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Term-by-term bosonization method

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We propose here a new method, called a term-by-term bosonization method, to be used to construct the boson expansion theory. It is trivial to justify it mathematically. Its use makes the whole procedure of the bosonization very easy and transparent.

[NUCLEAR STRUCTURE Boson expansion theories, Marumori-Yamamura-Tokunaga method, term-by-term bosonization method, fermion system, norm matrix, reduction to irreducible tensors, truncation to collective space.]

I. INTRODUCTION

We have shown in a series of papers[1] that the boson expansion theory (BET), which is a microscopic theory, is capable of explaining a variety of nuclear collective motions. The formalism we used for our calculations was given some time ago by Kishimoto and Tamura in Refs. 2 and 3 (to be referred to as KT-1 and KT-2, respectively) by extending the work of Belyaev and Zelevinsky[4] and of Sorensen.[5]

During the past three years or so, we undertook renewed formal investigations, with the purpose of putting our earlier formalism on a firmer basis. The first results of these renewed investigations were reported by Tamura, Weeks, and Pedrocchi (TWP).[6] Based on the ideas discussed in TWP, Kishimoto and Tamura (KT-3) (Ref. 7) very recently worked out a new formalism of the BET which we believe has given a convincing justification of our previous formalism, and in turn of our numerical analyses.[1] In developing the formalism of KT-3, we combined the ideas of TWP with the techniques of Marumori, Yamamura, and Tokunaga (MYT).[8]

The algebra in KT-3 was rather involved, and this was thought unavoidable, because quite general fermion systems were treated rigorously. However, very recently, we found that a new method, which may be called a term-by-term bosonization (TTB) method, could be introduced so as to simplify (at least part of) the KT-3 algebra significantly. The purpose of the present paper is to present and discuss this TTB method. We used the MYT method in KT-3, because it was found to be easier to use than the commutator method, which we had used earlier.[2,3] In the present paper, it will be seen that the TTB method is even easier to use.

In KT-3, we also discussed the case in which the starting fermion system was simplified. More specifically, we considered the case in which the fermion space was truncated to the one that was spanned by (products of) one kind of collective fermion pair of a quadrupole nature. We showed that the boson images of the fermion pair operators were then given in very simple (Taylor series) forms. In the present paper, we also discuss this truncated case, and show that the same results as were obtained in KT-3 are derived in a very simple and transparent manner, once the TTB method is employed.

In Sec. II, we first briefly recapitulate the essence of the KT-3 formalism, and then explain the TTB method by applying it to the bosonization of very general fermion systems. In doing this, we use the notation of KT-3 without redefining it, and thus the reader is recommended to go through Sec. II A of KT-3 for notation before proceeding to Sec. II of the present paper. The use of the TTB method for the case with truncated fermion systems is explained in Sec. III. We try to make this section as self-contained as possible, so as to allow the reader to get the whole picture of the BET with the TTB method at a glance. We can attempt this, because the algebra needed to complete the entire derivation of the formalism is rather short. Finally, in Sec. IV, we summarize and discuss what had been achieved in Secs. II and III.

II. INTRODUCTION OF THE TTB METHOD

The starting point of KT-3 was to introduce[8] the Usui operator U defined in the TDR (Tamm-Dancoff representation) as[9]

$$U=\sum_{n=0}^{N}\sum_{(a)} |\, n;a)(n;a\,| \ , \tag{1}$$

where

$$(a)\equiv\{a_1,a_2,\ldots,a_n\} \ ,$$

a_i denoting a TD component. Further,

$$\begin{aligned}|\, n;a) &=(1/\sqrt{n!})(A_a^\dagger)^n\,|\,0)\\ &\equiv(1/\sqrt{n!})A_{a_1}^\dagger\cdots A_{a_n}^\dagger\,|\,0) \ ,\end{aligned} \tag{2}$$

which is a normalized ideal-boson state, $A_{a_i}^\dagger$ being a boson creation operator, and $|0)$ denoting the boson vacuum.

The fermion state $|\, n;a\,\rangle$ is defined by

$$|\, n;a\,\rangle=\sum_{(b)}(Z_n^{-1})_{a;b}\,|\, n;b\,\rangle\rangle \ . \tag{3}$$

Here, by denoting by $B_{a_i}^\dagger$ the operator creating the a_i component of the fermion pair in the TDR, and by $|0\rangle$ the fermion vacuum, we first constructed

$$\begin{aligned}|\, n;a\,\rangle\rangle &=(1/\sqrt{n!})(B_a^\dagger)^n\,|\,0\rangle\\ &\equiv(1/\sqrt{n!})B_{a_1}^\dagger\cdots B_{a_n}^\dagger\,|\,0\rangle \ ,\end{aligned} \tag{4}$$

just as we did $|\, n;a)$. We then obtained $(Z_n^{-1})_{a,b}$ as the inverse of the square root of the norm matrix

$$(Z_n^2)_{a;b}=(Z_n^2)_{b;a}$$

defined by

$$\langle\langle n;a \mid m;b\rangle\rangle=(Z_n^2)_{a;b}\delta_{mn} . \tag{5}$$

Obviously, the states $|n;a\rangle\rangle$ are not orthonormal in general, but the states $|n;a\rangle$ are. Therefore, the U operator defined by (1) transforms an orthonormal fermion state into an orthonormal ideal-boson state.

