Dynamical groups of liquid drop models

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Two inequivalent quantum mechanical versions of the liquid drop model can be obtained, depending on whether the classical volume conservation condition is imposed to first or second order. These two models have dynamical groups IU(5) and U(6), respectively. The two models are related to each other through standard group contraction and group expansion procedures.

NUCLEAR STRUCTURE Liquid drop model, dynamical symmetry, first and second order volume conservation, inhomogeneous unitary group, contraction and expansion of groups.

I. INTRODUCTION

The rotating-vibrating liquid drop provides a very intuitive geometric basis for understanding collective nuclear motion. Traditionally, classical degrees of freedom are introduced to describe surface deformations, the volume is assumed conserved to lowest nontrivial (first) order, and the degrees of freedom are quantized.[1,2] If, however, volume is assumed conserved to second order, a different and inequivalent version of the quantized liquid drop model results.[3] In the present work, we show that these two distinct quantum versions of the liquid drop model are related by standard group contraction and group expansion procedures.

These two distinct models have as a common basis an expansion of the drop radius in terms of multipole moment degrees of freedom. The volume is then expressed in terms of these degrees of freedom.[1-3]

If volume conservation to first order is imposed, the $L=0$ amplitude is constant and the remaining degrees of freedom are independent. These amplitudes are quantized following a standard prescription.[1,2] If only the quadrupole excitation is important, the only quantum mechanical degrees of freedom are described by the operators $d_M, d_M^\dagger$ with commutation relations

$$[d_M, d_{M'}^\dagger] = \kappa I_{MM'}, \quad \kappa > 0 \ .$$

The dynamical group for this quantum mechanical system is the noncompact group IU(5).[4] It is generated by the 36 operators $d_M^\dagger d_{M'}$, $d_M^\dagger$, $d_{M'}$, and I. These operators act in the infinite-dimensional Hilbert space with basis states

$$|n_2, n_1, n_0, n_{-1}, n_{-2}\rangle, \ n_i \ (\text{integer}) \geq 0$$

(Fock space). Hamiltonians are constructed from the SO(3) scalar operators which can be constructed from the generators of IU(5). This kinematical framework is denoted $V1$ below.

Volume conservation to second order can be imposed by allowing the $L=0$ amplitude to vary.[3] The $L=0$ amplitude and the remaining amplitudes are not independent, but are constrained by a relation of the form

$$s^2 + \sum_{-2}^{+2} |d_M|^2 + \sum_{-3}^{+3} |f_M|^2 + \cdots = (\tfrac{1}{2})^2 \ . \tag{1}$$

These amplitudes are quantized following a standard prescription. If the quadrupole excitation is the only important $L \neq 0$ degree of freedom, the quantum mechanical degrees of freedom are described by the operators s, d_M, $s^\dagger$, and $d_M^\dagger$. The only nonzero commutation relations are $[d_M, d_{M'}^\dagger] = \kappa\delta_{MM'}$, $[s, s^\dagger] = \kappa I$, $\kappa > 0$. The dynamical group for this quantum mechanical system is the compact group U(6). It is generated by the 36 operators $d_M^\dagger d_{M'}$, $d_M^\dagger s$, $s^\dagger d_{M'}$, and $s^\dagger s$.[4] These operators act in the finite-dimensional Hilbert space with basis states

$$|n_s; n_2, n_1, n_0, n_{-1}, n_{-2}\rangle \ ,$$

where n_s, n_i (integer) ≥ 0,

$$n_s + \sum_{-2}^{+2} n_i = N \ ,$$

and N is related to κ. These spaces carry fully symmetric (boson) representations $\{N, \dot{0}\}$ of U(6). Hamiltonians are constructed from the SO(3) scalars that can be constructed from the generators of U(6). This kinematical framework is denoted $V2$ below.

