Radiative corrections to the end point of the tritium β decay spectrum

Wayne W. Repko*
Particle Theory Group, University of Texas, Austin, Texas 78712
and Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824

Chong-en Wu†
Guangxi University, Nanning, Guangxi, People's Republic of China
and Department of Physics and Astronomy, Michigan State University, East Lansing, Michigan 48824
(Received 1 August 1983)

We compute the effect of radiative corrections on the shape of the end point in tritium beta decay. In order to obtain a correction which is finite relative to the uncorrected spectrum, it is necessary to sum the contribution of soft real photons to all orders. A technique for performing this sum taking into account the constraints of energy conservation is presented. While radiative effects make a relatively small change in the electron spectrum, they could prove important in precise end point measurements designed to determine the neutrino mass.

[RADIOACTIVITY Calculation of radiative corrections to the ^{3}H β spectrum.]

I. INTRODUCTION

Measurements of the beta decay end point of tritium (^{3}H) using a gaseous source[1] of atomic ^{3}H and a solid source[2] of molecular ^{3}H are underway in an effort to measure the neutrino mass m_ν. These experiments should be capable of measuring m_ν if it exceeds 5—10 eV. They are therefore in a position to confirm the value $m_\nu=35$ eV obtained using a source consisting of valine molecules.[3]

Given the potential accuracy of these experiments, we previously examined the effect of Coulomb corrections, weak magnetism, atomic excitations, and nuclear recoil on the ^{3}H electron spectrum for finite neutrino mass.[4] In this paper we present an evaluation of radiative corrections to the beta spectrum. These corrections consist of contributions from virtual as well as real photons. The effect of radiative corrections on the beta decay rate has been known for some time.[5-7] Their effect on the end point shape involves a slight complication because the first order correction contains a logarithmic singularity relative to the uncorrected spectrum at the end point. This type of behavior is known to be a manifestation of the infrared singularity associated with the emission of soft real photons.[8] The singularity can be eliminated by including soft photon contributions to all orders in perturbation theory. When this is done, the logarithmic singularity is converted into a power behavior which is finite at the end point. Other first order terms which are finite at the end point can then be included to obtain the complete radiative correction to the spectrum. For ^{3}H, the size of these effects is quite small. Nonetheless, the corrections could prove important in precise measurements of the beta end point.

In Sec. II, the exponentiation of the soft real photon contribution is derived. The next section uses this result to obtain an expression for the spectrum which is finite at the end point. Finally, the case of ^{3}H is discussed.

II. EXPONENTIATION OF THE SOFT REAL PHOTON CONTRIBUTION

According to the general argument of Yennie, Frauschi, and Suura,[8] the rate for a beta decay accompanied by n soft photons has the form

$$d\tilde{\Gamma}_n=(2\pi)^4\delta^{(4)}(p-p'-p_e-p_\nu-k_1-\cdots-k_n)\frac{\langle |M_0|^2\rangle}{4E_eE_\nu}\frac{(4\pi\alpha)^n}{n!}e^{2I}$$
$$\times\left\langle\prod_{i=1}^{n}\left[\hat{\epsilon}_i\cdot\left[\frac{p_e}{p_e\cdot k_i}-\frac{(Z+1)p'}{p'\cdot k_i}+\frac{Zp}{p\cdot k_i}\right]\right]^2\right\rangle\left[\frac{1}{(2\pi)^3}\right]^{3+n}d^3p'd^3p_ed^3p_\nu\frac{d^3k_1}{2k_1}\cdots\frac{d^3k_n}{2k_n}\ . \qquad (1)$$

Here M_0 is the zeroth order beta decay amplitude; Z is the nuclear charge; the $\hat{\epsilon}_i$, $i=1,\ldots,n$, are photon polarization vectors; and the angular brackets denote a sum over final and average over initial spins. The factor $\exp(2I)$ represents[8] the contribution of soft *virtual* photons to all orders in the fine structure constant α.

When summing over photon spins, it is important to remember that the effect of nuclear recoil can be neglected for present purposes. Explicitly, we will neglect terms of order $|\vec{p}|/M$, where $\vec{p}$ is a typical beta decay momentum and M is a nuclear mass. With this in mind, it is convenient to sum over photon polarizations using the Coulomb gauge. This leads to scalar products of three-momenta divided by various masses. Neglect of recoil implies that only those terms associated with the electron contribute in Eq. (1). Summation over photon spins and integration over d^3p', d^3p_ν, and the directions of $\vec{k}_i$ relative to $\vec{p}_e$ yields

$$d\tilde{\Gamma}_n = d\Gamma_0 \left[\frac{2\alpha}{\pi} t(\beta) \right]^n \frac{e^{2I}}{n!} \times \frac{(\Delta - E_e - k_1 - \cdots - k_n)^2}{(\Delta - E_e)^2} \frac{dk_1}{k_1} \cdots \frac{dk_n}{k_n} , \quad (2)$$