It is easy to see that (1) results in two relations written as

$$|m;b\rangle=U^\dagger|m;b)$$

and

$$\langle n;a|=(n;a|U,$$

and thus that the following equality emerges:

$$\langle n;a|O_F|m;b\rangle=(n;a|O_B|m;b) . \tag{6}$$

In (6), O_F is a fermion operator, and its boson image O_B is given by

$$O_B=UO_FU^\dagger =\sum_{n,m}\sum_{a,b}|n;a)\langle n;a|O_F|m;b\rangle(m;b| . \tag{7}$$

The significance of obtaining the equality in (6) is the following.[6] It means that, once O_B is obtained, one can replace the calculation of any fermion matrix element (and thus of any physical quantity) by that of the corresponding boson matrix element. Since it turns out that the (numerical) evaluation of the boson matrix element is, in general, much easier than that of the fermion matrix element, it does make sense to bosonize the original fermion calculation, in the way as described by (6). We obtained O_B in the specific form given in (7) because we used the MYT method. We shall discuss this point further later.

After understanding in this way the basic framework of the formalism, we shall now go a little further into the technical details of the KT-3 algebra. As seen from (3) and (4), a fermion matrix element is always written as

$$\sum_{a',b'}(Z_n^{-1})_{a;a'}\langle\langle n;a'|O_F|m;b'\rangle\rangle(Z_m^{-1})_{b';b} .$$

Since

$$\langle\langle n;a'|O_F|m;b'\rangle\rangle$$

is always reduced to (a constant times) a norm matrix, it can be concluded that a fermion matrix element is always written, somewhat schematically, as $(Z^{-1})(Z^2)(Z^{-1})$. This shows that the essence of the algebra manipulating the fermion matrix elements lies in the manipulation of the norm matrix elements.

In KT-3, we first introduced a matrix (Y_n), related to (Z_n^2) as

$$(Z_n^2)_{a;a'}=\Delta_{aa'}-(Y_n)_{a;a'} , \tag{8}$$

and obtained it (with $a=\{1,\ldots,n\}$ and $a'=\{1',\ldots,n'\}$) as

$$(Y_n)_{a;a'}=\sum_{i=0}^{n}P_{a'}^{(i)}\Delta_{(i+1)\cdots n;(i+1)',\ldots,n'}(Y_i)^{(L)}_{1\cdots i;1'\cdots i'} . \tag{9}$$

Note that (9) is obtained from Eq. (4.8) of KT-3 by changing the summation index i there into $(n-i)$, and then noting that $Y_0=Y_1=0$. In (9), $(Y_i)^{(L)}$ is the linked-cluster part of (Y_i), Δ is the symmetrized products of the Kronecker deltas, and $P^{(i)}$ is the symmetrizer. See the beginning of Sec. IV A of KT-3 for their precise definition.

In KT-3, we then showed that $(Y_n^k)_{a;a'}$, with any integer $k\geq 2$, was brought into the form of (9), except that $(Y_i)^{(L)}$ there was replaced by $(Y_i^k)^{(L)}$. We further showed that (Z_n) and (Z_n^{-1}) were also brought into forms which were of the same structure as is (9), and eventually that the matrix element, e.g., of $B_e^\dagger$, one of the basic fermion pair operators, was obtained (with $a'=\{2',\ldots,n'\}$) as

$$\langle n;a|B_e^\dagger|n-1;a'\rangle=\sqrt{n}\sum_{i=1}^{n}P^{(i-1)}_{2'\cdots n'}\Delta_{(i+1)\cdots n;(i+1)'\cdots n'}(Z_i/Z_{i-1})^{(L)}_{1\cdots i;e2'\cdots i'} . \tag{10}$$

The algebra to go from (8) to (10) is of a purely fermion nature; bosons are yet to be introduced. In KT-3, we then inserted (10) into (7), and obtained $(B_e^\dagger)_B$, the boson image of $B_e^\dagger$, as

$$(B_e^\dagger)_B=\sum_{i=1}^{N}\sum_{a,a'}\left[\frac{1}{(i-1)!}(Z_i/Z_{i-1})^{(L)}_{a;ea'}\right](A_a^\dagger)^i(A_{a'})^{i-1} . \tag{11}$$

This completes the review of the essence of the KT-3 algebra, and below we will refer back to this review frequently. In doing this, we may refer to the part that goes from (8) to (10) simply as the *(purely) fermion part* of the KT-3 algebra, and to the part that goes from (10) to (11) as the *bosonization part.*

We emphasized above that, in the fermion part of the KT-3 algebra, expressions of the form of (9) were encountered repeatedly, and we wish to explain now why they were. To do this, we first note that $(Y_n)_{a;a'}$ on the left-hand side (lhs) of (9) is regarded as a tensor of rank n, and as such, is expected to be reduced to a sum of irreducible tensors, whose rank $i=0\sim n$. We then recognize that the linked-cluster factor $(Y_i)^{(L)}$ on the right-hand side (rhs) of (9) is precisely this irreducible tensor of rank i, and that it must be multiplied by the Δ factor whose rank equals $n-i$, so that each term maintains the original rank n. The original $(Y_n)_{a;a'}$ was completely symmetric with respect to any interchange of the indices $(a')=\{1',\ldots,n'\}$. In order to maintain this symmetry as well, each product $\Delta^{(n-i)}(Y_i)^{(L)}$ must further be preceded by the symmetrizer $P_{a'}^{(i)}$ $(=P_{a'}^{(n-i)})$. [To be more general, the rhs of (9) should have had the $P_a^{(i)}$ operator, to symmetrize the indices $(a)=\{1,\ldots,n\}$. In KT-3, however, we adopted the practice of avoiding this for simplicity; it did not lead to any error.] The structure of the rhs of (9) can be well understood in this way. The reason why we encountered expressions of this form repeatedly is simply because the KT-3 type of algebra forced us to treat a variety of reducible tensors.

