Faddeev equations including three-body forces in first order perturbation theory

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(Received 15 July 1983)

We propose a modification of the standard Faddeev equations which takes into account the effects of a three-body force in first order perturbation theory on the triton wave function and its binding energy. Furthermore, we report our results for the energy expectation value caused by the two-pion-exchange three-nucleon force.

[NUCLEAR STRUCTURE Faddeev equations, energy expectation value of the two-pion-exchange three-nucleon force.]

Since it is well known that the energetic effects of three-body forces in the triton (handled up to now) are small compared to the expectation value of realistic two-body forces, it seems justified to take into account only first order corrections. To that aim, we shall study the low order perturbation theory for the bound state Faddeev equations with respect to the three-body force. Though the results for the energy shift and the wave function are highly standard, the formulation within the Faddeev scheme, in which the actual calculations are carried through, seems to be new. We shall also present numerical results extending the ones found in Ref. 1.

The starting point is the Schrödinger equation in integral form:

$$\Psi=G_0\sum_{\mu=1}^{3}V_\mu\Psi+G_0W\Psi\ , \tag{1}$$

which includes the three-body force W. We decompose the wave function as

$$\Psi=\sum_{\mu=1}^{3}\psi_\mu+\psi_4\ , \tag{2}$$

with

$$\begin{aligned}\psi_\mu&=G_0V_\mu\Psi\ ,\\ \psi_4&=G_0W\Psi\ .\end{aligned} \tag{3}$$

Then, for identical particles, these components obey the following Faddeev equations:

$$\psi_1=G_0t_1(P\psi_1+\psi_4)\ , \tag{4a}$$

$$\psi_4=G_0t_4(1+P)\psi_1\ , \tag{4b}$$

where the permutation operator P is given by

$$P=P_{12}P_{23}+P_{13}P_{23}\ . \tag{5}$$

We denote the unperturbed solution ($W=0$) by ψ_0, which obeys

$$\psi_0=G_0tP\psi_0\ . \tag{6}$$

Then we introduce the first order corrections with respect to W for the Faddeev component $\psi\equiv\psi_1$, and the energy:

$$\begin{aligned}\psi&=\psi_0+\psi'\ ,\\ E&=E_0+\Delta E'\ .\end{aligned} \tag{7}$$

Expanding $G_0(E)$ and $t(E)$ in $\Delta E'$ to first order,

$$G_0(E_0+\Delta E')=G_0(E_0)-G_0(E_0)\Delta E'G_0(E_0)+0(\Delta E'^2)\ , \tag{8}$$

$$t(E_0+\Delta E')=t(E_0)-\Delta E't(E_0)G_0(E_0)G_0(E_0)t(E_0)+O(\Delta E'^2)\ ,$$

and inserting the first order expression for (4b) into (4a), we find the basic equation

$$\psi=\psi_0+\psi'=G_0tP\psi_0+G_0tP\psi'-\Delta E'G_0tG_0\psi_0-\Delta E'G_0\psi_0+G_0tG_0W(1+P)\psi_0\ . \tag{9}$$

Here, and in the following, G_0 and t are evaluated at the unperturbed energy E_0. Equation (9) determines ψ' and $\Delta E'$. Evaluating the matrix element $(\psi_0\,|\,PtP\,|\,\psi_0)$ and using (6), one gets the following expression for the energy shift $\Delta E'$:

$$\Delta E'=\frac{\langle\psi_0\,|\,W(1+P)\,|\,\psi_0\rangle}{\langle\psi_0\,|\,1+P\,|\,\psi_0\rangle}\ . \tag{10a}$$

This is identical to the standard form

$$\Delta E'=\frac{\langle\Psi_0\,|\,W\,|\,\Psi_0\rangle}{\langle\Psi_0\,|\,\Psi_0\rangle}\ , \tag{10b}$$

which follows from the connection of ψ_0 to the wave function Ψ_0:

$$\Psi_0=(1+P)\psi_0\ . \tag{11}$$

So, we end up with the following integral equation for the perturbed part of the Faddeev component:

$$\psi'=\phi+G_0tP\psi'\ , \tag{12}$$

where

$$\phi=G_0tG_0W\Psi_0-\Delta E'(G_0+G_0tG_0)\psi_0\ . \tag{13}$$

Comparing this with the unperturbed Faddeev equation (6), we recognize the same kernel, but now an inhomogeneous term is present. The driving term ϕ is orthogonal to the left-hand side eigenfunction of the kernel:

