πN S-wave field theoretical separable model

K. Nakano
Theoretische Kernphysik, Universität Hamburg, 2000 Hamburg 50, Federal Republic of Germany
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A πN S-wave separable model is developed by implementing most of the field theoretically required analytic properties of the scattering amplitude. In order to improve the low energy behavior of the solution, the S-wave interaction model of Drell *et al.* is incorporated. The resultant dispersion relation is solved using a technique of singular integral equations. Also proposed is an iterative procedure for solving singular integral equations.

[NUCLEAR REACTIONS πN S-wave separable model. Field theoretical properties with Drell *et al.* interaction model. Singular equations by Omnès's method.]

I. INTRODUCTION

Recently a new technique was developed for calculating form factors for a πN P-wave separable model (Ref. 1, hereafter called I). This approach utilizes many of the basic properties of the underlying field theory.[2] We now extend the same technique to calculate form factors for a πN S-wave separable model.

Although the general procedure is the same for both S and P waves, there exists one characteristic difference in the properties of the S wave amplitude,[3] the absence of the nucleon pole term and the accompanying coupling constant at zero energy. In dispersion theoretic approaches, a coupling constant at zero energy is not necessary in the derivation, and hence one arrives at a simpler form of the dispersion relations in the case of S waves. However, one feels less confident in form factors so obtained (in comparison with the P-wave case[1]), since in writing the dispersion relations one less condition from the field theoretic properties[2,3] is used. Furthermore, dispersion relations for S waves are not in the subtracted form, and hence low energy form factors may be unduely influenced by a variation of the high energy phase shift parametrization. A further disadvantage exists in the case of sign changing phase shifts; one ends up with an undetermined constant which is introduced through a rearrangement of the asymptotic behavior[1] of the energy dependent coupling constant.

In order to reduce these ambiguities in the current S-wave model, we appeal to a particular field theoretical model and borrow one of its properties to further specify the form of the dispersion relations. We use the S-wave interaction model proposed by Drell, Friedman, and Zachariasen,[3] particularly their amplitude $a_\alpha(\omega)$. Although the exact solution of $a_\alpha(\omega)$ cannot be obtained easily, its low energy behavior has the simple form

$$a_\alpha(\omega)=c_0+c_1\Gamma_\alpha\omega\ .$$

It has been shown that this low energy approximation is quite accurate.[2,3] Since the roles played by the denominator function $d_\alpha(\omega)$ of a separable model[1] and $-1/a_\alpha(\omega)$ in the t matrix[3] are similar, and since the relationship between the phase shifts and the form factor to $d_\alpha(\omega)$ is the same as the corresponding relationship to $-1/a_\alpha(\omega)$, we implement the requirement

$$d_\alpha(0)=-1/a_\alpha(0)=-1/c_0$$

in the dispersion relations. This is equivalent to the introduction of a coupling constant at zero energy in the P-wave case[1] and thus it eliminates the aforementioned defects of the simpler forms of dispersion relations. We reevaluate c_0 by using more recent and accurate data than were available previously.

In Sec. II the dispersion relations for the πN S-wave interaction will be given. These equations are regarded as integral equations and solved in Sec. III by Omnès's method[4] and by an iterative procedure based on Omnès's method. Numerical results and discussions are presented in the final section.

II. DISPERSION RELATIONS

Using the same notation as in I, the denominator functions for the πN S-wave interaction can be written for each channel,

$$d_1(z)=\underline{1}-\frac{1}{\pi}\int_\mu^\infty dx\,p(x)\left[\frac{v_1^2}{\hat{\eta}_1(x-z)}-\frac{w_1}{x+z}\right], \tag{1a}$$

$$d_3(z)=\frac{z-b}{z-\omega_0}\underline{1}+\frac{1}{\pi(z-\omega_0)}\int_\mu^\infty dx\,p(x)(v_3^2-u_3^2)$$
$$-\frac{1}{\pi(z-\omega_0)}\int_\mu^\infty dx\,p(x)\left[\frac{x-\omega_0}{\hat{\eta}_3(x-z)}v_3^2+\frac{x+\omega_0}{x+z}w_3\right], \tag{1b}$$