It is worthwhile to go a little further into the derivation of (9). It originated from the evaluation of $(Z_n^2)_{a;a'}$, as seen in (8), and the evaluation of $(Z_n^2)_{a;a'}$ means, as seen from (4) and (5), the evaluation of the matrix element written in full as

$$\langle 0\,|\,B_n\cdots B_1B_{1'}^\dagger\cdots B_{n'}^\dagger\,|\,0\rangle/n!\,.$$

The standard way of evaluating this matrix element is to move to the right the annihilation operators $B_1,\ldots,B_n$ one by one, in the course using the commutation relations written as

$$[B_1,B_2^\dagger]=\delta_{1,2}-\sum_p P_{1,2}^{(p)}C_p^\dagger\,, \qquad (12a)$$

$$[C_p^\dagger,B_1^\dagger]=\sum_2 P_{1,2}^{(p)}B_2^\dagger\,, \qquad (12b)$$

which are nothing but those given in Eq. (2.7) of KT-3. If the number of times with which the first term on the rhs of (12a) is used equals $n-i$, it results in the $\Delta^{(n-i)}$ factor. It must then be multiplied by $(Y_i)^{(L)}$, which results from the use of the second term on the rhs of (12a), together with that of (12b).

We have thus seen that $(Y_i)^{(L)}=0$, if $P^{(p)}=0$. If $P^{(p)}$ indeed vanishes, it follows from (12) that B_1 and $B_2^\dagger$ behave as if they were pure boson operators. This means that the nonvanishing $P^{(p)}$, and thus the nonvanishing $(Y_i)^{(L)}$, describe the deviation of the fermion dynamics (or statistics) from that of bosons, and in this sense the $(Y_i)^{(L)}$ factor in (9) may very well be called the *dynamical factor*.

We have thus made it clear why we encounter expressions of the form

$$(D_n)_{a;a'}=\sum_{i=0}^{n}P_{a'}^{(i)}\Delta_{(i+1)\cdots n;(i+1)'\cdots n'} \times(D_i)^{(L)}_{1\cdots i;1'\cdots i'}\,. \qquad (9')$$

We introduced the notation $(D_i)^{(L)}$ to signify that it is a dynamical factor. An example of (9′) is, of course, Eq. (9). Another example is (10), although in it there appears a symmetrizer $P^{(i-1)}$, rather than $P^{(i)}$, because there the index e is treated differently from the $a'=\{2'\cdots n'\}$ indices.

After making these preparations, we now begin the construction of the TTB method. It will be done by noting first that the following equalities hold:

$$(n;a\,|\,\frac{1}{i!}\sum_{b,b'}(D_i)^{(L)}_{b;b'}(A_b^\dagger)^i(A_{b'})^i\,|\,n;a')=\theta(n-i)P_{a'}^{(i)}(D_i)^{(L)}_{1\cdots i;1'\cdots i'}\Delta_{(i+1)\cdots n;(i+1)'\cdots n'},\quad (n\geq 0)\,, \qquad (13a)$$

$$(n;a\,|\,\frac{1}{(i-1)!}\sum_{b,b'}(D_i)^{(L)}_{b;eb'}(A_b^\dagger)^i(A_{b'})^{i-1}\,|\,n-1;a')=\theta(n-i)\sqrt{n}\,P_{a'}^{(i-1)}(D_i)^{(L)}_{1\cdots i;e2'\cdots i'}\Delta_{(i+1)\cdots n;(i+1)'\cdots n'}\,,\quad (n\geq 1)\,, \qquad (13b)$$

$$(n;a\,|\,\frac{1}{(i-2)!}\sum_{b,b'}(D_i)^{(L)}_{b;efb'}(A_b^\dagger)^i(A_{b'})^{i-2}\,|\,n-2;a')=\theta(n-i)\sqrt{n(n-1)}\,P_{a'}^{(i-2)}(D_i)^{(L)}_{1\cdots i;ef3'\cdots i'}\Delta_{(i+1)\cdots n;(i+1)'\cdots n'}\,,\quad (n\geq 2)\,. \qquad (13c)$$

Clearly the rhs of (13a) is the ith term of (9′) [or of (9)], while that of (13b) is encountered in the ith term of (10). Had we considered, e.g., a direct bosonization (which we do not do normally) of the $B_e^\dagger B_f^\dagger$ term in the Hamiltonian, we might have encountered terms such as that appearing on the rhs of (13c).

Since we have explained in detail the structure and the origin of the terms that appear on the rhs of (13), the reader will see clearly that they more or less exhaust the types of terms that are to be encountered in the KT-3 type of fermion algebra. Thus, what is shown in (13) is that each of such terms can be replaced by an appropriate boson matrix element, and this is what we mean by the term-by-term bosonization (TTB). As will be seen soon below, the essence of the TTB method is to take advantage of knowing (13).

