Anharmonicities in $K^\pi=0^+$ and $K^\pi=0^-$ coupled vibrational spectra of deformed nuclei discussed in a simple model

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(Received 5 April 1983)

The multiphonon method—i.e., exact diagonalization in the restricted space of collective phonons of different types—is tested in a simple model allowing an exact solution for a many body system of fermions interacting via pairing plus quadrupole and octupole forces. It appears satisfactory for the description of the anharmonicities of the lowest-lying vibrational states with $K^\pi=0^+$ and 0^- in deformed nuclei. In particular, it allows electromagnetic transitions between the different one phonon states which cannot be described in any harmonic treatments such as the Tamm-Dancoff or random-phase approximations.

[NUCLEAR STRUCTURE Multiphonons $K=0^+$ and $K=0^-$ states of deformed nuclei discussed in a simple model.]

I. INTRODUCTION

The spectroscopic properties of the lowest lying vibrational states in deformed even-even nuclei can, in general, be described with the use of quasibosons, i.e., coherent mixtures of two nucleonic quasiparticle states. The higher excited states of vibrational nature, which may exhibit strong anharmonicities, need more elaborate techniques taking the Pauli principle properly into account.

The multiphonon method (MPM) has been developed to achieve this requirement. The first version of the MPM (Refs. 1 and 2) was built for one type of phonon. It was applied to the study of vibrations built on quadrupole $K^\pi=0^+$ phonons in rare earth nuclei[3] and on octupole $K^\pi=0^-$ phonons in the heavy nuclei.[4] It has also been compared to boson expansion techniques[3] and checked in a simple model.[5,6] A generalization of the MPM to several phonons of different types has recently been obtained.[7]

The aim of the present work is to check the validity of this second version of the MPM in a simple model allowing an exact solution and to study the importance of the coupling of phonons of different types. In Sec. II we present this model. In Secs. III—V we describe, respectively, the standard treatments of the pairing (BCS), the random phase approximation (RPA), and the Tamm-Dancoff calculation (TDA). The generalized version of the MPM (Ref. 7) is sketched in Sec. VI. The results of our calculation using these different approaches are discussed in Sec. VII. Finally, conclusions are drawn in the last section.

II. THE MODEL AND ITS EXACT SOLUTION

Our model is a natural generalization of that used in Refs. 5 and 6. It consists of two $j=\frac{3}{2}$ multiplets with different parities, located at the same energy. Both of them are split by a prolate deformation into two doubly degenerate levels. These multiplets are repeated Ω times (Fig. 1). The total number of states is thus 8Ω.

Consequently, each state is characterized by ρ, the index of the shell ($\rho=1$ for the lower shell, and $\rho=2$ for the upper one), τ, the label of the multiplet ($1\leq\tau\leq\Omega$), and the parity π. We allow an even number N of identical nucleons ($0<N<8\Omega$) to move in this system.

The model Hamiltonian is written as

$$H=H_{\rm sp}+H_P+H_Q+H_{\rm oct}\ , \tag{2.1}$$

where $H_{\rm sp}$ is the single particle part, H_P the usual monopole pairing contribution, and H_Q and $H_{\rm oct}$ the quadrupole and octupole parts of a long range two-body force. Explicitly,

$$H_{\rm sp}=\sum_{\rho\tau\pi} e_\rho (a^\dagger_{\rho\tau\pi}\, a_{\rho\tau\pi}+a^\dagger_{\rho\bar\tau\pi}\, a_{\rho\bar\tau\pi})\ , \tag{2.2}$$

where $a^\dagger_{\rho\tau\pi}$ and $a_{\rho\tau\pi}$ are the creation and annihilation operators for nucleons in the state $\rho\tau\pi$, and $\rho\bar\tau$ is the time reversed state of $\rho\tau$:

$$H_P=-GP^\dagger P\ , \tag{2.3}$$

where

$$P^\dagger=\sum_{\rho\tau\pi} a^\dagger_{\rho\tau\pi}\, a^\dagger_{\rho\bar\tau\pi} \tag{2.4}$$