where b is a complex constant yet to be determined and $\underline{1}$ has dimensions of μc^2. The indices 1 and 3 refer to the

isospin $T=\frac{1}{2}$ and $\frac{3}{2}$ states, respectively. The $T=\frac{3}{2}$ state phase shifts change sign at $\omega_0=1275$ MeV. The on-shell t-matrix element is parametrized by

$$T_\alpha(\omega)=-\frac{m+\sqrt{\mu^2+k^2}}{\pi mk\sqrt{\mu^2+k^2}}\hat{\eta}_\alpha\sin\hat{\delta}_\alpha e^{i\hat{\delta}_\alpha}$$
$$=-\frac{m+\sqrt{\mu^2+k^2}}{\pi mk\sqrt{\mu^2+k^2}}\frac{v_\alpha(\omega)^2}{d_\alpha(\omega)}\ . \tag{2}$$

We note that the momentum dependence[5] of the t matrix is slightly different from the P-wave case.[1] Equation (1) is the simpler form of the dispersion relation to which we referred in the Introduction. This equation can be further modified by a subtraction at some point where the value of $d(z)$ is known. In I this was done at $z=0$, since $d_\alpha(z)$ was required[6,7] to have a nucleon pole for the P wave at $z=0$, i.e., $d_\alpha(0)=0$. For the S wave a nucleon pole does not exist[3]; however, we can also choose $z=0$ as a subtraction point. This is because we can find a simple low-energy parametrization of the S-wave scattering amplitude which easily suggests the value of $d_\alpha(0)$. Specifically, we turn to the S-wave interaction model proposed by Drell *et al.*[3] According to this model a low energy scattering amplitude can be accurately approximated by

$$a_\alpha(\omega)=c_0^\alpha+c_1^\alpha\Gamma^\alpha\omega\ , \tag{3a}$$

where c_0 and c_1 are constants to be determined from experimental data and

$$\Gamma^\alpha=-1 \text{ for } \alpha=1$$
$$=\tfrac{1}{2} \text{ for } \alpha=3\ . \tag{3b}$$

Because of the relationship

$$a_\alpha(\omega)=-\frac{\sin\delta_\alpha e^{i\delta_\alpha}}{kv^2} \tag{4a}$$

[Eq. (43) of Ref. 3] and

$$d_\alpha(\omega)=\frac{kv_\alpha^2 e^{-i\hat{\delta}_\alpha}}{\hat{\eta}_\alpha\sin\hat{\delta}_\alpha}\ , \tag{4b}$$

which can be easily derived from Eq. (2), we are able to equate at low energies

$$d_\alpha(\omega)=-\frac{\mu c^2}{c_0^\alpha+c_1^\alpha\Gamma^\alpha\omega}\ . \tag{5}$$

Thus this assumption gives the value

$$d_\alpha(0)=-\frac{\mu c^2}{c_0^\alpha}\ . \tag{6}$$

This result is not as precise as in the P-wave case, but it is accurate to the same degree as Eq. (3a) is able to reproduce the phase shifts. In order to determine the value of c_0, we equate Eqs. (3a) with (4a) after expanding the latter in the low energy limit. Thus we have [$v(0)=1$]

$$\delta_\alpha(\omega)=-\frac{k}{\mu c}(c_0^\alpha+c_1^\alpha\Gamma^\alpha\omega),\quad \omega\to 0\ . \tag{7}$$

By using a recent parametrization of the S-wave phase shifts of Rowe *et al.*,[8] we obtain

$$c_0^\alpha=-0.255 \text{ for } \alpha=1$$
$$=0.157 \text{ for } \alpha=3\ . \tag{8}$$