Quantum mechanical systems with kinematical frameworks $V1$ and $V2$ are derived from the classical liquid drop picture with volume conservation imposed to first and second order, respectively. In view of this similarity in origins, we expect the quantum frameworks $V1$ and $V2$ to be closely related. We establish below the relations between $V1$ and $V2$ at the three levels of structure discussed in Ref. 3: (1) commutation relations, (2) Hilbert space, and (3) physical operators.

II. $V2 \rightarrow V1$

A. Commutation relations

The dynamical group U(6) of $V2$ goes, in the limit of a simple Inönü-Wigner contraction,[4,5] to the dynamical group IU(5) of $V1$. To carry out this contraction, we perform a simple scale change $d_M \to d_M$, $s \to cs$ on the operators. This results in the following change of basis in the u(6) algebra

u(6)		iu(5)
$d_M^\dagger d_{M'} \to d_M^\dagger d_{M'}$		$d_M^\dagger d_{M'}$
$d_M^\dagger s \to d_M^\dagger (cs)$	$\xrightarrow{c\to 0}$	$d_M^\dagger$
$s^\dagger d_{M'} \to (cs)^\dagger d_{M'}$		$d_{M'}$
$s^\dagger s \to c^* c s^\dagger s$		I

As long as $c \neq 0$, the change of basis is nonsingular and the structure of the dynamical algebra is unchanged. For $c=0$, the change of basis is singular, and the transformed operators obey iu(5) commutation relations.[4,5]

B. Hilbert space

As the parameter $c \to 0$, the u(6) operators are allowed to act in larger and larger Hilbert spaces $\{N,\dot{0}\}$, with $N \to \infty$. Since the limit of $cs, cs^\dagger$ is I, the matrix elements

$$cs = c\sqrt{n_s}, \quad cs^\dagger = c\sqrt{n_s+1}$$

must have the limit 1 as $N \to \infty$. This is possible if n_i is finite ($i = 0, \pm 1, \pm 2$),

$$n_s = N - \sum_{i=-2}^{+2} n_i \simeq N \to \infty \ ,$$

and $Nc^2 = 1$.[4–6] In this limit ($n_i \ll n_s \simeq N \to \infty$) the basis states have the limit

$$| n_s; n_2, n_1, n_0, n_{-1}, n_{-2} \rangle \to | n_2, n_1, n_0, n_{-1}, n_{-2} \rangle \ ,$$

and the Hilbert space becomes a Fock space with five independent degrees of freedom.

C. Physical operators

The contraction limit of any u(6) operator is obtained by replacing the operators $cs, cs^\dagger$ by the c number 1. For example, the rotationally invariant operator

$$[(d^\dagger s)^{(2)} (s^\dagger \tilde{d})^{(2)}]^{(0)}$$

in u(6) contracts to the rotationally invariant operator

$$[(d^\dagger)^{(2)} (\tilde{d})^{(2)}]^{(0)}$$

in iu(5),

$$c^2 [(d^\dagger s)^{(2)} (s^\dagger \tilde{d})^{(2)}]^{(0)} \xrightarrow{\text{contract}} [(d^\dagger)^{(2)} (\tilde{d})^{(2)}]^0 = d^\dagger \cdot d \ .$$

The contraction of all SO(3) scalars in $V2$ up to fourth order in the operators $s, d, s^\dagger d^\dagger$ to the corresponding SO(3) scalars in $V1$ is summarized in Table I. Here n (n') describes the degree of the operators, and Δ (Δ') is the difference between the number of creation and annihilation operators in the SO(3) invariant. There are nine u(6) operators of degree $n \leq 4$ and thirteen iu(5) operators of degree $n' \leq 4$. Two of the nine u(6) operators [$s^\dagger s$ and $(s^\dagger s)^2$ have trivial contractions to I in iu(5). Two other u(6) operators, $d^\dagger \cdot d$ and $(d^\dagger \cdot d) s^\dagger s$, contract to the same operator $d^\dagger \cdot d$ in iu(5). The nine u(6) operators with $n \leq 4$ give rise, under contraction, to six distinct nontrivial iu(5) operators with $n' \leq 4$. The remaining seven operators in iu(5) with $n' \leq 4$ are obtained by contraction from u(6) operators with $n > 4$. The three iu(5) operators with $n' = 4$, $|\Delta'| = 2$ are not independent, but are proportional to each other. This is the case also for the iu(5) operators with $n' = 4$, $|\Delta'| = 4$. Similar remarks hold for the u(6) operators with $n = 6$, $n = 8$. As a result, both models $V1$ and $V2$ have nine *independent* SO(3) invariant operators of degree less than or equal to four.