It is easy to prove (13). Let us take first (13a), and note that its lhs can be rewritten, if use is made of (2), as

$$\frac{1}{i!n!}\sum_{b,b'}(D_i)^{(L)}_{b;b'}(0\,|\,(A_a)^n(A_b^\dagger)^i(A_{b'})^i(A_{a'}^\dagger)^n\,|\,0)\,. \qquad (14)$$

We then note that

$$(A_b)^i(A_a^\dagger)^n=\sum_{k=0}^{\min(i,n)}k![P_a^{(k)}P_b^{(k)}]\Delta^{(k)}_{\bar{a};\bar{b}}(A_{\tilde{a}}^\dagger)^{n-k}(A_{\tilde{b}})^{i-k}\,, \qquad (15)$$

where $\bar{a}=\{1,\ldots,k\}$, $\tilde{a}=\{k+1,\ldots,n\}$, and similarly for $\bar{b}$ and $\tilde{b}$. If (15) is used in (14), we see that only the $k=i$ term survives, and that (14) is reduced to the expression on the rhs of (13a). Thus (13a) has been proved. The proof of (13b) and (13c) is done similarly. We merely note that the $\sqrt{n}$ factor in (13b) emerged there when we rewrote the factor $1/\sqrt{n!}(n-1)!$, originating from the use of (2), as $\sqrt{n}/n!$. Similarly the factor $\sqrt{n(n-1)}$ in (13c) appeared when we replaced $1/\sqrt{n!}(n-2)!$ by $\sqrt{n(n-1)}/n!$.

Since the equalities in (13) have been proved, we now begin to present examples of their application, thus hopefully demonstrating how useful the TTB method can be. We first note that, if we set $D_i=Z_i/Z_{i-1}$, the rhs of (13b) equals the ith term on the rhs of (10). Therefore, the use of (13b) in (10) (after the above replacement of D_i is made) is seen to give rise immediately to the following equality:

$$\langle n;a\,|\,B_e^\dagger\,|\,n-1;a'\rangle = \left\{ n;a \left| \sum_{i=1}^{n} \sum_{b,b'} \frac{1}{(i-1)!} (Z_i/Z_{i-1})^{(L)}_{b;eb'} (A_b^\dagger)^i (A_{b'})^{i-1} \right| n-1;a' \right\} . \tag{16}$$

The operator part in the boson matrix element on the rhs of (16) is, however, exactly equal to what was given as $(B_e^\dagger)_B$ in (11).

We have thus shown that, once the fermion part of the algebra is completed, so that Eq. (10) has been derived, the bosonization part of the algebra can be completed with trivial ease. In fact, all we need to do is to take a glance at the equalities in (13). This is certainly a dramatic simplification of the bosonization procedure. One sees this clearly by comparing the above presentation with the rather lengthy algebra which we had to go through in KT-3 in obtaining (11) from (10).

As we remarked above, the MYT type of procedure followed in KT-3 was to insert (10) into (7), and then obtain (11); and this procedure involved lengthy algebra. In the bosonization procedure with the TTB method shown above, however, we had no recourse to (7). As soon as (10) was obtained, we replaced each term on its rhs by a boson matrix element, and then summed the operands so as to obtain (11).

We presented the above discussion by taking the point of view that the use of (7) was characteristic of the MYT method. Actually, however, (7) is a relation which is much more general. It is simply equivalent to (6), as seen from the fact that the $(n;a||m;b)$ element of (7) is nothing but (6). We can also show that the use of (7) is not, by itself, responsible for making the bosonization procedure lengthy.

Let us set $O_F = B_e^\dagger$ in (7), and insert (16) into it. We then have

$$(B_e^\dagger)_B = \left[\sum_{n,a} |\,n;a)(n;a\,| \right] \{\ \} \left[\sum_{m,a'} |\,m;a')(m;a'\,| \right] . \tag{17}$$

In (17) we denoted, for simplicity, the operand of (16), i.e., the rhs of (11), by $\{\ \}$. Also, when (16) was inserted into (7), there appeared a factor $|\,n-1;a')(n-1;a'\,|$, but it was replaced in (17) by the last [ ] factor there; only the $m=n-1$ term survives the m sum in it. As seen from its explicit form in (16), the $\{\ \}$ factor in (17) is independent of the summation indices n, m, a, and a', and thus the two [ ] factors in (17) can be replaced by one, due to the completeness relation. Therefore, (17) simply reduces to (11), showing indeed that the use of (7) does not make the bosonization procedure lengthy, if its use is made in combination with (16). We needed the TTB method, however, in obtaining (16).

As is well known, the MYT method uses two techniques importantly. One is the introduction of the Usui operator, which was instrumental in deriving (7), as was explained above. The second of the MYT techniques is the so-called boson expansion of the vacuum projector $|\,0)(0|$, which is written as [cf., e.g., Eq. (5.9b) of KT-3]

$$|\,0)(0\,| = \sum_k [(-)^k/k!] \sum_a (A_a^\dagger)^k (A_a)^k . \tag{17'}$$

The bosonization in KT-3 was done by inserting (10) and (17′) into (7), and became a lengthy procedure. We could not avoid doing this, because we were not aware of the possibility of using (16).

