PHYSICAL REVIEW C VOLUME 28, NUMBER 1 JULY 1983

Inverse problem for the half-off-shell T matrix

B. Talukdar, K. Niyogi,* and D. K. Ghosh
Department of Physics, Visva-Bharati University, Santiniketan 731235, West Bengal, India
(Received 13 November 1981)

The inverse scattering problem for the off-shell T matrix is formulated within the framework of a combined variable-phase—off-shell scattering theory. A simple expression is constructed for the s-wave half-off-shell T matrix. The input information consists of binding energies and phase shifts at all energies.

[NUCLEAR REACTIONS Scattering theory, off-shell generalization of the variable-phase approach, inverse problem for the T matrix.]

In this report we consider the so-called "inverse" scattering problem[1] and derive a method to generate the two-body half-off-shell T matrix using scattering and bound state information. We believe that our result will represent the half-off-shell T operator to a reasonable approximation. This study is expected to play a role in theories[2] where interparticle forces are eliminated in favor of appropriate T matrices. Our treatment of the problem will be based on an off-energy-shell generalization of the variable-phase approach (VPA) to potential scattering derived by one of us.[3]

Despite its remarkable success in dealing with the usual scattering problem,[4] which consists in constructing the two-body observables from a given law of interaction, the traditional or on-shell VPA has hardly been used to solve the inverse scattering problem except that Babikov[5] cited a very instructive example in respect of this. In the following we quote[4] some of the results of the on-shell VPA which will be useful for our future reference.

Consider the s-wave radial Schrödinger equation for the central potential $v(r)$:

$$\left[\frac{d^2}{dr^2}+k^2-v(r)\right]u(k,r)=0 \ , \tag{1}$$

where the on-shell momentum $k=E^{1/2}$. Introducing the Green's function[1]

$$G(r,r')=\frac{1}{k}\sin k(r-r'), \quad r'<r$$
$$=0, \quad r'>r \ , \tag{2}$$

we convert Eq. (1) to a Volterra integral equation

$$u(k,r)=\sin kr+\frac{1}{k}\int_0^r \sin k(r-r')\,v(r')u(k,r')\,dr' \ . \tag{3}$$

The on-shell VPA is developed by setting

$$\alpha(k,r)\cos\delta(k,r)=1+\frac{1}{k}\int_0^r dr'v(r')\cos kr'u(k,r') \ , \tag{4a}$$

$$\alpha(k,r)\sin\delta(k,r)=-\frac{1}{k}\int_0^r dr'\,v(r')\sin kr'u(k,r') \ . \tag{4b}$$

Here $\delta(k,r)$ and $\alpha(k,r)$ stand for the so-called phase and amplitude functions. The phase function $\delta(k,r)$ satisfies the nonlinear differential equation

$$\delta'(k,r)=-k^{-1}v(r)\sin^2[kr+\delta(k,r)] \ , \tag{5}$$

with the initial condition

$$\delta(k,0)=0 \ . \tag{6}$$

The phase shift $\delta(k)$ for scattering on $v(r)$ is obtained by using the limiting condition

$$\delta(k)=\lim_{r\to\infty}\delta(k,r) \ . \tag{7}$$

The function $\delta(k,s)$ at a distance s from the origin is just the phase shift[4] induced by $v(r)$ amputed of all parts extending beyond s. As opposed to Eq. (5) the amplitude function $\alpha(k,r)$ satisfies a linear differential equation with $\alpha(k,0)=1$. The function $\alpha(k,s)$ is related to the modulus of the Jost function[6] produced by $v(r)$ truncated at s. In terms of $\alpha(k,r)$ and $\delta(k,r)$, the regular solution of the radial Schrödinger equation is written as

$$u(k,r)=\alpha(k,r)\sin[kr+\delta(k,r)] \ . \tag{8}$$

Also we note that[4]

$$\exp\left\{-k\int_s^r dt\cot[kt+\delta(k,t)]\right\}=u(k,s)/u(k,r) \ . \tag{9}$$

The interpolating function for the scattering length satisfies the equation

$$a'(r)=v(r)[r-a(r)]^2 \ , \tag{10}$$

with boundary conditions $a(0)=0$ and $a(\infty)=A_0$, the scattering length. Babikov[5] assumes an interesting form for $a(r)$ given by

$$a(r)=A_0 r(r+A_0)^{-1} \ . \tag{11}$$

Equations (10) and (11) give $v(r)=A_0{}^2 r^{-4}$. Thus in this pedagogic example the dipole polarization potential appears to be determined from knowledge of the interpolating function for the scattering length at all r.

In contrast to the on-shell VPA, the objects of interest of the combined variable-phase—off-shell scattering theory are the interpolating T matrix functions.[3] The off-shell T matrix function $T(p,q,k^2;r)$ satisfies the equation

$$T'(p,q,k^2;r)=\frac{2}{\pi pq}v(r)[\sin pr-\tfrac{1}{2}\pi pT(k,p,k^2;r)e^{ikr}][\sin qr-\tfrac{1}{2}\pi qT(k,q,k^2;r)e^{ikr}] \ , \tag{12}$$

with boundary conditions $T(p,q,k^2;0)=0$ and $T(p,q,k^2;\infty)=$the T matrix. Here p and q are two different off-shell momenta and, $T(k,p,k^2;r)$ and $T(k,q,k^2;r)$ are the half-off-shell T matrix functions. The half-off-shell and on-shell T matrix functions satisfy the equations

$$T'(k,q,k^2;r)=\frac{2}{\pi kq}v(r)[\sin kr-\tfrac{1}{2}\pi kT(k,k,k^2;r)e^{ikr}]\,[\sin qr-\tfrac{1}{2}\pi qT(k,q,k^2;r)e^{ikr}] \ , \tag{13}$$

$$T'(k,k,k^2;r)=\frac{2}{\pi k^2}v(r)[\sin kr-\tfrac{1}{2}\pi kT(k,k,k^2;r)e^{ikr}]^2 \ . \tag{14}$$