By using this result in Eq. (1), we obtain improved dispersion relations

$$-\frac{c_0^1}{\mu c^2}d_1(z)\equiv g_1(z)=1+\frac{zc_0^1}{\pi\mu c^2}\int_\mu^\infty\frac{dx}{x}p\left[\frac{v_1^2}{\hat{\eta}_1(x-z)}+\frac{w_1}{x+z}\right]=1+\frac{z}{\pi}\int_\mu^\infty\frac{dx}{x}\left[\frac{\mathrm{Im}g_1(x)}{x-z}-\frac{\mathrm{Im}g_1(-x)}{x+z}\right]\ , \tag{9a}$$

$$\frac{z-\omega_0}{\omega_0}\frac{c_0^3}{\mu c^2}d_3(z)\equiv g_3(z)=1+\frac{c_0^3}{\omega_0}z-\frac{z}{\omega_0}\frac{c_0^3}{\pi\mu c^2}\int_\mu^\infty\frac{dx}{x}p\left[\frac{x-\omega_0}{x-z}\frac{v_3^2}{\hat{\eta}_3}-\frac{x+\omega_0}{x+z}w_3\right]$$
$$=1+\frac{c_0^3}{\omega_0}z+\frac{z}{\pi}\int_\mu^\infty\frac{dx}{x}\left[\frac{\mathrm{Im}g_3(x)}{x-z}-\frac{\mathrm{Im}g_3(-x)}{x+z}\right]\ . \tag{9b}$$

Equation (9) and the corresponding crossed channel equation constitute a set of coupled singular integral equations. In the next section we solve these equations by Omnès's method.[4]

Once solutions for g are obtained, the form factors can be calculated from either Eq. (2) or the imaginary part of Eq. (9). For the direct channel we have

$$v_1(\omega)^2=\frac{\hat{\eta}_1(\omega)\sin\hat{\delta}_1(\omega)\mu}{kc_0^1}e^{i\hat{\delta}_1(\omega)}g_1(\omega)\ , \tag{10a}$$

$$v_3(\omega)^2=-\frac{\omega_0}{\omega_0-\omega}\frac{\hat{\eta}_3(\omega)\sin\hat{\delta}_3(\omega)\mu}{kc_0^3}e^{i\hat{\delta}_3(\omega)}g_3(\omega)\ . \tag{10b}$$

III. SINGULAR INTEGRAL EQUATIONS

In order to solve Eq. (9), we introduce[1,7] the phases across the unitarity cut $\hat{\delta}$ and crossing cut φ:

$$d_\alpha(\omega)=|d_\alpha(\omega)|e^{-i\hat{\delta}_\alpha(\omega)}\ , \tag{11a}$$

$$d_\alpha(-\omega)=|d_\alpha(-\omega)|e^{-i\varphi_\alpha(\omega)}\ . \tag{11b}$$

Since the phases of d_α and g_α are the same, Eq. (9) can be written in the following way:

$$g_1(z)=1+\frac{z}{\pi}\int_\mu^\infty\frac{dx}{x}\frac{\sin\varphi_1 e^{i\varphi_1}}{x+z}g_1(-x)$$
$$-\frac{z}{\pi}\int_\mu^\infty\frac{dx}{x}\frac{\sin\hat{\delta}_1 e^{i\hat{\delta}_1}}{x-z}g_1(x)\ . \tag{12}$$

We have explicitly written the results only for the $\alpha=1$ channel; the extension to the $\alpha=3$ case is obvious. Equation (12) is in the same form as the previous P_{33} wave interaction[9] and can be solved by Omnès's method,[4] which converts the singular integral into a Hilbert problem.[10] The final solution is not unique, as any entire function may be added to it. However, we follow the usual custom and neglect this possibility. The coupled channel integral equation (12) and its crossed counterpart can be converted into Fredholm-type integral equations[4,9]:

$$g_1(\omega)\equiv e^{-i\hat{\delta}_1(\omega)}f_1(\omega)\,,$$

$$f_1(\omega)=\cos\hat{\delta}_1(\omega)-\frac{\omega e^{-p(\omega)}}{\pi}\mathrm{P}\int_\mu^\infty \frac{dx}{x}\frac{\sin\hat{\delta}_1(x)}{x-\omega}e^{p(x)}$$
$$+\frac{\omega}{\pi}\int_\mu^\infty \frac{dx}{x}\sin\varphi_1(x)f_1(-x)\left[\frac{\cos\hat{\delta}_1(\omega)}{\omega+x}-\frac{e^{-p(\omega)}}{\pi}\mathrm{P}\int_\mu^\infty dy\frac{\sin\hat{\delta}_1(y)e^{p(y)}}{(y+x)(y-\omega)}\right], \tag{13a}$$

where

$$p(x)=\frac{P}{\pi}\int_\mu^\infty dy\frac{\hat{\delta}_1(y)}{y-x}\,, \tag{13b}$$

and P indicates that the Cauchy principal value is to be used. Equations (13) are nonsingular, coupled integral equations and can be easily solved by standard techniques.

