what has been carried out in detail in the preceding section. Recall that when we add to (17) the same expression with the meson momenta $\vec{p}$ and $\vec{q}$ interchanged, we obtain the production amplitude for $V+\theta\to N+2\theta$. On the energy shell $E=M_N+\omega_p+\omega_q$, so that the first argument of $F(z,E)$ in (17) becomes ω_q, the energy of one of the final state θ's. This gives the physical interpretation of the variable z in (18). The analytic structure of $F(z,E)$ in z for fixed E is rather simple; a pole with a residue related to the amplitude for $V+\theta\to V+\theta$ and a right-hand cut for $z\geq\mu$ [see Eqs. (20) and (21)]. By multiplying $F(z,E)$ by the denominator function of the $N-\theta$ T matrix [see Eqs. (9) and (22)], the pole in $F(z,E)$ is removed and a simpler discontinuity relation is obtained [see Eqs. (23) and (24)]. Applying Cauchy's theorem leads to the integral equation (27). In this equation appears the subtraction "constant" $G(\infty,E)$, which, as pointed out previously, is owing to the presence of the absorption channel $(V+\theta\to W\to V+\theta)$. This unknown function of E [see Eqs. (31) and (40) and the remarks following Eq. (40)] has a simple pole at $E=M_W$ and a right-hand cut for

$E \geq M_V + \mu$, and is determined by the unitarity relation for the elastic $V-\theta$ scattering amplitude.

As these remarks should make it clear, by combining subenergy unitarity and analyticity with unitarity and analyticity in the total energy E, we have been led in an unambiguous fashion to Eqs. (36), (37), (41), and (67). With these equations the amplitudes for $V+\theta \rightarrow V+\theta$ and $V+\theta \rightarrow N+2\theta$ can be calculated, as well as the W propagator.

The question arises as to whether or not the techniques developed here are adequate for obtaining three-particle equations for physical systems. The answer is a tentative yes. A derivation of three-particle equations for the π-N system based on the Chew-Low model[10] has almost been completed. Using the techniques of E, the derivation without the inclusion of the absorption channel ($\pi+N \rightarrow N \rightarrow \pi+N$) was completed some time ago. The motivation of the present work was supplied by the need to extend these techniques so as to incorporate the effect of this channel.

It should also be possible to apply the methods developed here to the cloudy bag model[11] of the pion-nucleon system. In this model, as dictated by the quark picture, the Δ resonance plays just as elementary a role as the nucleon. It should be possible to treat the Δ channel ($\pi+N \rightarrow \Delta \rightarrow \pi+N$) just as the W channel has been handled here. We have talked about the W particle as if it were stable, but this need not be the case.

Both the Chew-Low model[10] and the cloudy bag model[11] neglect nucleon recoil. It is of course desirable to develop equations for the π-N system which do not contain this approximation, and especially to see if the techniques used here can be extended to the relativistic domain. Some time ago relativistic three-body equations were developed for the π-N system by Aaron, Amado, and Young.[12] In that work the existence of a linear scattering integral equation was assumed, whereas using the techniques developed here and by other authors,[7,13] it should be possible to derive equations from an underlying quantum field theory Hamiltonian. At the very least, the structure of the equations found here should provide the necessary phenomenology to extend the existing equations[12] so as to include the effect of the direct nucleon pole and the Δ resonance.

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[1]T. D. Lee, Phys. Rev. 95, 1329 (1954); S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961); E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill, New York, 1962).

[2]R. D. Amado, Phys. Rev. 122, 696 (1961).

[3]M. G. Fuda, Phys. Rev. C 25, 1972 (1982), referred to as *L*.

[4]R. D. Amado, Phys. Rev. 132, 485 (1963); C. Lovelace, Phys. Rev. 135, B1225 (1964).

[5]M. G. Fuda, Phys. Rev. C 26, 204 (1982), referred to as *E*.

[6]M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. 124, 1258 (1961).

[7]R. D. Amado, Phys. Rev. Lett. 33, 333 (1974); Phys. Rev. C 11, 719 (1975); 12, 1354 (1975).

[8]J. B. Bronzan, Phys. Rev. 139, B751 (1965); F. S. Chen-Cheung, Phys. Rev. 152, 1408 (1966).

[9]T. Mizutani and D. S. Koltun, Ann. Phys. (N.Y.) 109, 1 (1977); I. R. Afnan and B. Blankleider, Phys. Rev. C 22, 1638 (1980); Y. Avishai and T. Mizutani, Nucl. Phys. A338, 377 (1980); S. Morioka and I. R. Afnan, Phys. Rev. C 26, 1148 (1982).

[10]G. F. Chew, Phys. Rev. 94, 1748 (1954); G. F. Chew and F. E. Low, *ibid.* 101, 1570 (1956); G. C. Wick, Rev. Mod. Phys. 27, 339 (1955).

[11]G. A. Miller, A. W. Thomas, and S. Thebérge, Phys. Lett. 91B, 192 (1980); S. Thebérge, A. W. Thomas, and G. Miller, Phys. Rev. D 22, 2838 (1980).

[12]R. Aaron, R. D. Amado, and J. E. Young, Phys. Rev. 174, 2022 (1968).

[13]J. J. Brehm, Ann. Phys. (N.Y.) 108, 454 (1977); I. J. R. Aitchison and J. J. Brehm, Phys. Rev. D 17, 3072 (1978); 20, 1119 (1979); 20, 1131 (1979); 21, 718 (1980); Phys. Lett. 84B, 349 (1979).