with

$$0 \le k_1 + k_2 + \cdots + k_n \le \Delta - E_e , \quad (3)$$

and $m_\nu = 0$. Here, $d\Gamma_0$ denotes the uncorrected spectrum for a $\frac{1}{2}^+ \to \frac{1}{2}^+$ transition

$$d\Gamma_0 = \frac{G_F^2 \cos^2\theta_c}{2\pi^3} (f_V^2 + 3f_A^2)(\Delta - E_e)^2 p_e E_e dE_e , \quad (4)$$

and $t(\beta)$, $\beta = |\vec{p}_e| / E_e$, is

$$t(\beta) = \frac{1}{2\beta} \ln \frac{(1+\beta)}{(1-\beta)} - 1 = \frac{1}{\beta} \tanh^{-1}\beta - 1 . \quad (5)$$

It is clear from Eq. (2) that the remaining integration over the photon phase space is inferred divergent. This divergence will ultimately cancel with a corresponding divergence in exp($2I$), but it must be regularized to demonstrate the cancellation. It proves convenient to replace the integral over photon phase space by the regularized integral

$$\rho_n(\epsilon) = \mu^{-n\epsilon} \int_0^{\Delta - E_e} dk_1 \cdots \int_0^{\Delta - E_e - k_1 - \cdots - k_{n-1}} dk_n k_1^{\epsilon - 1} k_2^{\epsilon - 1} \cdots k_n^{\epsilon - 1} \frac{(\Delta - E_e - k_1 - \cdots - k_n)^2}{(\Delta - E_e)^2} , \quad (6)$$

where μ is introduced so that $\rho_n(\epsilon)$ has the correct dimensions. After the term in Eq. (6) which is singular for $\epsilon = 0$ is cancelled, the remaining expression will be finite in the limit $\epsilon \to 0$ and independent of μ. By successive changes of variable of the form

$$k_i = (\Delta - E_e - k_1 - \cdots - k_{i-1}) x_i , \quad (7)$$

it is not difficult to obtain

$$\rho_n(\epsilon) = \left[\frac{(\Delta - E_e)^\epsilon}{\mu^\epsilon} \Gamma(\epsilon) \right]^n \frac{\Gamma(3)}{\Gamma(3 + n\epsilon)} , \quad (8)$$

where $\Gamma(z)$ is the usual gamma function. Notice that the infrared singularity has been replaced by a pole in $\Gamma(\epsilon)$. Were it not for the factor $\Gamma(3 + n\epsilon)$ in the denominator, the resulting expression for $d\tilde{\Gamma}_n$ would be such as to give an exponential when $d\Gamma = \sum_n d\tilde{\Gamma}_n$ is computed. The exponentiation can be achieved by making use of the integral representation[9]

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \oint_C \frac{dt}{t^z} e^t . \quad (9)$$

For real z, C is a contour beginning at $-\infty$ below the real axis extending to the right of z and returning to $-\infty$ above the real axis. If z is an integer, the contour can be chosen to be a circle around z, and the formula is obvious. With the aid of Eq. (9), we can write

$$d\Gamma = \sum_n d\tilde{\Gamma}_n = d\Gamma_0 \frac{\Gamma(3)}{2\pi i} \oint_C \frac{dt}{t^3} e^t \exp\left\{ 2I + \frac{2\alpha}{\pi} t(\beta) \frac{(\Delta - E_e)^\epsilon}{\mu^\epsilon t^\epsilon} \Gamma(\epsilon) \right\} . \quad (10)$$

The terms in Eq. (10) which are singular or constant as $\epsilon \to 0$ are

$$\exp\left\{ \frac{2\alpha}{\pi} t(\beta) \left[\Gamma(\epsilon) + \ln \frac{(\Delta - E_e)}{\mu} - \ln t \right] \right\} , \quad (11)$$

and hence we can write the factors associated with soft photon corrections to beta decay as

$$d\Gamma = d\Gamma_0 \left[\frac{\Delta - E_e}{m_e} \right]^{(2\alpha/\pi) t(\beta)} \frac{\Gamma(3)}{\Gamma\left[3 + \frac{2\alpha}{\pi} t(\beta) \right]} \exp\left\{ 2I + \frac{2\alpha}{\pi} t(\beta) \left[\Gamma(\epsilon) - \ln \frac{m_e}{\mu} \right] \right\} . \quad (12)$$

In obtaining Eq. (12), the integral representation Eq. (9) was used to complete the integration over dt.