In the fermion part of the KT-3 algebra, we encountered a variety of reducible tensors, and then had to deal with their products, i.e., with expressions that may be written as

$$\sum_{a''} (D_n)_{a;a''} (E_n)_{a'';a'} . \tag{18}$$

In KT-3, we introduced a theorem called theorem I, so that the algebra involved in (18) could be handled systematically. This theorem was proved in Appendix B of KT-3, and it may be interesting to ask whether the use of the TTB method simplifies its proof. In Appendix B of KT-3, we first expressed (D_n) and (E_n) as in (9′), and then performed the sum over a'' in (18).

To use the TTB method in (18) means to replace first, by using (13a), the terms on the rhs of (9′), and in a similar expression for (E_n), by boson matrix elements, and then insert the resultant sums into (18). Since the whole algebra is thus reduced to that of purely boson nature, one might expect that the proof of theorem I is indeed simplified. Actually, this is *not* the case. The reason is that we encounter a step where we have to use (15), which, as seen, produces the $P^{(i)}$ and the $\Delta^{(n-i)}$ type of factors, thus bringing back the algebra very close to what we have been calling purely fermion-type algebra.

It is thus seen that, even when the TTB method is used, the fermion part of the KT-3 algebra is not so much simplified, although, as we saw above, the bosonization part is dramatically simplified. We have arrived at this conclusion because we have been treating the general fermion problem rigorously. The matter changes drastically once we begin treating the truncated fermion problem, as we shall see in the next section.

In concluding this section, we want to make a comment about the finiteness (or the infiniteness) of the boson expansion. We again take $(B_e^\dagger)_B$ as an example, and note first that in (11) we had a sum over i, ranging from 1 to N. The corresponding sum in (16), on the other hand, had n as its upper limit. The sum in (16), however, can be extended to N, or even to ∞, without changing the value of the rhs of (16). We thus see that, formally, the expansion of $(B_e^\dagger)_B$ can be infinite. In its practial use, however, it is finite, since N is finite. No one will ever think of a fermion problem to start with in which $N=\infty$. [See Eq. (1) and Ref. 9; see also Ref. 6 for a related discussion.]

III. USE OF THE TTB METHOD FOR A TRUNCATED FERMION SYSTEM

In Sec. II, we developed the TTB method for very general fermion systems. In practice, however, we work under a much more restrictive framework, by considering, e.g., a space spanned by products of only one kind of pair of a collective nature.[1–3] In the present section, we will discuss the use of the TTB method under such a restriction.

The BET under this restriction was also discussed in KT-3, and it was in particular shown that the boson expansion then became a power series expansion, having a smallness parameter called y_k. (See Sec. VII of KT-3 for details.) Since this parameter y_k is used importantly throughout the present section, we shall explain its origin explicitly. It is done by calculating the norm matrix (Z_2^2) as follows:

$$(Z_2^2)_{12;1'2'}=\langle 0\,|\,B_2B_1B_{1'}^{\dagger}B_{2'}^{\dagger}\,|\,0\rangle/2!$$
$$=\Delta_{12;1'2'}-\tfrac{1}{2}\sum_P P_{1;1'}^{(p)}P_{2';2}^{(p)} . \qquad (19a)$$

That (19a) is true is easily confirmed by using (12).

Somewhat more generally, however, we can write $(Z_2^2)_{12;1'2'}$ also as

$$(Z_2^2)_{12;1'2'}=\Delta_{12;1'2'}-(Y_2)_{12;1'2'} , \qquad (19b)$$

as seen from (8). We thus have

$$(Y_2)_{12;1'2'}=\tfrac{1}{2}\sum_p P_{1,1'}^{(p)}P_{2',2}^{(p)} . \qquad (20a)$$

This can also be written as

$$(Y_2)_{12;1'2'}=\sum_k (y_k/2)\sum_q (12\,|\,q)(1'2'\,|\,q) , \qquad (20b)$$

defining y_k. In (20b), $(12\,|\,q)$ abbreviates the Clebsch-Gordan coefficient $(2\mu_1 2\mu_2\,|\,kq)$. Note that, throughout this section, the indices $1,2,\ldots,$ stand for the magnetic quantum numbers $\mu_1,\mu_2,\ldots,$ because we consider only a collective component of quadrupole nature.

In Sec. VII of KT-3, we showed that

$$(Z_i/Z_{i-1})^{(L)}=O(y_k^{i-1}) .$$

In particular we found that

$$(Z_1/Z_0)_{1;e}^{(L)}=\Delta_{1,e} , \qquad (21)$$

$$(Z_2/Z_1)_{12;e2'}^{(L)}=-(1/2)(Y_2)_{12;e2'} , \qquad (22)$$

which permitted us to obtain $(B_e^{\dagger})_B$ [from (11)] as

$$(B_e^{\dagger})_B=A_e^{\dagger}-(1/2)\sum_{12;2'}(Y_2)_{12;e2'}A_1^{\dagger}A_2^{\dagger}A_{2'} \qquad (23)$$

to $O(y_k)$. Throughout the rest of the present subsection we consider every quantity only to $O(y_k)$. In the terminology of KT-1, we are thus satisfied with developing the fourth order BET. This restriction is made only to make our presentation easier to follow. Its extension to higher order is straightforward.