Equation (12) and its crossed counterpart are almost symmetric with respect to each other in their energy dependence, the only difference being in the phases. The crossing phases are assumed to be determined by the static crossing relation[1,9]

$$\frac{1}{d_\alpha(-\omega)}=\sum_{\beta=1}^{2}\frac{A_{\alpha\beta}}{d_\beta(\omega)}\,, \tag{14a}$$

specifically,

$$\tan\varphi_1(\omega)=\frac{B_1(\omega)}{D_1(\omega)}\,,$$

$$B_1(\omega)=\frac{A_{11}}{f_1(\omega)}\sin\hat{\delta}_1(\omega)-\frac{\omega-\omega_0}{\omega_0}\frac{c_0^3}{c_0^1}\frac{A_{12}}{f_3(\omega)}\sin\hat{\delta}_3(\omega)\,,$$
$$D_1(\omega)=\frac{A_{11}}{f_1(\omega)}\cos\hat{\delta}_1(\omega)-\frac{\omega-\omega_0}{\omega_0}\frac{c_0^3}{c_0^1}\frac{A_{12}}{f_3(\omega)}\cos\hat{\delta}_3(\omega)\,, \tag{14b}$$

where the crossing matrix A is given by[11]

$$A=\tfrac{1}{3}\begin{bmatrix}-1 & 4\\ 2 & 1\end{bmatrix}. \tag{14c}$$

Equation (14a) is a complex expression and hence provides essentially two conditions, one for $\varphi(\omega)$ and another for $f(-\omega)=e^{i\varphi(\omega)}g(-\omega)$. Since Eq. (9) was derived by only requiring it to be crossing symmetric, without specifying its exact content as in Eq. (14a), and since Eq. (12) is solved by using only half of the crossing relations, Eq. (14b), the solution $f(-\omega)$ obtained by solving the coupled integral equations (12) may be different from that which is obtained from Eq. (14). Therefore it is interesting to compare the two sets of solutions. Since Eq. (14) does not constitute coupled equations, we will develop an iterative method to obtain solutions.

(a) The direct part of the first set of iterative equations can be obtained from Eq. (9) by neglecting the contribution from the crossed channel. The solution for this approximate equation can be obtained from Eq. (13) by setting $f(-\omega)=0$. Then the crossed part of the iterative solution can be obtained by substituting the approximate direct channel solution so obtained into the crossed counterpart of Eq. (13), the angle φ being calculated through Eq. (14b) using the approximate direct channel solution. The next step of the iteration procedure begins with the substitution of the approximate crossed channel solution $f(-\omega)$ into Eq. (13). Thus one is not required to solve any integral equations. After each iteration the solution is explicitly given. We found that three iterations were sufficient. (b) The second set of iterative solutions can be obtained by a similar procedure, except that each time $f(-\omega)$ is to be calculated from Eq. (14a) by

$$f_1(-\omega)=\frac{D(\omega)\cos\varphi_1(\omega)+B(\omega)\sin\varphi_1(\omega)}{B(\omega)^2+D(\omega)^2}\,. \tag{14d}$$

Here we also found that three iterations were sufficient. This second type of iterative procedure has been extensively used by Ernst *et al.*[12] in solving the Chew-Low equation.[2]

As is clear in the above procedure, the direct channel solution of these iterative methods does not require crossing phases. Hence one need not base the initial solution on the static solution[9] as in the case of Omnès's solution, Eq. (13), but need only assume that the contribution from the crossed channel is smaller[9] than that from the direct channel.