and

$$H_Q=-\tfrac{1}{2}\chi_2\hat Q^2\ , \tag{2.5}$$

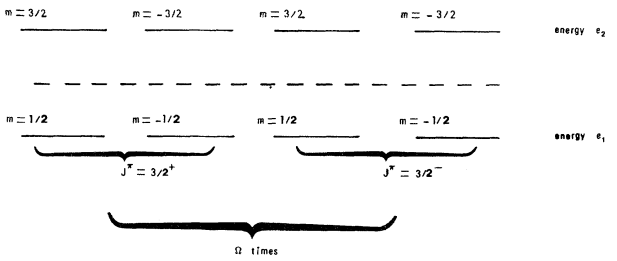

FIG. 1. Individual levels considered in this paper. The reference energy is chosen in the middle of the degenerate shells.

$$H_{\text{oct}}=-\tfrac{1}{2}\chi_3\hat{R}^2 . \tag{2.6}$$

In order to make the model soluble, we have supposed that the quadrupole operator $\hat{Q}$ and the octupole operator $\hat{R}$ have matrix elements only between states of the same shell, and these are assumed to be equal for different multiplets. This ansatz is written

$$\begin{aligned} Q_{\rho\tau\pi,\rho'\tau'\pi}&=q_\rho\delta_{\rho\rho'}\delta_{\tau\tau'} , \\ R_{\rho\tau\pi,\rho'\tau'\bar{\pi}}&=r_\rho\delta_{\rho\rho'}\delta_{\tau\tau'} . \end{aligned} \tag{2.7}$$

In relation (2.7), as well as in all of what follows, the indices π and $\bar{\pi}$ in the same formula refer to different parities ($\pi\bar{\pi}=-1$).

Furthermore, if we take care of the property

$$\mathrm{Tr}\hat{Q}=\mathrm{Tr}\hat{R}=0 \tag{2.8}$$

only one parameter is needed for the quadrupole or octupole operator. Consequently, one gets

$$\hat{Q}=\sum_{\pi\rho\tau} q_\rho(a^\dagger_{\rho\tau\pi}a_{\rho\tau\pi}+a^\dagger_{\rho\bar{\tau}\pi}a_{\rho\bar{\tau}\pi}) , \tag{2.9}$$

$$\hat{R}=\sum_{\pi\rho\tau} r_\rho(a^\dagger_{\rho\tau\pi}a_{\rho\tau\bar{\pi}}+a^\dagger_{\rho\bar{\tau}\pi}a_{\rho\bar{\tau}\bar{\pi}}) , \tag{2.10}$$

where according to (2.8) we can set

$$q_1=-q_2=q, \quad r_1=-r_2=r . \tag{2.11}$$

To obtain the exact solution it is convenient to introduce the following operators,

$$\begin{aligned} K_+(\rho)&=\sum_{\tau,\pi} a^\dagger_{\rho\tau\pi}a^\dagger_{\rho\bar{\tau}\pi} , \\ K_-(\rho)&=[K_+(\rho)]^\dagger , \\ K_0(\rho)&=\tfrac{1}{2}\sum_{\tau,\pi}(a^\dagger_{\rho\tau\pi}a_{\rho\tau\pi}+a^\dagger_{\rho\bar{\tau}\pi}a_{\rho\bar{\tau}\pi})-\Omega , \end{aligned} \tag{2.12}$$

of positive parity, and

$$\begin{aligned} L_+(\rho)&=\sum_{\tau,\pi} a^\dagger_{\rho\tau\pi}a^\dagger_{\rho\bar{\tau}\bar{\pi}} , \\ L_-(\rho)&=[L_+(\rho)]^\dagger , \\ L_0(\rho)&=\tfrac{1}{2}\sum_{\tau,\pi}(a^\dagger_{\rho\tau\pi}a_{\rho\tau\bar{\pi}}+a^\dagger_{\rho\bar{\tau}\pi}a_{\rho\bar{\tau}\bar{\pi}}) , \end{aligned}$$

of negative parity.