$$\tilde{\psi}_0=\tilde{\psi}_0G_0tP\ , \tag{14}$$

which obviously is given as

$$\tilde{\psi}_0=\psi_0 PtP \ . \tag{15}$$

Using (6) and (10a), one immediately verifies

$$\langle \tilde{\psi}_0 \mid \phi \rangle = 0 \ . \tag{16}$$

Therefore, the solution of (12) with the property

$$\langle \tilde{\psi}_0 \mid \psi' \rangle = 0 \tag{17}$$

exists. One can construct ψ' via the Neumann series:

$$\psi'=\phi+K\phi+K^2\phi+\cdots \ , \tag{18}$$

with $K=G_0tP$. Clearly each term has the property of being orthogonal to $\tilde{\psi}_0$. This construction is very convenient from a practical point of view, since it parallels exactly what is being done in solving the unperturbed problem (6). The only difference is that in the latter case an arbitrary driving term (with a nonzero component in the direction ψ_0) is used to create the Neumann series, whereas here a specific one is required. If it were not that for realistic two-nucleon forces (which include repulsive parts) a negative eigenvalue of K exists, which lies outside the unit circle, then that series (18) would even converge. Converging or not, one can sum it up by standard Padé methods.

It is now interesting to establish the link with the usual wave function correction of first order. We introduce the projection operator

$$\Lambda \equiv 1 - \mid \psi_0 \rangle \langle \tilde{\psi}_0 \mid \ , \tag{19}$$

choosing the normalization

$$\begin{aligned}\langle \tilde{\psi}_0 \mid \psi_0 \rangle &= \langle \psi_0 \mid PtP \mid \psi_0 \rangle = \langle \psi_0 \mid PVP + PtG_0VP \mid \psi_0 \rangle \\ &= \langle \psi_0 \mid (1+P)VP \mid \psi_0 \rangle = 1 \ . \end{aligned} \tag{20}$$

Then we can solve (12) formally using an inverse in a restricted space

$$\psi'=\lambda\psi_0+(1-G_0tP)^{-1}\Lambda\phi \ . \tag{21}$$

The first term, $\lambda\psi_0$, is undetermined, but λ has to be of first order in W. Now we shall show that this form can be rewritten into the usual one for the first order wave function correction. One has

$$\begin{aligned}(1-G_0tP)^{-1}\Lambda &= [E_0-H_0-V(1+P)]^{-1} \\ &\quad \times (E_0-H_0-V)\Lambda \end{aligned} \tag{22}$$

and

$$\begin{aligned}(E_0-H_0-V)\Lambda &= [1-VP \mid \psi_0 \rangle \langle \psi_0 \mid (1+P)] \\ &\quad \times (E_0-H_0-V) \\ &\equiv \tilde{\Lambda}(E_0-H_0-V) \ . \end{aligned} \tag{23}$$

Furthermore, it is easy to see that

$$(E_0-H_0-V)\phi = VG_0W(1+P)\psi_0 - \Delta E'\psi_0 \ . \tag{24}$$

Therefore, we get the intermediate result

$$\begin{aligned}\psi' &= \lambda\psi_0+[E-H_0-V(1+P)]^{-1}\tilde{\Lambda} \\ &\quad \times [VG_0W(1+P)\psi_0-\Delta E'\psi_0] \ . \end{aligned} \tag{25}$$

Since we aim to build up the wave function, we apply $(1+P)$ from the left and use

$$\begin{aligned}(1+P)[E-H_0-V(1+P)]^{-1} &= [E-H_0-(1+P)V]^{-1} \\ &\quad \times (1+P) \ . \end{aligned} \tag{26}$$

Further,

$$\begin{aligned}(1+P)\tilde{\Lambda} &= (1-(1+P)VP \mid \psi_0 \rangle \langle \psi_0 \mid)(1+P) \\ &\equiv \tilde{\tilde{\Lambda}}(1+P) \ , \end{aligned} \tag{27}$$

and we get

$$\begin{aligned}(1+P)\psi' &= \lambda\Psi_0+[E_0-H_0-(1+P)V]^{-1}\tilde{\tilde{\Lambda}} \\ &\quad \times \left[\sum_\mu V_\mu G_0 W \Psi_0 - \Delta E' \Psi_0\right] \ . \end{aligned} \tag{28}$$

Since the resolvent operator acts on a totally antisymmetric wave function, we can replace it by

$$G' = \left[E_0-H_0-\sum V_\mu\right]^{-1}\tilde{\tilde{\Lambda}} \ . \tag{29}$$