IV. RESULTS AND DISCUSSIONS

The only input for the calculation, the empirical complex phase shifts, are taken from the Rowe *et al.* parametrization[8] for $T\pi(\mathrm{lab})<400$ MeV, the Almehed-Lovelace compilation[13] for 400 MeV $<T\pi<1436$ MeV, and the Landau-Tabakin Regge tabulation[14] for $T\pi>1440$ MeV. The real part of phase shifts are uniformly damped at higher energy by multiplying the factor

$$1-\frac{\omega-1000\ \mathrm{MeV}}{9340\ \mathrm{MeV}-1000\ \mathrm{MeV}}\quad\text{for }\omega>1000\ \mathrm{MeV}\,. \tag{15}$$

The damping ensures an acceptable solution. Furthermore, the phase shifts are cut off above $\omega=9340$ MeV

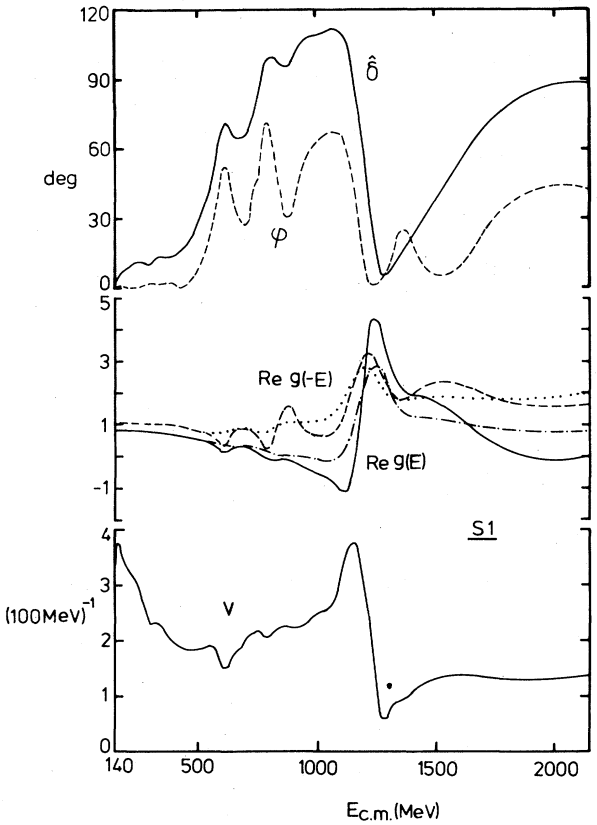

FIG. 1. Calculated results by Omnès's method and by the first iterative procedure. See the text. Upper graph: phases $\hat{\delta}$ (solid line) and φ (dashed line) versus center of mass energy ω. Middle graph: self-consistency check for the real part of Eq. (12). The solid and dashed dotted lines are the left- and right-hand sides of Reg(ω); the dashed and dotted lines are the left- and right-hand sides of Reg$(-\omega)$. Lower graph: redefined form factor V.

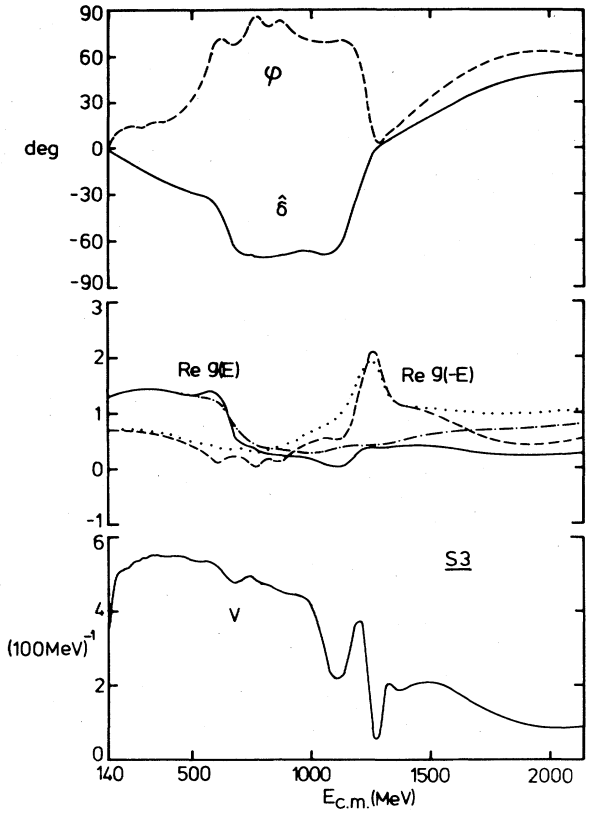

FIG. 2. Same as in Fig. 1.