With these operators the model Hamiltonian attains the simple expression

$$\begin{aligned} H=&(e_2-e_1)[K_0(2)-K_0(1)]-G[K_+(1)+K_+(2)][K_-(1)+K_-(2)] \\ &-2\chi_2q^2[K_0(2)-K_0(1)]^2-2\chi_3r^2[L_0(2)-L_0(1)]^2 , \end{aligned} \tag{2.13}$$

where e_2-e_1 measures the separation in energy of the two shells; the reference energy has been chosen to have $\frac{1}{2}(e_1+e_2)=0$. It is easy to see that the combinations

$$\begin{aligned} \hat{I}_i(\rho)&=\tfrac{1}{2}[K_i(\rho)+L_i(\rho)] , \\ \hat{J}_i(\rho)&=\tfrac{1}{2}[K_i(\rho)-L_i(\rho)], \quad \text{where } i=0,\pm \end{aligned} \tag{2.14}$$

introduce quasispin operators which form together the algebra of the

$$\mathrm{SU2}_{I(1)}\times\mathrm{SU2}_{I(2)}\times\mathrm{SU2}_{J(1)}\times\mathrm{SU2}_{J(2)}$$

group. We note that these operators mix parities.

No closed expressions can be given for the eigenenergies and eigenstates of (2.13). But numerical solutions can easily be found by diagonalizing H in the basis of the states

$$|I(1),I_0(1);I(2),I_0(2);J(1),J_0(1);J(2),J_0(2)\rangle , \tag{2.15}$$

where $I(\rho)$ $[I(\rho)+1]$ and $J(\rho)$ $[J(\rho)+1]$ are the eigenvalues of total quasispins $\hat{I}^2(\rho)$ and $\hat{J}^2(\rho)$ whereas $I_0(\rho)$ and $J_0(\rho)$ are the eigenvalues of $\hat{I}_0(\rho)$ and $\hat{J}_0(\rho)$.

We remind the reader of the usual conditions,

$$\begin{aligned} &-I(\rho)\le I_0(\rho)\le I(\rho) , \\ &-J(\rho)\le J_0(\rho)\le J(\rho) , \\ &0\le I(\rho)\le\frac{\Omega}{2} , \\ &0\le J(\rho)\le\frac{\Omega}{2} . \end{aligned} \tag{2.16}$$

As can be seen from (2.13) and (2.14), H commutes with $\hat{I}^2(\rho)$ and $\hat{J}^2(\rho)$ so that $I(\rho)$ and $J(\rho)$ are good quantum numbers.

In the following, and for reasons which will become clear later on, we restrict ourselves to the states where

$$I(\rho)=J(\rho)=\frac{\Omega}{2}, \quad \rho=1,2 . \tag{2.17}$$

The space spanned by these states will be referred to as the "collective" space $\mathscr{E}_c$. We emphasize that, in our model, this space is completely decoupled by H from the other parts of the total Fock space $\mathscr{E}$.

Since we are interested in a definite nucleus, the total number N of particles is fixed. It is easy to write the number operator $\hat{N}$ in terms of (2.14),

$$\hat{N}=4\Omega+2\sum_\rho[\hat{I}_0(\rho)+\hat{J}_0(\rho)] . \tag{2.18}$$

We then have the additional condition for the basis states (2.15),

$$I_0(1)+I_0(2)+J_0(1)+J_0(2)=\frac{N}{2}-2\Omega \tag{2.19}$$

which allows a further restriction to a subspace $\mathscr{E}_{NC}$ of the collective space $\mathscr{E}_C$.

A final restriction to a subspace $\mathscr{E}_{NCP}$ of $\mathscr{E}_{NC}$ of a given parity can be achieved. To this purpose a new and simple transformation of the basis (2.15) (which does not preserve parity) is required. One can show that the states

$$|I_0(1),I_0(2),J_0(1),J_0(2),\pm\rangle = F\{|I_0(1),I_0(2),J_0(1),J_0(2)\rangle \pm |J_0(1),J_0(2),I_0(1),I_0(2)\rangle\},$$

where

$$F=\begin{cases}\frac{1}{2} & \text{when } I_0(\rho)=J_0(\rho),\ \rho=1,2\\ \frac{1}{\sqrt{2}} & \text{otherwise},\end{cases}$$

restore parity.