The projection operator $\tilde{\tilde{\Lambda}}$ excludes the ground state Ψ_0, as is explicitly seen as follows:

$$\begin{aligned}\frac{1}{\langle \Psi_0 \mid \Psi_0 \rangle}\langle \Psi_0 \mid \tilde{\tilde{\Lambda}} &= \frac{1}{\langle \Psi_0 \mid \Psi_0 \rangle}(\langle \Psi_0 \mid - \langle \Psi_0 \mid (1+P)VP \mid \psi_0 \rangle \langle \psi_0 \mid) \\ &= \frac{1}{\langle \Psi_0 \mid \Psi_0 \rangle}(\langle \Psi_0 \mid -3\langle \psi_0 \mid (1+P)VP \mid \psi_0 \rangle \langle \psi_0 \mid) = \frac{1}{\langle \Psi_0 \mid \Psi_0 \rangle}(\langle \Psi_0 \mid -3\langle \psi_0 \mid) \ . \end{aligned} \tag{30}$$

Obviously the application onto a totally antisymmetric state yields zero. So, we find

$$(1+P)\psi' = \lambda\Psi_0 + G'\sum_\mu V_\mu G_0 W \Psi_0 \ , \tag{31}$$

where G' acts in a space orthogonal to the unperturbed ground state Ψ_0. Finally, we use

$$G'\sum_\mu V_\mu G_0 = G' - \left[1 - \mid \Psi_0 \rangle \frac{1}{\langle \Psi_0 \mid \Psi_0 \rangle} \langle \Psi_0 \mid \right] G_0 \ , \tag{32}$$

and add the additional first order Faddeev component form (4b):

$$\begin{aligned}(1+P)\psi' + \psi_4' &= G'W\Psi_0 + \lambda\Psi_0 \\ &\quad + \mid \Psi_0 \rangle \frac{1}{\langle \Psi_0 \mid \Psi_0 \rangle} \langle \Psi_0 \mid G_0 W \mid \Psi_0 \rangle \\ &\equiv \Psi_F' \ . \end{aligned} \tag{33}$$

On the other hand, the standard result for the first order wave function correction is, of course,

$$\Psi' = G'W\Psi_0 \ , \tag{34}$$

where one excludes by definition an admixture of the unperturbed state. Using an obvious choice of λ, the admixture of Ψ_0 in Ψ'_F can be eliminated. We shall comment below that this is unnecessary.

The presented link of the Faddeev scheme to the result gained directly from the Schrödinger equation is more transparent and simpler to achieve using the Faddeev equations in differential form. Then the set (4) reads

$$\begin{aligned}(E-H_0-V)\psi &= VP\psi+V\psi_4 ,\\ (E-H_0-W)\psi_4 &= W(1+P)\psi ,\end{aligned} \tag{35}$$

or in first order in W,

$$(E_0+\Delta E'-H_0-V)(\psi_0+\psi')=VP(\psi_0+\psi')+V\psi'_4 , \tag{36a}$$

$$(E_0-H_0)\psi'_4=W(1+P)\psi_0 . \tag{36b}$$

The zeroth order parts drop out and one has

$$[E_0-H_0-V(1+P)]\psi'=V\psi'_4-\Delta E'\psi_0 . \tag{37}$$

Again the shift $\Delta E'$ can be deduced together with the prescription to calculate ψ'. Let us now apply $(1+P)$ from the left:

$$[E_0-H_0-(1+P)V](1+P)\psi'=(1+P)V\psi'_4-\Delta E'\Psi_0 . \tag{38}$$

Because of the antisymmetry of $(1+P)\psi'$ and ψ'_4, this can be rewritten into

$$\left[E_0-H_0-\sum_\mu V_\mu\right](1+P)\psi'=\sum_\mu V_\mu\psi'_4-\Delta E'\Psi_0 , \tag{39}$$

which has the solution

$$(1+P)\psi'=\lambda\Psi_0+G'\sum_\mu V_\mu\psi'_4 , \tag{40}$$

identical to (31).

We get back to the standard procedure if instead of solving (39), we first add (36b) to (39). Then one finds the Schrödinger equation for the first order corrections:

$$\left[E_0-H_0-\sum_\mu V_\mu\right][(1+P)\psi'+\psi'_4]=W\Psi_0-\Delta E'\Psi_0 . \tag{41}$$

The solution is

$$(1+P)\psi'+\psi'_4=G'W\Psi_0+\lambda'\Psi_0 , \tag{42}$$

where λ' is of the order W and is usually set to zero.