A similar analysis can be carried out for transition operators with specific J^π assignments. Table II contains a list of the operators in $V1$ and $V2$ of lowest degree which can be constructed to have the same J^π assignments. This table also indicates the relations among these operators.

III. $V1 \to V2$

A. Commutation relations

The dynamical group IU(5) of $V1$ can be expanded to U(6) following standard group theoretical procedures.[4] In essence, this involves judicious replacement of the c number 1 by creation and/or annihilation operators $\sigma, \sigma^\dagger$:

iu(5)		u(6)
$d_M^\dagger d_{M'}$		$d_M^\dagger d_{M'}$
$d_M^\dagger$	$\longrightarrow$	$d_M^\dagger \sigma$
$d_{M'}$		$\sigma^\dagger d_{M'}$
I		$\sigma^\dagger \sigma$

The operators $\sigma, \sigma^\dagger$ obey $[\sigma, \sigma^\dagger] = \lambda I, \lambda \neq 0$. All 36 operators commute with

$$\sigma^\dagger \sigma + \sum_{M=-2}^{+2} d_M^\dagger d_M \ .$$

The operator $d^\dagger$ is sometimes given a Holstein-Primakoff representation,

$$d^\dagger \simeq (N - \Sigma d_M^\dagger d_M)^{1/2} d^\dagger \ .$$

However, this cannot be done without saying something about the Hilbert space (N) in which these operators act (next level of structure).

B. Hilbert space

The introduction of the operators $\sigma, \sigma^\dagger$ introduces a new quantum number, n_s, required for labeling basis states, $| n_M \rangle \to | n_s; n_M \rangle$. However, the Hilbert space is partitioned into finite dimensional invariant subspaces, as can be seen by the invariance of the number operator

$$\sigma^\dagger \sigma + \sum_{-2}^{+2} d_M^\dagger d_M \ ,$$

whose eigenvalue

$$N = n_s + \sum_{-2}^{+2} n_M$$

remains constant within any invariant subspace. Each such subspace carries a fully symmetric representation $\{N,\dot{0}\}$ of U(6).

C. Physical operators

The iu(5) operators with J^π are easily expanded to u(6) operators with identical spin-parity assignments by includ-

TABLE I. Contraction-expansion relation among rotationally invariant operators. The nine SO(3) invariant scalars in u(6) of degree $n \le 4$ and the thirteen SO(3) invariant scalars in iu(5) of degree $n' \le 4$ are listed. Only nine of these thirteen operators are independent. The contraction-expansion relations among them are indicated. The seven simplest operators in u(6) which contract to the extra iu(5) operators of degree $n'=4$ are also indicated. Here Δ [in u(6)] and Δ' [in iu(5)] indicate the total number of excitations created or annihilated by the operator. $\longrightarrow$ trivial contraction; $\longleftrightarrow$ identical in $V1$ and $V2$; ----> identical contracted limit as a simpler operator; $\rightleftarrows$ unique relation; $\longleftarrow$ iu(5) operator with $n' \le 4$ obtained from u(6) operator with $n > 4$.