This procedure can be carried through when $d\Gamma_0$ is replaced by $d\Gamma_0 + d\Gamma_1$, where $d\Gamma_1$ represents the first order corrections to the spectrum which are infrared *finite*.[8] To complete the calculation of the corrected spectrum, it is necessary to obtain these terms and show that the $\Gamma(\epsilon)$ cancels. This is done in the next section.

III. CORRECTED ELECTRON SPECTRUM

The ϵ independence of $d\Gamma$ relies on the cancellation of the $\Gamma(\epsilon)$ term by a similar term in $2I$. To see how this comes about, recall that I is given by

$$I=-\frac{1}{2(2\pi)^4 i}\int\frac{d^4k}{k^2}e^2\left\{(Z+1)\left[\frac{(2p_e+k)_\mu}{k^2+2p_e\cdot k}+\frac{(2p'-k)_\mu}{k^2-2p'\cdot k}\right]^2-Z\left[\frac{(2p_e+k)_\mu}{k^2+2p_e\cdot k}-\frac{(2p+k)_\mu}{k^2+2p\cdot k}\right]^2+Z(Z+1)\left[\frac{(2p'-k)_\mu}{k^2-2p'\cdot k}-\frac{(2p-k)_\mu}{k^2-2p\cdot k}\right]^2\right\}. \quad (13)$$

It should be noted that each squared factor is gauge invariant. The infrared contribution to I is associated with the pole in the photon propagator. Because nuclear recoil can be neglected, it is again convenient to use the Coulomb gauge. In this gauge, the infrared term can be obtained by considering the transverse part of the propagator, since the longitudinal part has no pole. The infrared contribution comes entirely from the electron, and I has the regulated form

$$I=-\frac{\alpha}{\pi}t(\beta)\left[\frac{m_e}{\mu}\right]^\epsilon\frac{\Gamma(\epsilon)}{\Gamma(1+\epsilon)}+\frac{\alpha}{\pi}F\,, \quad (14)$$

where F is infrared finite. By expanding the ϵ dependence of I, the exponential factor in Eq. (12) becomes

$$\exp\left\{\frac{2\alpha}{\pi}F-\frac{2\alpha}{\pi}\gamma_E\right\}, \quad (15)$$

where γ_E is Euler's constant. This shows that $d\Gamma$ is infrared finite. After expanding all terms which are finite at the end point to the first order in α, we find

$$d\Gamma=d\Gamma_0\left\{1+\frac{2\alpha}{\pi}[-\tfrac{3}{2}t(\beta)+F+G]\right\}\left[\frac{\Delta-E_e}{m_e}\right]^{(2\alpha/\pi)t(\beta)}, \quad (16)$$

where $d\Gamma$ has been expressed as

$$d\Gamma_1=\frac{2\alpha}{\pi}G\,d\Gamma_0\,. \quad (17)$$

The first order expressions for $d\Gamma$ which appear in the literature[5–7] also expand the last factor in Eq. (16). This, of course, leads to the troublesome logarithmic end point singularity mentioned in the Introduction. However, it is easy to identify $F+G$ in this manner. Having done so, Eq. (16) provides an expression for the radiative correction to the spectrum which is finite at the end point. Using the results of Ref. 7, the finite corrections to the electron spectrum can be obtained by absorbing constant terms into the coupling constants f_V and f_A. Those correction terms depending on the electron energy are retained as part of the spectrum. The spectrum corrections obtained in this manner are identical to those which correct the vector coupling. These corrections have been shown to be structure independent.[10,11] The corrected spectrum is

$$d\Gamma=d\Gamma_0'(W-\epsilon)^{(2\alpha/\pi)t(\beta)}\left[1+\frac{2\alpha}{\pi}\left\{t(\beta)\left[\ln 2-\frac{3}{2}+\frac{(W-\epsilon)}{\epsilon}\right]+\tfrac{1}{4}[t(\beta)+1]\left[2(1+\beta^2)-2\ln\frac{2}{1-\beta}+\frac{(W-\epsilon)^2}{6\epsilon^2}\right]+\frac{1}{2\beta}\left[L(\beta)-L(-\beta)+L\left[\frac{2\beta}{1+\beta}\right]+\tfrac{1}{2}L\left[\frac{1-\beta}{2}\right]-\tfrac{1}{2}L\left[\frac{1+\beta}{2}\right]\right]\right\}\right], \quad (18)$$

where $W=\Delta/m_e$, $\epsilon=E_e/m_e$, and the prime on $d\Gamma_0$ indicates that f_V and f_A have radiative corrections included. The function $L(\beta)$ in Eq. (18) is the Spence function defined by

$$L(\beta)=\int_0^\beta\frac{dt}{t}\ln(1-t)\,. \quad (19)$$

IV. CONCLUSIONS

The result Eq. (18) holds quite generally. Since $t(\beta)$ can become large, it is possible to obtain a rather substantial power law correction to the shape of the end point. However, in practice, even for large maximum electron energies $\sim$20 MeV the exponent of $(W-\epsilon)$ in Eq. (18) is only $\sim 7\alpha/\pi$. Hence, the shape of the spectrum is not drastically altered, although the finite corrections are not negligible.