Note that both $T(k,q,k^2;r)$ and $T(k,k,k^2;r)$ satisfy boundary conditions similar to those prescribed for $T(p,q,k^2;r)$. We have chosen to work with the normalization

$$T(k,k,k^2;r)=-\frac{2}{\pi k}\sin\delta(k,r)e^{i\delta(k,r)} \tag{15}$$

for the on-shell T matrix function. Eliminating $v(r)$ from Eqs. (13) and (14) and making use of (15), the equation for the half-off-shell T matrix can be integrated to get

$$\begin{aligned}T(k,q,k^2;r)=&-\frac{2}{\pi}e^{i\delta(k,r)}\{k^{-1}\cos qr\sin[kr+\delta(k,r)]-q^{-1}\sin qr\cos[kr+\delta(k,r)]\}\\&+\frac{2}{\pi kq}e^{i\delta(k,r)}(k^2-q^2)\sin[kr+\delta(k,r)]\exp[-f(k,r)]\int\sin qr\exp[f(k,r)]\,dr\\&+C\sin[kr+\delta(k,r)]e^{-i\delta(k,r)}\exp[-f(k,r)] \ ,\end{aligned} \tag{16}$$

where C is a constant of integration and

$$f(k,r)=k\int\cot[kr+\delta(k,r)]\,dr \ .$$

Here we shall restrict ourselves to off-shell amplitudes for scattering on a potential $v(r)=0$ for $r>R$. Since our aim is to derive equations which make no mention of the potential, it is rather unfortunate that we have to make this assumption, and we hope that more delicate analysis would show it to be unnecessary. By invoking the boundary conditions at $r=0$ and $r=R$ and by comparing the on-shell version of the result thus obtained with the value in Eq. (15) at $r=R$, it can be seen that $C=0$.

We have

$$\begin{aligned}T(k,q,k^2)=&-\frac{2}{\pi}e^{i\delta(k)}F(k,q,R)+\frac{2}{\pi kq}e^{i\delta(k)}(k^2-q^2)\sin[kR+\delta(k)]\\&\times\lim_{s\to 0}\left[\exp\left(-k\int_s^R\cot[kr+\delta(k,r)]\,dr\right)\int_s^R\sin qr\exp\left(k\int_s^r\cot[kt+\delta(k,t)]\,dt\right)dr\right] \ ,\end{aligned} \tag{17}$$

where $\delta(k)=\delta(k,R)$ and

$$\begin{aligned}F(k,q,R)=&\,k^{-1}\cos qR\sin[kR+\delta(k)]\\&-q^{-1}\sin qR\cos[kR+\delta(k)] \ .\end{aligned} \tag{18}$$

With the help of Eq. (9), the limit in Eq. (17) can be calculated to write

$$T(k,q,k^2)=-\frac{2}{\pi}e^{i\delta(k)}G(k,q,R) \tag{19}$$

and

$$G(k,q,R)=F(k,q,R)-\frac{k^2-q^2}{kq\,\alpha(k)}\int_0^R\sin qr\,u(k,r)\,dr \ , \tag{20}$$

where $\alpha(k)=\alpha(k,R)$.

In the on-shell limit $q\rightarrow k$, the second term in Eq. (20) goes to zero, while the first term in conjunction with Eq. (19) gives the usual on-shell T matrix. Thus one would expect that dominant contribution to $T(k,q,k^2)$ will come from the term $F(k,q,R)$ when $k^2\approx q^2$. Equation (3) shows that the function $u(k,r)$ satisfies an inhomogeneous Volterra equation. Replacement of Eq. (20) by its first Born term $\sin kr$ gives us

$$G(k,q,R)=F(k,q,R)-\frac{1}{2kq\,\alpha(k)}\{(k+q)\sin[(k-q)R]-(k-q)\sin[(k+q)R]\} \ . \tag{21}$$

Since $\alpha(k)$ is related to the modulus of the Jost function $f(k)$ we can write[1]

$$\alpha(k)e^{-i\delta(k)}=\prod_n\left[1-\frac{E_n}{E}\right]\exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dk'\,\delta(k')}{k-k'}\right] , \tag{22}$$

where E_n stands for the bound state energy and is associated with the simple zeros of $f(k)$ analytically continued in the upper half of the complex k plane. Equations (19), (21), and (22) taken together represent our desired approximate solution of the inverse problem for the half-off-shell T matrix.

This work is based in part on a thesis to be submitted by one of the authors (D.K.G.) to the Visva-Bharati University.

*Permanent address: Department of Physics, Bolpur College, Bolpur 731204, West Bengal, India.

[1] R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

[2] L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Sov. Phys. JETP 12, 1014 (1961)]; A. S. Reiner, Physica (Utrecht) 26, 700 (1960); W. G. Gibson, Phys. Rev. A 6, 2469 (1972); J. F. Reading and J. L. Seigal, Phys. Rev. B 5, 556 (1972); W. J. Titus, Am. J. Phys. 41, 512 (1973).

[3] B. Talukdar, Phys. Lett. 80A, 365 (1980); B. Talukdar, N. Mallick, and D. Roy, J. Phys. G 7, 1103 (1981).

[4] F. Calogero, *Mathematics in Science and Engineering* (Academic, New York, 1967), Vol. 35.

[5] V. V. Babikov, Usp. Fiz. Nauk 92, 3 (1967) [Sov. Phys. Usp. 92, 271 (1967)].

[6] R. G. Newton, J. Math. Phys. (N.Y.) 1, 319 (1960).