It is convenient to introduce here an operator Ω defined as

$$\Omega=\sum_{121'2'}(Y_2)A_1^{\dagger}A_2^{\dagger}A_{1'}A_{2'}$$
$$=\sum_{kq}(y_k/2)\sum_{121'2'}(12\,|\,q)(1'2'\,|\,q)A_1^{\dagger}A_2^{\dagger}A_{1'}A_{2'} . \qquad (24)$$

It satisfies a commutation relation given as

$$[\Omega,A_e^{\dagger}]=2\sum_{122'}(Y_2)_{12;e2'}A_1^{\dagger}A_2^{\dagger}A_{2'} . \qquad (25)$$

Because of (25), we can rewrite (23) as

$$(B_e^{\dagger})_B=A_e^{\dagger}-\tfrac{1}{4}[\Omega,A_e^{\dagger}] . \qquad (26)$$

After these preparations, we now begin to deal with

$$(Z_n^2)_{a;a'}=\langle 0\,|\,B_n\cdots B_1B_{1'}^{\dagger}\cdots B_{n'}^{\dagger}\,|\,0\rangle/n! . \qquad (27)$$

The straightforward way to evaluate (27) is to move the annihilation operators $B_1,B_2,\ldots,$ to the right one by one, by using the commutation relations (12); see above. Instead of doing this fully, we consider moving B_1 only, obtaining the result

$$B_1B_{1'}^{\dagger}\cdots B_{n'}^{\dagger}\,|\,0\rangle=P_{a'}^{(1)}\Delta_{1,1'}B_{2'}^{\dagger}\cdots B_{n'}^{\dagger}\,|\,0\rangle-2P_{a'}^{(2)}\sum_g (Y_2)_{1g;1'2'}B_g^{\dagger}B_{3'}^{\dagger}\cdots B_{n'}^{\dagger}\,|\,0\rangle . \qquad (28)$$

Then we have

$$(Z_n^2)_{a;a'}=\frac{1}{n}P_{a'}^{(1)}\Delta_{1;1'}(Z_{n-1}^2)_{2\cdots n;2'\cdots n'}-\frac{2}{n}P_{a'}^{(2)}\sum_g (Y_2)_{1g;1'2'}(Z_{n-1}^2)_{2\cdots n;g3'\cdots n'} . \qquad (29)$$

Equation (29) is still exact, but here we intend to introduce an approximation. This approximation replaces the $(Z_{n-1}^2)_{2\cdots n;g3'\cdots n'}$ factor in the second term of (29) by $\Delta_{2\cdots n;g3'\cdots n'}$. The justification of this replacement is that the second term of (29) already has a factor (Y_2), which is $O(y_k)$, and thus the other factor needs to be evaluated only to $O(y_k^0)$. By making this replacement, and then summing over g, we arrive at

$$(Z_n^2)_{a,a'}=\frac{1}{n}P_{a'}^{(1)}\Delta_{1;1'}(Z_{n-1}^2)_{2\cdots n;2'\cdots n'}-\frac{2}{n}P_{a'}^{(2)}(Y_2)_{12;1'2'}\Delta_{3\cdots n;3'\cdots n'} , \qquad (30)$$

which can be regarded as a recurrence formula [correct to $O(y_k)$] for (Z_n^2). By using this recurrence formula, it is a straightforward procedure to find that $(Z_n^2)_{a,a'}$ is finally expressed as

$$(Z_n^2)_{a,a'}=\Delta_{a,a'}-P_{a'}^{(2)}(Y_2)_{12;1'2'}\Delta_{3\cdots n;3'\cdots n'} . \qquad (31)$$

At this stage we begin to use the TTB method, and recognize that the second term of (31) is nothing but the $(n;a\,||\,n;a')$ element of Ω, Ω being defined in (24). [This is the same as to say that we use (13a) with $i=2$.] We thus have

$$(Z_n^2)_{a,a'}=\Delta_{a,a'}-\tfrac{1}{2}\Omega_{a,a'} , \qquad (32)$$

where

$$\Omega_{a,a'}=(n;a\mid\Omega\mid n;a') . \tag{33}$$

With this knowledge of $(Z_n^2)_{a,a'}$, we can construct orthonormal states $|n;a\rangle$ and $|n-1;a'\rangle$ as

$$\begin{aligned}|n;a\rangle&=\sum_a(Z_n^{-1})_{a,\bar a}\,|n;a\rangle\!\rangle=|n;\bar a\rangle\!\rangle+\tfrac14\sum_{\bar a}\Omega_{a;\bar a}\,|n;\bar a\rangle\!\rangle ,\\ |n-1;a'\rangle&=|n-1;a\rangle\!\rangle+\tfrac14\sum_{\bar a}\Omega_{a';\bar a'}\,|n-1;\bar a'\rangle\!\rangle ,\end{aligned} \tag{34}$$

where $\Omega_{a';\bar a'}=(n-1;a'\mid\Omega\mid n-1;\bar a')$.