We have computed the solutions in several cases, corresponding to different values of the parameters. For the dynamical ones, one can introduce the three dimensionless quantities

$$e=\frac{e_2-e_1}{G},\quad x=\frac{2\chi_2 q^2}{G},\quad y=\frac{2\chi_3 r^2}{G}. \tag{2.20}$$

In what follows, all energies will be expressed in G units. For the definition of the system we have two other parameters: the degeneracy Ω and the number of particles N, which we shall also replace, when more convenient, by

$$\nu=\frac{N}{4\Omega}-1 \tag{2.21}$$

which measures the deviation from a half-filled system ($-1\le\nu\le+1$).

Once one has the eigenstates, one easily calculates the $E2$ (and $E3$) transitions, i.e., matrix elements of the quadrupole operator (2.9) [octupole operator (2.10)] expressed in terms of the quasispins generators (2.14).

We would like to point out the order of the matrices which we need to diagonalize. Despite the successive restrictions leading to space $\mathscr{E}_{CNP}$, the matrices may have, for some sets of parameters, (N,Ω), very large orders. As an illustration we give, in Table I, some values of the encountered dimensions, for different sets (N,Ω). These values should be compared to

$$\begin{bmatrix}4\Omega\\ N/2\end{bmatrix},$$

the dimension of the total space $\mathscr{E}_N$ collective and noncollective. For the parameter set ($N=28$, $\Omega=8$) we used as a standard case, this total dimension is 471 435 600.

III. THE TREATMENT OF THE PAIRING

To study the pairing in our model, we generalize the improved BCS treatment of Ref. 5. First we introduce a parity dependent special Bogolyubov-Valatin transformation with coefficients $u_{\rho\tau\pi}$ and $v_{\rho\tau\pi}$ leading to quasiparticles $\alpha_{\rho\tau\pi}$.

Then we consider the Hamiltonian $\mathscr{H}=H-\lambda\hat{N}$ which we write in the familiar form

$$\mathscr{H}=U+\mathscr{H}_{11}+\mathscr{H}_{20}+\mathscr{H}_{\text{int}}. \tag{3.1}$$

Our "improved" BCS treatment considers the pairing, quadrupole, and octupole parts of the Hamiltonian (2.1) on an equal footing. Furthermore, since only matrix elements of the form (2.7) are taken into account, our treatment is completely equivalent to a Hartree-Fock-Bogolyubov (HFB) procedure. It leads to a state-dependent energy gap, which has the form

$$\Delta_{\rho\tau\pi}=\sum_{\rho'\tau'\pi'} u_{\rho'\tau'\pi'}v_{\rho'\tau'\pi'}+\frac{x}{2}u_{\rho\tau\pi}v_{\rho\tau\pi}+\frac{y}{2}u_{\rho\tau\bar{\pi}}v_{\rho\tau\bar{\pi}}, \tag{3.2}$$

where π' stands for one of the two opposite parities π and $\bar{\pi}$. The pairing coefficients follow from the condition $\mathscr{H}_{20}\equiv 0$, which, according to the Thouless theorem, is equivalent to minimizing the constant term U in (3.1). They are

$$\begin{bmatrix}u^2_{\rho\tau\pi}\\ v^2_{\rho\tau\pi}\end{bmatrix}=\frac{1}{2}\left[1\pm\frac{\tilde{e}_{\rho\tau\pi}}{E_{\rho\tau\pi}}\right], \tag{3.3}$$

where

$$E_{\rho\tau\pi}=(\tilde{e}^2_{\rho\tau\pi}+\Delta^2_{\rho\tau\pi})^{1/2} \tag{3.4}$$

is the quasiparticle energy, and

TABLE I. Values of the dimension of the energy matrix one has to diagonalize in the exact calculation for different sets of parameters (N,Ω).