($E_{\text{c.m.}}=10$ GeV). The phase shifts $\hat{\delta}$ vary rapidly near ω_0 and the $T=\frac{3}{2}$ phase shifts also change sign. This variation has the effect of producing kinky structures in the solution. Therefore the phase shifts near ω_0 were artificially smoothed. This has a negligible effect on lower energy solutions. The crossing phase always has an ambiguity of $\pm\pi$,[1] as is clear from the definition of Eq. (14b). Hence we first set $\varphi(\mu)=0$ and choose $\varphi(\omega)$ by adjusting $\pm\pi$ so that it varies smoothly as a function of its argument. This phase was also damped at higher energy by the same factor, Eq. (15).

The off-shell t-matrix element and form factors were redefined in the following way for convenience:

$$T_\alpha(k',k;\omega)=-\frac{1}{\pi m}\left[\frac{(m+\sqrt{\mu^2+k'^2})(m+\sqrt{\mu^2+k^2})}{\sqrt{\mu^2+k'^2}\sqrt{\mu^2+k^2}}\right]^{1/2}\frac{v_\alpha(k')v_\alpha(k)}{d_\alpha(\omega)}$$

$$\equiv-\frac{V_\alpha(k')V_\alpha(k)}{d_\alpha(\omega)}\,, \tag{16a}$$

$$V_1(k)=\left[\frac{\mu(m+\sqrt{\mu^2+k^2})}{\pi mk\sqrt{\mu^2+k^2}}\frac{\hat{\eta}_1(\omega)\sin\hat{\delta}_1(\omega)f_1(\omega)}{-c_0^1}\right]^{1/2}, \tag{16b}$$

$$V_3(k)=\left[\frac{\mu(m+\sqrt{\mu^2+k^2})}{\pi mk\sqrt{\mu^2+k^2}}\frac{\hat{\eta}_3(\omega)\sin\hat{\delta}_3(\omega)f_3(\omega)}{c_0^3}\frac{\omega_0}{\omega-\omega_0}\right]^{1/2}. \tag{16c}$$

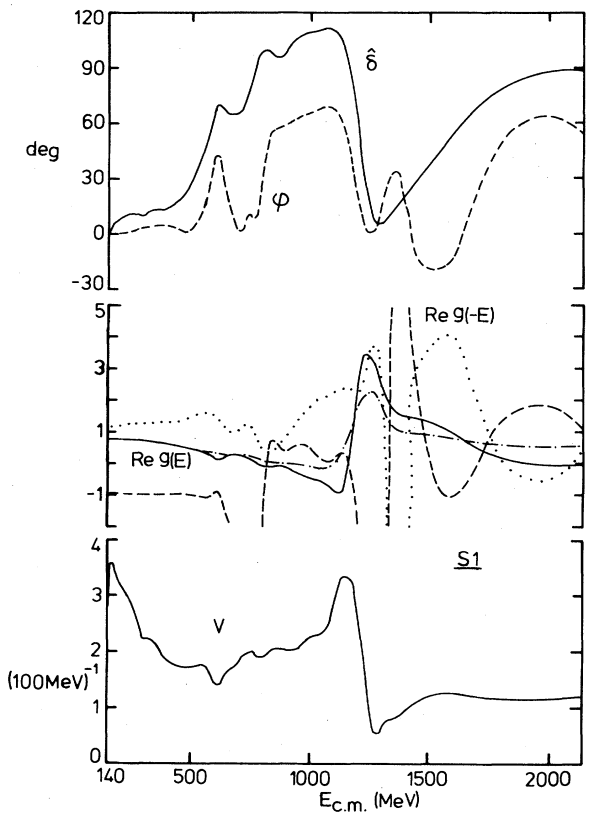

FIG. 3. Calculated results by the second iterative method. See the caption to Fig. 1.