n	Δ	u(6)		iu(5)	n'	$\lvert\Delta'\rvert$
2	0	$s^\dagger s$	$\longrightarrow$	I	0	0
2	0	$d^\dagger\cdot d$	$\longleftrightarrow$	$d^\dagger\cdot d$	2	0
4	0	$(d^\dagger\cdot d)(s^\dagger s)$	$--->$	$d^\dagger\cdot d$	2	0
4	0	$(s^\dagger s)^2$	$\longrightarrow$	I	0	0
4	0	$(d^\dagger d^\dagger)^{(0)}(ss)^{(0)}+\text{h.c.}$	$\rightleftarrows$	$(d^\dagger d^\dagger)^{(0)}+\text{h.c.}$	2	2
4	0	$[(d^\dagger d^\dagger)^{(2)}(\tilde{d}s)^{(2)}]^{(0)}+\text{h.c.}$	$\rightleftarrows$	$[(d^\dagger d^\dagger)^{(2)}\tilde{d}]^{(0)}+\text{h.c.}$	3	1
4	0	$[(d^\dagger d^\dagger)^{(L)}(\tilde{d}\tilde{d})^{(L)}]^{(0)}+\text{h.c.}$ $L=0,2,4$	$\longleftrightarrow$	$[(d^\dagger d^\dagger)^{(L)}(\tilde{d}\tilde{d})^{(L)}]^{(0)}$ $L=0,2,4$	4	0
6	0	$[(d^\dagger d^\dagger)^{(2)}(d^\dagger s)^{(2)}]^{(0)}ss+\text{h.c.}$	$\longleftarrow$	$[(d^\dagger d^\dagger)^{(2)}d^\dagger]^{(0)}+\text{h.c}$	3	3
6	0	$(d^\dagger d^\dagger)^{(L)}(d^\dagger\tilde{d})^{(L)}]^{(0)}ss+\text{h.c.}$ $L=0,2,4$	$\longleftarrow$	$[(d^\dagger d^\dagger)^{(L)}(d^\dagger\tilde{d})^{(L)}]^{(0)}+\text{h.c.}$ $L=0,2,4$	4	2
8	0	$[(d^\dagger d^\dagger)^{(L)}(d^\dagger d^\dagger)^{(L)}]^{(0)}(ss)^2$ $L=0,2,4$	$\longleftarrow$	$[(d^\dagger d^\dagger)^{(L)}(d^\dagger d^\dagger)^{(L)}]^{(0)}+\text{h.c.}$ $L=0,2,4$	4	4

ing appropriate $\sigma,\sigma^\dagger$ operators. The expansion is not unique (e.g., $d^\dagger \to d^\dagger s$ and $d^\dagger \to d^\dagger s s^\dagger s$, etc.). The nonuniqueness is drastically reduced by choosing the u(6) operator of lowest degree (smallest n) which can be obtained from an iu(5) operator by expansion. In this case $n=n'+|\Delta'|$. For example,

$$[(d^\dagger d^\dagger)^{(2)}(\tilde{d})^{(2)}]^{(0)} \xrightarrow{\text{expand}} [(d^\dagger d^\dagger)^{(2)}(\tilde{d}\sigma)^{(2)}]^{(0)} .$$

The u(6) operator in this expansion is unique, up to rearrangement into the form in which it is bilinear in the u(6) operators,

$$[(d^\dagger\tilde{d})^{(2)}(d^\dagger\sigma)^{(2)}]^{(0)} .$$

Table I summarizes the expansion of SO(3) scalars of degree $n' \le 4$ in iu(5) to SO(3) scalars in u(6). The expansions include the operators of degree six and eight in u(6) which contract to the seven SO(3) invariants in iu(5) described above.

Table II summarizes the expansion of transition operators with transformation properties J^π in iu(5) to corresponding operators in u(6). We note, in particular, that iu(5) has only one operator $d^\dagger+\tilde{d}$ with $J^\pi=2^+$ of lowest degree $n'=1$, while u(6) has two of lowest degree $n=2$, $d^\dagger s+s^\dagger\tilde{d}$ and $(d^\dagger\tilde{d})^{(2)}$. Under expansion we have the following:

$$d^\dagger+\tilde{d} \xrightarrow{\text{expand}} d^\dagger\sigma+\sigma^\dagger\tilde{d}+\kappa(d^\dagger\tilde{d})^{(2)} .$$