Ω \ $N/2$		3	4	5	6	7	8	9	10	11	12	13	14
4	0^+	10	19	26	36	40	45	40	36	26	19	10	6
	0^-	10	16	26	32	40	40	40	32	26	16	10	4
6	0^+	10	19	28	44	58	77	90	106	112	119	112	106
	0^-	10	16	28	40	58	72	90	100	112	112	112	100
8	0^+	10	19	28	44	60	85	108	138	162	191	210	232
	0^-	10	16	28	40	60	80	108	132	162	184	210	224
10	0^+	10	19	28	44	60	85	110	146	180	223	260	304
	0^-	10	16	28	40	60	80	110	140	180	216	260	296
12	0^+	10	19	28	44	60	85	110	146	182	231	278	336
	0^-	10	16	28	40	60	80	110	140	182	224	278	328

$$\tilde{e}_{\rho\tau\pi}=e_\rho-\lambda-\chi_2\frac{Q}{G}q_\rho-v^2_{\rho\tau\pi}-\frac{x}{4}(u^2_{\rho\tau\pi}-v^2_{\rho\tau\pi})$$
$$-\frac{y}{4}(u^2_{\rho\tau\bar{\pi}}-v^2_{\rho\tau\bar{\pi}}) \tag{3.5}$$

is the self-consistent energy, while

$$Q=2\sum_{\rho\tau\pi}q_\rho v^2_{\rho\tau\pi} \tag{3.6}$$

denotes the expectation value of the quadrupole moment.

The gap equation is

$$\Delta_{\rho\tau\pi}=\tfrac{1}{2}\sum_{\rho'\tau'\pi'}\frac{\Delta_{\rho'\tau'\pi'}}{E_{\rho'\tau'\pi'}}+\frac{x}{4}\frac{\Delta_{\rho\tau\pi}}{E_{\rho\tau\pi}}+\frac{y}{4}\frac{\Delta_{\rho\tau\bar{\pi}}}{E_{\rho\tau\bar{\pi}}}\ . \tag{3.7}$$

One can easily prove that all BCS quantities are in fact independent of the quantum number τ. We further make the ansatz that they are also independent of the parity π; consequently, only one index ρ will be needed. We then solve the BCS equations, i.e., the gap equation (3.7) and the equation for the number of particles,

$$N=2\Omega\sum_\rho\left[1-\frac{\tilde{e}_\rho}{E_\rho}\right]\ . \tag{3.8}$$

We use the same procedure as in Ref. 5, and introduce two quantities,

$$z_1=a-\frac{\Delta_2}{\Delta_1},\quad z_2=a-\frac{\Delta_1}{\Delta_2} \tag{3.9}$$

satisfying the relation

$$(a-z_1)(a-z_2)=1,\quad \text{with } a=1+\frac{x+y}{4\Omega}\ . \tag{3.10}$$

The BCS equations may then be reduced to

$$z_2^2(a-z_1)\{[2Kz_1z_2-(z_1+z_2)]^2-[2z_1z_2d+2\nu z_1(1-Kz_2)]^2\}$$
$$=z_1^2(a-z_2)\{[2Kz_1z_2-(z_1+z_2)]^2-[2z_1z_2d-2\nu z_2(1-Kz_1)]^2\}\ , \tag{3.11}$$

where

$$K=\frac{(8\Omega-1)x-y+2}{4\Omega(a^2-1)}$$

and

$$d=\frac{e}{2\Omega(a^2-1)}\ .$$

To solve the system of equations (3.10) and (3.11) we use the numerical method of Ref. 5.

The treatment of the pairing for a given system (i.e., for a definite N and Ω) and a given force (x and y fixed) consists in solving this system. Once z_1 and z_2 are obtained, one calculates E_ρ, $\tilde{e}_\rho$, Δ_ρ, u_ρ, and v_ρ, and finally one determines the chemical potential λ, the expectation value Q of the quadrupole moment, and the energy $U_{\rm BCS}$ of the ground state. For this purpose, one needs Eqs. (3.3) and (3.6) and the following relations:

$$\frac{1}{E_\rho}=\frac{z_\rho}{\Omega(a^2-1)}\ ,$$
$$\tilde{e}_1=\frac{-2\Omega\nu(a^2-1)(Kz_2-1)+ez_2}{2Kz_1z_2-(z_1+z_2)}\ , \tag{3.12}$$
$$\tilde{e}_2=\frac{-2\Omega\nu(a^2-1)(Kz_1-1)-ez_1}{2Kz_1z_2-(z_1+z_2)}\ ,$$
$$\lambda=\tfrac{1}{2}[(K'z_1-1)\tilde{e}_1+(K'z_2-1)\tilde{e}_2-1]\ , \tag{3.13}$$

where

$$K'=\frac{2-x-y}{4\Omega(a^2-1)}\ ,$$

and

$$U_{\rm BCS}=-\frac{\chi_2Q^2}{2}-2\Omega(x+y)\sum_\rho u_\rho^2v_\rho^2-4\Omega^2\left[\sum_\rho u_\rho v_\rho\right]^2$$
$$+4\Omega\sum_\rho e_\rho v_\rho^2-2\Omega\sum_\rho v_\rho^4\ . \tag{3.14}$$

IV. RANDOM PHASE APPROXIMATION

To get a rather compact form of the quasiparticle interaction term

$$\mathscr{H}_{\rm int}=\mathscr{H}_{40}+\mathscr{H}_{31}+\mathscr{H}'_{22}+\mathscr{H}''_{22} \tag{4.1}$$

we introduce the following "elementary" operators:

$$\gamma^\dagger_{\rho\tau\pi}=\alpha^\dagger_{\rho\tau\pi}\alpha^\dagger_{\rho\bar{\tau}\pi}\ ,$$
$$\lambda^\dagger_{\rho\tau\pi}=\alpha^\dagger_{\rho\tau\pi}\alpha^\dagger_{\rho\bar{\tau}\bar{\pi}}$$

and, since every BCS quantity depends only on the shell index ρ, we have the following "collective" operators:

$$\Gamma^\dagger_\rho=\sum_{\tau,\pi}\gamma^\dagger_{\rho\tau\pi}\ ,\quad \Lambda^\dagger_\rho=\sum_{\tau,\pi}\lambda^\dagger_{\rho\tau\pi}\ ,$$
$$\mathscr{N}_\rho=\sum_{\tau,\pi}(\alpha^\dagger_{\rho\tau\pi}\alpha_{\rho\tau\pi}+\alpha^\dagger_{\rho\bar{\tau}\pi}\alpha_{\rho\bar{\tau}\pi})\ , \tag{4.2}$$
$$\mathscr{L}_\rho=\sum_{\tau,\pi}(\alpha^\dagger_{\rho\tau\pi}\alpha_{\rho\tau\bar{\pi}}+\alpha^\dagger_{\rho\bar{\tau}\pi}\alpha_{\rho\bar{\tau}\bar{\pi}})\ .$$

These operators obey the following algebra:

$$[\Gamma_\rho,\Gamma^\dagger_{\rho'}]=[\Lambda_\rho,\Lambda^\dagger_{\rho'}]=\delta_{\rho\rho'}(2\Omega-\mathscr{N}_\rho)\ ,$$
$$[\mathscr{N}_\rho,\Gamma^\dagger_{\rho'}]=[\mathscr{L}_\rho,\Lambda^\dagger_{\rho'}]=2\delta_{\rho\rho'}\Gamma^\dagger_\rho\ ,$$
$$[\mathscr{L}_\rho,\Gamma^\dagger_{\rho'}]=[\mathscr{N}_\rho,\Lambda^\dagger_{\rho'}]=2\delta_{\rho\rho'}\Lambda^\dagger_\rho\ , \tag{4.3}$$
$$[\Lambda_\rho,\Gamma^\dagger_{\rho'}]=[\Gamma_\rho,\Lambda^\dagger_{\rho'}]=-\delta_{\rho\rho'}\mathscr{L}_\rho\ ,$$
$$[\mathscr{N}_\rho,\mathscr{N}_{\rho'}]=[\mathscr{N}_\rho,\mathscr{L}_{\rho'}]=[\mathscr{L}_\rho,\mathscr{L}_{\rho'}]=0\ .$$