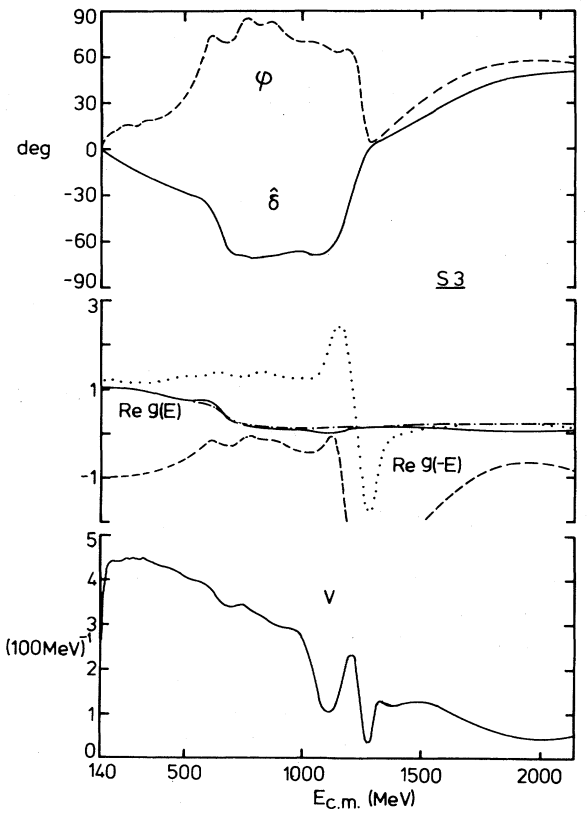

FIG. 4. Same as in Fig. 3.

In Figs. 1 and 2 calculated results using Omnès's method are displayed. In each graph the upper and lower parts show the phases $\hat{\delta}$ and φ, and the redefined form factor V_α, Eq. (16). The middle part compares the real part of both sides of Eq. (12). Since the form factors are determined by the experimental phase shifts and the solution for $f(\omega)$, the only test to examine the goodness of a solution is to see if a solution obtained from solving Eq. (12) can really reproduce itself when it is substituted back into the original equation. The imaginary part of Eq. (12) gives an identity relationship. As one can see, the solution is self-consistent as well as crossing symmetric for the energy region $\omega < 500$ MeV, the physically most interesting domain. However, at high energies there is a considerable discrepancy. Particularly around 1200–1300 MeV there is rapid variation, reflecting the rapid change in the input phase $\hat{\delta}$. Also sizeable fluctuations in $\hat{\delta}$ produce unavoidable wiggles in the solution. Hence it is clear that the solution would become much smoother and self-consistency improved if the phase shifts in the high energy domain were parametrized by a smooth form as in the low energy case.[8]

The results of the first set of iterative solutions are almost identical to those of Figs. 1 and 2. Except for small changes in the crossing phases at higher energies, they appear as in Figs. 1 and 2. This fact demonstrates that the alternative iterative procedure discussed in the previous section gives a very accurate approximation.

On the contrary, the second iterative procedure which uses the crossing relation only does not give a self-consistent solution. As Figs. 3 and 4 exhibit, even though good self-consistency is obtained for the direct part of the $S3$ wave, in general self-consistency and hence crossing symmetry are badly broken. In particular, the crossing solution shows excessive variation when input phases undergo sizeable changes. This is clear since the second iterative method does not include the smoothing process of integration as in the first method. The integrated real crossed solution of the second iterative procedure has a phase difference of π from the real part of the crossed solution itself for a wide energy region. Since the latter is proportional to $\cos\varphi$, it appears that the change in definition of φ by $\varphi \rightarrow \pi - \varphi$ may solve the major part of the discrepancy. However, it so happens that whenever $\cos\varphi$ changes sign, the integrated crossed solution also changes its sign. Thus we end up with a similar discrepancy for the crossed part, as is depicted in Figs. 3 and 4.

In summary we have developed a πN S-wave separable model which incorporates many of the field theoretical properties required for the scattering amplitude, and demonstrated its adequacy in the low energy domain, $\omega < 500$ MeV. Also an effective iterative method was developed, dispensing with a subtle interchangeability[4] of two semi-infinite integrals and having to choose the initial crossing phase. This work follows the previous work for the P-wave interaction,[1,9] and it is hoped that the tech-

nique of treating the singular integral equations may be extended into other areas, since the inverse scattering problem[1,5–7] does not necessarily guarantee that a solution obtained by that method will always reproduce the input phase shifts.

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