Altogether, we find the wave function up to first order in W:

$$\Psi=\Psi_0(1+\lambda)+\Psi' . \tag{43}$$

The indeterminacy in λ disappears into an unobservable overall phase after normalization. One has up to first order (now we assume $\langle\Psi_0\,|\,\Psi_0\rangle=1$):

$$\begin{aligned}\frac{|\Psi\rangle}{\sqrt{\langle\Psi\,|\,\Psi\rangle}}&=\frac{(1+\lambda)\Psi_0+\Psi'}{\sqrt{1+2\,\mathrm{Re}\lambda}}=(1-\mathrm{Re}\lambda)(1+\lambda)\Psi_0+\Psi'\\ &=(1+i\,\mathrm{Im}\lambda)\Psi_0+\Psi'=e^{i\,\mathrm{Im}\lambda}(\Psi_0+\Psi')+O(\lambda^2) .\end{aligned} \tag{44}$$

As a consequence, the nonorthogonality of Ψ'_F with respect to Ψ_0, which shows up using the simple algorithm of the Faddeev scheme as given in (18), does not need to

TABLE I. Components of the triton wave function in the j-j coupling scheme. The quantities p_α give the percentages.

α	l	s	j	λ	jj	t	p_α
1	0	0	0	0	$\frac{1}{2}$	1	0.443
2	0	1	1	0	$\frac{1}{2}$	0	0.449
3	2	1	1	0	$\frac{1}{2}$	0	0.033
4	0	1	1	2	$\frac{3}{2}$	0	0.011
5	2	1	1	2	$\frac{3}{2}$	0	0.002
6	1	0	1	1	$\frac{1}{2}$	0	0.001
7	1	0	1	1	$\frac{3}{2}$	0	0.003
8	1	1	0	1	$\frac{1}{2}$	1	0.011
9	1	1	1	1	$\frac{1}{2}$	1	0.012
10	1	1	1	1	$\frac{3}{2}$	1	0.005
11	1	1	2	1	$\frac{3}{2}$	1	0.001
12	3	1	2	1	$\frac{3}{2}$	1	0.004
13	1	1	2	3	$\frac{5}{2}$	1	0.011
14	3	1	2	3	$\frac{5}{2}$	1	0.000
15	2	0	2	2	$\frac{3}{2}$	1	0.004
16	2	0	2	2	$\frac{5}{2}$	1	0.005
17	2	1	2	2	$\frac{3}{2}$	0	0.001
18	2	1	2	2	$\frac{5}{2}$	0	0.001

be removed if one needs only the first order correction in Ψ.

Now, we want to present our results for the energy expectation value of the Tucson two-pion-exchange three-body force.[2,3] The present work is an extension of a first study[1] where we computed the energy expectation value of the 2PE-3BF, including the contributions from channels 1–5. For the sake of completeness we shall briefly describe our procedure to calculate ΔE in first order perturbation.

Starting from the Faddeev equations for the bound state (6) in a partial wave decomposition with the standard Jacobi momenta

$$\vec{p}=\tfrac{1}{2}(\vec{k}_2-\vec{k}_3)\ , \qquad \vec{q}=\tfrac{2}{3}[\vec{k}_1-\tfrac{1}{2}(\vec{k}_2+\vec{k}_3)]\ , \tag{45}$$

and the basis states (see Ref. 1 for notation)

$$|pq\alpha\rangle=|pq(ls)j(\lambda\tfrac{1}{2})jj\mathscr{F}(t\tfrac{1}{2})T\rangle\ , \tag{46}$$

we gain the triton wave function Ψ_0 from a five channel solution of (6) with the Reid potential[4] by

$$\Psi_0=(1+P)\psi_0\ . \tag{47}$$

Although the Faddeev component ψ_0 was restricted to the ${}^1S_0$, ${}^3S_1$-${}^3D_1$ channels, the permutation operator P introduces infinitely many partial wave states. From these we took into account the 18 states with $j\le 2$, whose quantum numbers are given in Table I together with the computed probabilities of the wave function.

TABLE II. Main contributions to the energy shift (52) in MeV.