Contraction of the u(6) operator gives uniquely

$$d^\dagger s+s^\dagger\tilde{d}+\kappa(d^\dagger\tilde{d})^{(2)} \to c(d^\dagger s+s^\dagger\tilde{d})+\kappa c(d^\dagger\tilde{d})^{(2)} \xrightarrow{c\to 0} d^\dagger+\tilde{d}$$

since $c(d^\dagger\tilde{d})\to 0$. The presence of two distinct 2^+ operators in u(6) models but of only one 2^+ operator of lowest degree in iu(5) models must be considered a consequence of the volume conservation, finite versus infinite N differences between these two models. The $E2$ operator in iu(5) models can assume the form $d^\dagger+\tilde{d}+\kappa(d^\dagger\tilde{d})^{(2)}$ if the simplicity (lowest n') assumption is relaxed. However, once relaxed, the "rigidity" of the model disappears, and the way is opened to introduce an endless list of parameters to fit data.

IV. REMARKS

The quantum mechanical models $V1$ and $V2$ are related to each other through nontrivial group deformation processes: contraction ($V2 \to V1$) and expansion ($V1 \to V2$). The quantized liquid drop model derived under the assumption of volume conservation to second order has a kinematical framework identical to the interacting boson model.[7] These two models are therefore identical at the

TABLE II. Contraction-expansion relations among SO(3) operators with J^π. The lowest degree transition operators in u(6) and iu(5) with specific spin-parity assignments are listed. These operators are related by the group contraction-expansion procedure. $\longrightarrow$ trivial contraction; $\longleftrightarrow$ identical in $V1$ and $V2$; ....> missing in $V1$ because $d^\dagger+\tilde{d}$ has lower degree; $\rightleftarrows$ unique relation; <-x- not present in u(6) in lowest order.

J^π	u(6)		iu(5)
0^+	$s^\dagger s$	$\longrightarrow$	I
	$d^\dagger\cdot d$	$\longleftrightarrow$	$d^\dagger\cdot d$
		<-x-	$(d^\dagger d^\dagger)^{(0)}+\text{h.c.}$
1^+	$(d^\dagger\tilde{d})^{(1)}$	$\longleftrightarrow$	$(d^\dagger\tilde{d})^{(1)}$
2^+	$d^\dagger s+s^\dagger\tilde{d}$	$\rightleftarrows$	$d^\dagger+\tilde{d}$ $(n'=1)$
	$(d^\dagger\tilde{d})^{(2)}$	.....>	$(n'=2)$
		--x-->	$(d^\dagger d^\dagger)^{(2)}+\text{h.c.}$
3^+	$(d^\dagger\tilde{d})^{(3)}$	$\longleftrightarrow$	$(d^\dagger\tilde{d})^{(3)}$
4^+	$(d^\dagger\tilde{d})^{(4)}$	$\longleftrightarrow$	$(d^\dagger\tilde{d})^{(4)}$
		<--x--	$(d^\dagger d^\dagger)^{(4)}+\text{h.c.}$

level of quantum mechanical models. One can therefore hope that the physical pictures giving rise to these models, surface deformations of an incompressible (to second order) fluid and correlated fermion pairs, will eventually also be shown to be equivalent in the sense that each is responsible for the other. Furthermore, the expansion and contraction relation between quantum liquid drop models $V1$ and $V2$ engenders an identical relation between the Bohr-Mottelson liquid drop model and the interacting boson model.

Similar analysis can be carried out when other degrees of freedom (octopole, hexadecapole) are important. The models $V2$ have one more degree of freedom (i.e., $L=0$) than the models $V1$, but there is one constraint [Eq. (1)] relating this additional degree with the other degrees. Both models therefore have the same number of independent degrees of freedom; the range of variability of these degrees (either classical or quantum) is unbounded in $V1$ and bounded in $V2$. For example, if the $L=2$ and $L=3$ modes are important, $V1$ has dynamical group $IU(5+7=12)$ and $V2$ has dynamical group $U(1+5+7=13)$.

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