With these notations the different parts of $\mathscr{H}$ are

$$\mathscr{H}_{11}=\sum_{\rho} E_{\rho}\mathscr{N}_{\rho} ,$$

$$\mathscr{H}_{40}=\sum_{\rho\rho'}\left[u_{\rho}^{2}v_{\rho'}^{2}+\frac{x}{4}(-1)^{\rho+\rho'+1}\zeta_{\rho}\zeta_{\rho'}\right]\Gamma_{\rho}^{\dagger}\Gamma_{\rho'}^{\dagger}-\sum_{\rho\rho'}(-1)^{\rho+\rho'}\frac{y}{4}\zeta_{\rho}\zeta_{\rho'}\Lambda_{\rho}^{\dagger}\Lambda_{\rho'}^{\dagger}+\text{H.c.} ,$$

$$\mathscr{H}_{31}=\sum_{\rho\rho'}\left[\tfrac{1}{2}\eta_{\rho}\zeta_{\rho'}+\frac{x}{2}(-1)^{\rho+\rho'+1}\zeta_{\rho}\eta_{\rho'}\right]\Gamma_{\rho}^{\dagger}\mathscr{N}_{\rho'}-\sum_{\rho\rho'}(-1)^{\rho+\rho'}\frac{y}{2}\zeta_{\rho}\eta_{\rho'}\Lambda_{\rho}^{\dagger}\mathscr{L}_{\rho'}+\text{H.c.} , \tag{4.4}$$

$$\mathscr{H}'_{22}=-\sum_{\rho\rho'}\left[u_{\rho}^{2}u_{\rho'}^{2}+v_{\rho}^{2}v_{\rho'}^{2}(-1)^{\rho+\rho'}\frac{x}{2}\zeta_{\rho}\zeta_{\rho'}\right]\Gamma_{\rho}^{\dagger}\Gamma_{\rho'}-\sum_{\rho\rho'}(-1)^{\rho+\rho'}\frac{y}{2}\zeta_{\rho}\zeta_{\rho'}\Lambda_{\rho}^{\dagger}\Lambda_{\rho'} ,$$

$$\mathscr{H}''_{22}=\sum_{\rho}\tfrac{1}{4}[\zeta_{\rho}^{2}+(x+y)\eta_{\rho}^{2}]\mathscr{N}_{\rho}-\sum_{\rho\rho'}\left[\frac{\zeta_{\rho}\zeta_{\rho'}}{4}+(-1)^{\rho+\rho'}\frac{x}{4}\eta_{\rho}\eta_{\rho'}\right]\mathscr{N}_{\rho}\mathscr{N}_{\rho'}-\sum_{\rho\rho'}\frac{y}{4}(-1)^{\rho+\rho'}\eta_{\rho}\eta_{\rho'}\mathscr{L}_{\rho}\mathscr{L}_{\rho'}$$

with $\eta_{\rho}=u_{\rho}^{2}-v_{\rho}^{2}$ and $\zeta_{\rho}=2u_{\rho}v_{\rho}$. Here H.c. denotes the Hermitian conjugate operators.

The RPA introduces new operators,

$$B_{j}^{\dagger}=\sum_{\rho\tau\pi}(X_{\rho\tau\pi}^{j}\gamma_{\rho\tau\pi}^{\dagger}-Y_{\rho\tau\pi}^{j}\gamma_{\rho\tau\pi}) \tag{4.5}$$

for positive parity, and

$$B_{j}'^{\dagger}=\sum_{\rho\tau\pi}(X'^{j}_{\rho\tau\pi}\lambda_{\rho\tau\pi}^{\dagger}-Y'^{j}_{\rho\tau\pi}\lambda_{\rho\tau\pi})$$

for negative parity. It requires the equations of motion

$$[\mathscr{H},B_{j}^{\dagger}]=\omega_{j}B_{j}^{\dagger} ,$$
$$[\mathscr{H},B_{j}'^{\dagger}]=\omega_{j}'B_{j}'^{\dagger} . \tag{4.6}$$

The index j labels the solutions we look for. In writing explicitly these expressions we assume, as is usual in RPA, that operators Γ and λ obey boson commutation relations. To be consistent with the HFB character of the pairing treatment, our linearization procedure of (4.6) consists in retaining in the commutators all terms in $\gamma^{\dagger}$, γ, $\lambda^{\dagger}$, and λ.