By having (34), we are now ready to perform the evaluation of the matrix element $\langle n;a\mid B_e^\dagger\mid n-1;a'\rangle$. It is done as follows:

$$\begin{aligned}\langle n;a\mid B_e^\dagger\mid n-1;a'\rangle&=\langle\!\langle n;a\mid B_e^\dagger\mid n-1a'\rangle\!\rangle+\tfrac14\Omega_{a;\bar a}\langle\!\langle n;\bar a\mid B_e^\dagger\mid n-1;a'\rangle\!\rangle+\tfrac14\Omega_{a';\bar a'}\langle\!\langle n;a\mid B_e^\dagger\mid n-1;\bar a'\rangle\!\rangle\\ &=\sqrt{n}\,\{(Z_n^2)_{a;ea'}+\tfrac14\Omega_{a;\bar a}(Z_n^2)_{\bar a;ea'}+\tfrac14\Omega_{a';\bar a'}(Z_n^2)_{a;e\bar a'}\}\\ &=\sqrt{n}\,\{\Delta_{\bar a;ea'}-\tfrac12\Omega_{a;ea'}+\tfrac14\Omega_{a;\bar a}\Delta_{\bar a;ea'}+\tfrac14\Delta_{a;e\bar a'}\Omega_{\bar a';a'}\}\\ &=\sqrt{n}\,\{\Delta_{a;ea'}-\tfrac14\Omega_{a;ea'}+\tfrac14(n;a\mid A_e^\dagger\mid n-1;\bar a')(n-1;\bar a'\mid\Omega\mid n-1;a')/\sqrt{n}\,\}\\ &=\sqrt{n}\,\Delta_{a;ea'}-\tfrac14(n;a\mid\Omega A_e^\dagger\mid n-1;a')+\tfrac14(n;a\mid A_e^\dagger\Omega\mid n-1;a')\\ &=(n;a\mid\{A_e^\dagger-\tfrac14[\Omega,A_e^\dagger]\}\mid n-1;a')\\ &=(n;a\mid(B_e^\dagger)_B\mid n-1;a');\ \text{to order } O(y_k) .\end{aligned} \tag{35}$$

In (35), summation over dummy indices $\bar a$ and $\bar a'$ was assumed. Otherwise, every step of the algebra is very elementary, and the reader will be able to follow it without any further elaboration. We thus have reproduced (26).

We may also consider here the bosonization of $C_p^\dagger$, and find first that

$$\langle n;a\mid C_p^\dagger\mid n;a'\rangle=\langle\!\langle n;a\mid C_p^\dagger\mid n;a'\rangle\!\rangle+\tfrac14\sum_{a''}[\Omega_{a'',a}\langle\!\langle n;a''\mid C_p^\dagger\mid n,a'\rangle\!\rangle+\langle\!\langle n;a\mid C_p^\dagger\mid n;a''\rangle\!\rangle\Omega_{a'';a'}] , \tag{36a}$$

where (34) was used again. We can rewrite the first term, by going through algebra which is now familiar to us, as

$$\langle\!\langle n;a\mid C_p^\dagger\mid n;a'\rangle\!\rangle=(1/n)\sum_{g,g'}P_a^{(1)}P_{a'}^{(1)}[\delta_{1;g}\delta_{1';g'}P_{g';g}^{(p)}\Delta_{a;a'}]-\tfrac14[P_{a'}^{(1)}\sum_g P_{1';g}^{(p)}\Omega_{a;g\bar a'}+P_a^{(1)}\sum_g P_{g;1}^{(p)}\Omega_{g\bar a;a'}] . \tag{36b}$$

[In (36b), $\bar a=\{2,\ldots,n\}$ with $a=\{1,2,\ldots,n\}$.] Both (36a) and (36b) are correct to $O(y_k)$. In evaluating the matrix elements in the second term of (36a), we obtain them only to $O(y_k^0)$, since they are multiplied with Ω. We then find that this second term cancels the second term of (36b). Thus

$$\langle n;a\mid C_p^\dagger\mid n;a'\rangle=(1/n)\sum_{g;g'}P_a^{(1)}P_{a'}^{(1)}[\delta_{1;g}\delta_{1';g}P_{g';g}^{(p)}\Delta_{a;a'}] , \tag{36c}$$

to which we apply the TTB method obtaining, in agreement with KT-3, that

$$(C_p^\dagger)_B=\sum_{1;1'}P_{1';1}^{(p)}A_1^\dagger A_{1'} . \tag{37}$$

Our program to apply the TTB method to the truncated fermion system has thus been completed very successfully. We nevertheless feel it desirable to go one step further, because we have used the M representation throughout, as seen from the forms of the states given in Eqs. (2)—(4); there the total angular momentum I was not a good quantum number. For practical uses, it is desirable to have a BET valid in the I representation.

An orthonormal boson state in the I representation may be written as[10] $|n;\gamma vIM)$, where v is the seniority, M is the projection of I, and γ is the extra quantum number. Below, we abbreviate $|n;\gamma vIM)$ as $|n;\alpha)$. As seen also in Ref. 10, $|n;\alpha)$ can be constructed very explicitly, by introducing the coefficients of fractional parentage (cfp's).