In the case of ^3H decay, $\beta\ll 1$. The magnitude of the correction can be obtained by expanding the Spence functions in Eq. (18) to obtain

$$d\Gamma=d\Gamma_0'(W-\epsilon)^{(2\alpha/\pi)t(\beta)}\left[1+\frac{2\alpha}{\pi}\left\{t(\beta)\left[\ln 2-\frac{3}{2}+\frac{(W-\epsilon)}{\epsilon}\right]+\tfrac{1}{4}[t(\beta)+1]\left[2(1+\beta^2)+2\ln(1-\beta)+\frac{(W-\epsilon)^2}{6\epsilon^2}\right]\right.\right.$$
$$\left.\left.-\,2+\tfrac{1}{2}\beta-\tfrac{17}{36}\beta^2+\tfrac{5}{6}\beta^3\right\}\right]. \qquad (20)$$

Experiments to measure the neutrino mass typically look for departures from the simple straight line behavior of the Kurie plot. With this in mind, we have plotted the ratio of the corrected Kurie plot K_γ to the uncorrected version K_0 in Fig. 1. The figure shows that radiative corrections make a percent or so change in the spectrum. However, the variation of the ratio in the last 100 eV of the electron spectrum is very slight even to within 1 eV of the end point. It would appear that radiative corrections do not have a substantial effect on the determination of the end point in ^{3}H beta decay. The major corrections to the spectrum are, therefore, those discussed previously.[4]

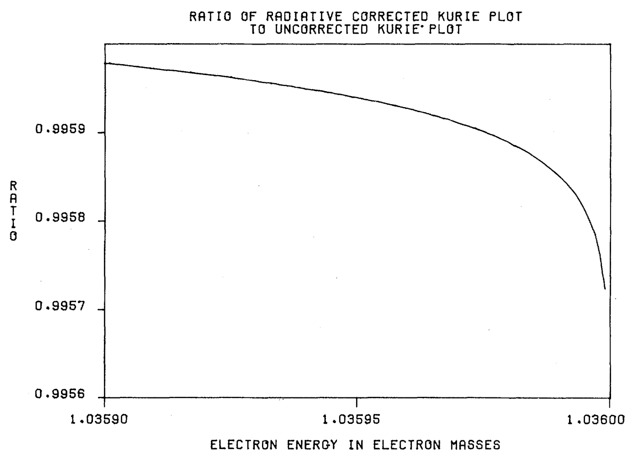

FIG. 1. Ratio of the corrected Kurie plot K_γ to the uncorrected Kurie plot K_0 for E_e within 100 eV of the end point.

ACKNOWLEDGMENTS

We are pleased to acknowledge numerous conversations with Professor D. A. Dicus and Professor O. Fackler. The first author wishes to thank Professor S. Weinberg and the Particle Theory Group of the University of Texas for their support and hospitality. The second author wishes to acknowledge the support of the Department of Physics of Michigan State University. This work was supported in part by the National Science Foundation under Grant No. PHY-81-05020.

*Permanent address: Department of Physics and Astronomy, Michigan State University, East Lansing, MI 48824.

†Permanent address: Guangxi University, Nanning, Guangxi, People's Republic of China.

[1]R. G. H. Robertson *et al.*, in Proceedings of the 1982 Conference on Neutrinos, Balaton, Hungary, 1982, Los Alamos National Laboratory Report LA-UR-82-1728, 1982.

[2]O. Fackler, private communication.

[3]V. A. Lubimov, E. G. Novikov, V. Z. Nozik, E. F. Tretyakov, and V. S. Kosik, Phys. Lett. 94B, 266 (1980).

[4]C.-E. Wu and W. W. Repko, Phys. Rev. C 27, 1754 (1983).

[5]S. M. Berman, Phys. Rev. 112, 267 (1958).

[6]T. Kinoshita and A. Sirlin, Phys. Rev. 113, 1652 (1959).

[7]S. M. Berman and A. Sirlin, Ann. Phys. (N.Y.) 20, 20 (1962).

[8]D. Yennie, S. Frauschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961).

[9]P. Dennery and A. Krzywicki, in *Mathematics for Physicists* (Harper and Row, New York, 1967), p. 94.

[10]E. S. Abers, D. A. Dicus, R. E. Norton, and H. R. Quinn, Phys. Rev. 167, 1461 (1968).

[11]D. A. Dicus and R. E. Norton, Phys. Rev. D 1, 1360 (1970).