α'	$\alpha=1$	$\alpha=2$
1	−0.164	(+0.145)
2	+0.145	−0.139
3	−0.207	−1.058
4	−0.276	+1.592
5	+0.215	+0.003
6	+0.038	−0.456
7	−0.032	+0.338
8	0.	−0.031
9	−0.404	−0.029
10	−0.517	+0.097
11	0.	+0.009
12	0.	+0.051
13	0.	+0.183
14	0.	−0.000
15	−0.099	+0.087
16	−0.010	+0.012
17	+0.170	+0.002
18	+0.330	−0.062
$\sum_{\alpha'}$	−0.811	+0.643−(0.145)=0.498

The part of the 2PE-3BF which singles out particle 1 is given by[2,3]

$$\langle\vec{p}_1\vec{p}_2\vec{p}_3|W_1|\vec{p}\,'_1\vec{p}\,'_2\vec{p}\,'_3\rangle=(2\pi)^{-6}\delta^{(3)}[(\vec{p}_1+\vec{p}_2+\vec{p}_3)-(\vec{p}\,'_1+\vec{p}\,'_2+\vec{p}\,'_3)]\frac{g^2}{4m^2}\frac{H(\vec{Q}^2)}{\vec{Q}^2+\mu^2}\frac{H(\vec{Q}\,'^2)}{\vec{Q}\,'^2+\mu^2}\vec{\sigma}_2\cdot\vec{Q}\vec{\sigma}_3\cdot\vec{Q}\,'$$
$$\times\{\vec{\tau}_2\cdot\vec{\tau}_3[a+b\vec{Q}\cdot\vec{Q}\,'+c(\vec{Q}^2+\vec{Q}\,'^2)]+i\vec{\tau}_3\times\vec{\tau}_2\cdot\vec{\tau}_1\vec{\sigma}_1\cdot\vec{Q}\times\vec{Q}\,'(d_3+d_4)\}\ , \tag{48}$$

where the pion momenta $\vec{Q}$ and $\vec{Q}\,'$ read in terms of the Jacobi momenta (45)

$$\vec{Q}=(\vec{p}-\vec{p}\,')-\tfrac{1}{2}(\vec{q}-\vec{q}\,')\ , \qquad \vec{Q}\,'=(\vec{p}-\vec{p}\,')+\tfrac{1}{2}(\vec{q}-\vec{q}\,')\ , \tag{49}$$

and the strength parameters are given as

$$a=1.13\,\mu^{-1}\ ,\quad b=-2.58\,\mu^{-3}\ ,\quad c=1.00\,\mu^{-3}\ ,\quad d_3+d_4=-0.753\,\mu^{-3}\ . \tag{50}$$

We used a form factor of the form

$$H(\vec{Q}^2)=\left[\frac{\Lambda^2-\mu^2}{\Lambda^2+\vec{Q}^2}\right]^2\ , \tag{51}$$

with $\Lambda=17\ \mathrm{fm}^{-2}$.

After a partial wave decomposition of the 2PE-3BF (Refs. 3 and 5), the energy expectation value is

$$\Delta E=\sum_{\alpha\alpha'}\Delta E^{\alpha\alpha'}=\sum_{\alpha\alpha'}3\int dp\,p^2dq\,q^2dp'p'^2dq'q'^2\Psi_\alpha(pq)\Psi_{\alpha'}(p'q')W_1^{\alpha\alpha'}(pqp'q')\ . \tag{52}$$

As a look at Table I reveals, the main contributions to ΔE should come from combinations of channels 1 and 2 with the other ones because channels 1 and 2 build up about 90 percent of the wave function's norm. Our results coincide with these considerations. So, we shall present in detail only the contributions to ΔE with α or α' equal to 1 or 2 (see Table II). The numbers are meant to be $\Delta E^{\alpha\alpha'}+\Delta E^{\alpha\alpha}$ for $\alpha\neq\alpha'$ and $\Delta E^{\alpha\alpha}$ otherwise, all in MeV.

The remaining contributions sum up to $+0.155$ MeV so that we end with a negligible net result of -0.158 MeV additional attraction for the 2PE-3BF.

The conclusions we draw from our results, as presented in Table II, are as follows: Regarding the unexpected large contributions from combinations of channels 1 and 2 with channels which negligibly contribute to the wave function's norm [e.g., (1,17), (1,18), (2,6), and (2,7)], one has to include even components of the wave function beyond the 18 channels considered by us to come to a final conclusion about the 2PE-3BF energy expectation value.

The slow convergence of ΔE in the channel decomposition is a new feature caused by the three-body force. Either a very tedious summation over many partial waves has to be performed, or better yet, by a new technique, an exact evaluation of the permutation operator occurring in (10a) has to be conducted. We are presently studying that second possibility. Also, the inclusion of further three-body forces and other models for the πN amplitude in the 2PE-3BF is planned.

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