Indeed, we can write $|n;\alpha)$ as[10]

$$|n;\alpha)=\sum_\beta[1/\sqrt{n}\,]c_{n,\alpha;n-1,\beta}(2\mu_nI'M'\mid IM)A_{\mu_n}^\dagger\mid n-1;\beta) , \tag{38}$$

where cfp is written as $c_{n,\alpha;n-1,\beta}$. They satisfy the relation that

$$\sum_\beta c_{n,\alpha;n-1,\beta}c_{n,\alpha';n-1,\beta}=\delta_{\alpha,\alpha'} \tag{39}$$

resulting in

$$(n;\alpha\mid n;\alpha')=\delta_{\alpha,\alpha'} .$$

As seen in (38), each time we express an orthonormal state with a given boson number in terms of those with the boson number less by one, there emerges a factor which may somewhat symbolically be written as Σ(cfp$\times$CG), where CG stands for the Clebsch-Gordan coefficient. Since we do not need to know this factor explicitly, in the argument that follows, we shall denote it simply as Σ. It is then seen that (38) can be rewritten as

$$|n;\alpha)=[1/\sqrt{n!}](\Sigma)^{n-1}A_n^\dagger\cdots A_1^\dagger|0)\ . \tag{40}$$

We are now ready to construct corresponding fermion states. We shall first introduce a state $|n;\alpha\rangle\rangle$ defined as

$$|n;\alpha\rangle\rangle=[1/\sqrt{n!}](\Sigma)^{n-1}B_n^\dagger\cdots B_1^\dagger|0\rangle\ . \tag{41}$$

Since there is the $(\Sigma)^{n-1}$ factor, $|n;\alpha\rangle\rangle$ is totally symmetric with respect to any interchange of the $B^\dagger$ factors. It is not orthonormal, however, and we want to construct a corresponding orthonormal state, by obtaining, as usual, the Z_n^2 matrix first. Its calculation can be done as follows:

$$\begin{aligned}(Z_n^2)_{\alpha;\alpha'}&=\langle\langle n;\alpha|n;\alpha'\rangle\rangle\\&=(\Sigma)^{n-1}(\Sigma')^{n-1}\langle 0|B_n\cdots B_1B_{1'}^\dagger\cdots B_{n'}^\dagger|0\rangle/n!\\&=(\Sigma)^{n-1}(\Sigma')^{n-1}[\Delta_{a;a'}-P_{a'}^{(2)}(Y_2)_{12;1'2'}\Delta_{3\cdots n;3'\cdots n'}]\\&=(\Sigma)^{n-1}(\Sigma')^{n-1}[(n;a\,|n;a')-\tfrac{1}{2}(n;a\,|\Omega|n;a')]\\&=(n;\alpha|n,\alpha')-\tfrac{1}{2}(n;\alpha|\Omega|n;\alpha')\ .\end{aligned} \tag{42}$$

In obtaining the third equality of (42), relation (31) was used, while the fourth equality of (42) resulted just as (32) did from (31). The final equality in (42) is the consequence of (40).

The result of the algebra in (42) is summarized as

$$(Z_n^2)_{\alpha;\alpha'}=\Delta_{\alpha;\alpha'}-\tfrac{1}{2}\Omega_{\alpha;\alpha'}\ , \tag{43}$$

which is the same as is (32), except for the difference in the indices. Therefore, just as (32) resulted in (35), Eq. (43) results in

$$\langle n;\alpha|B_e^\dagger|n-1;\alpha'\rangle=(n;\alpha|(B_e^\dagger)_B|n-1\alpha') \tag{44}$$

with the same $(B_e^\dagger)_B$ as appeared in (26). In other words, we found that the bosonization is the same, irrespective of whether we work it out in the M or I representation. This result is not totally unexpected. What we found above has, nevertheless, the following significance.

It is our fundamental point of view that any BET, exact or truncated, must always be traceable back to its original fermion problem. Our realistic calculations were done in the I representation,[1-3] but so far we have not clarified (even in KT-3) what the fermion system we actually started with was. We now know, however, that our very basic starting point was the states in (41). They were of course fermion states. Yet the boson cfp's already appeared there.

IV. SUMMARY AND DISCUSSIONS

We shall begin the summary of what we have done above by recalling the basic ideas that were set forth in KT-3,[7] as well as in TWP.[6] As was stressed in these references, it is our fundamental point of view that it makes sense (to intend) to bosonize a fermion system, if (and only if) this system is constructed by taking as the basis states those that are given in (4). In (4), the pair creation operators $B^\dagger$ are those in TDR (or in some other equivalent representation), and thus states in (4) are not orthonormal in general. Out of these states, however, new states can be constructed as in (3), which are now orthonormal. Once these orthonormal states are obtained, we can construct matrix elements of any operator we want.

Actual calculation of these fermion matrix elements is, however, rather cumbersome in general, and this is where the BET comes in. It serves to establish a method to obtain the boson image O_B of a fermion operator O_F, so that the equality in (6) is satisfied. Once (6) is established, the calculation of the fermion matrix element is replaced by that of the boson matrix element, which is much easier to perform. Different methods differ in the procedure of obtaining O_B, and we showed above that the TTB method makes it very simple. The essence of this method is to use the equalities in (13), whenever such a use simplifies the algebra.

So far we have not derived any new formulae to be used for practical calculations, beyond those which were given, e.g., in KT-3. However, since we have found that the new TTB method is so easy to use, we may now attempt to apply it to problems where the application of older methods has been considered rather difficult. An interesting problem is to bosonize directly a fermion system whch is constructed, not based on Tamm-Dancoff pairs, as has been the case in the above, as well as in KT-3, but on RPA (random-phase approximation) pairs. We are currently working[11] on this problem by extending the technique used in Sec. III.

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