We remind the reader that this method differs from the usual RPA in respect to the fact that the contribution of $\mathscr{H}''_{22}$ is not neglected.

In the case of negative parity, by solving the RPA equations, we find a number of degenerate solutions the X' and Y' amplitudes of which are τ dependent and which correspond to states of noncollective nature (pure two quasiparticle excitations). Besides them, there exist, in general, *two* collective solutions for which the amplitudes $X'_{\rho\tau\pi}$ and $Y'_{\rho\tau\pi}$ depend only on the shell index ρ. Explicitly, one can show, after a straightforward but lengthy calculation, that the collective energies obey the following equation:

$$\omega'^{4}-\omega'^{2}[E_{1}^{+}E_{1}^{-}+E_{2}^{+}E_{2}^{-}-2\Omega y(\zeta_{1}^{2}E_{1}^{+}+\zeta_{2}^{2}E_{2}^{+})]$$
$$+E_{1}^{+}E_{2}^{+}[E_{1}^{-}E_{2}^{-}-2\Omega y(\zeta_{1}^{2}E_{2}^{-}+\zeta_{2}^{2}E_{1}^{-})]=0 , \tag{4.7}$$

where

$$E_{\rho}^{+}=2E_{\rho}-\tfrac{1}{2}(x+y) ,$$
$$E_{\rho}^{-}=2E_{\rho}-\zeta_{\rho}^{2}-\tfrac{1}{2}(x+y)(\eta_{\rho}^{2}-\zeta_{\rho}^{2}) .$$

In general, or at least for all values of the parameters we introduced, E_{ρ}^{+} is a positive quantity, and Eq. (4.7) admits two real solutions, ω_{1}' and ω_{2}'.

Explicitly,[8] one gets, for the lowest lying state,

$$\omega_{1}'=[\tfrac{1}{2}(\epsilon-\sqrt{D})]^{1/2} , \tag{4.8}$$

where

$$\epsilon=\sum_{\rho}E_{\rho}^{+}(E_{\rho}^{-}-k_{\rho}^{2}) ,$$

$$k_{\rho}=\sqrt{2\Omega y}\,\zeta_{\rho},$$

$$D=\epsilon^{2}+4E_{1}^{+}E_{2}^{+}k_{1}^{2}k_{2}^{2} .$$

The corresponding wave function coefficients are

$$X'_{\rho\tau\pi}=X'_{\rho}=\frac{1}{4}\left[1+\frac{\omega_{1}'}{M_{\rho}}\right]Z_{\rho} ,$$
$$Y'_{\rho\tau\pi}=Y'_{\rho}=\frac{1}{4}\left[1-\frac{\omega_{1}'}{M_{\rho}}\right]Z_{\rho} , \tag{4.9}$$

where

$$M_{\rho}=E_{\rho}^{+}E_{\rho}^{-}-\omega'^{2} ,$$
$$Z_{\rho}=(-1)^{\rho}\left\{\frac{E_{\rho}^{+}}{\Omega\omega_{1}'}[1+(-1)^{\rho+1}\beta]\right\}^{1/2}$$

and

$$\beta=\frac{\epsilon'}{\sqrt{D}} ,$$
$$\epsilon'=\sum_{\rho}(-1)^{\rho}E_{\rho}^{+}(E_{\rho}^{-}-k_{\rho}^{2}) .$$

For the second collective state, one obtains, similarly,

$$\omega_{2}'=[\tfrac{1}{2}(\epsilon+\sqrt{D})]^{1/2} , \tag{4.10}$$
$$X'_{\rho\tau\pi}=X'_{\rho}=\frac{1}{4}\left[1+\frac{\omega_{2}'}{M_{\rho}}\right]Z'_{\rho} ,$$
$$Y'_{\rho\tau\pi}=Y'_{\rho}=\frac{1}{4}\left[1-\frac{\omega_{2}'}{M_{\rho}}\right]Z'_{\rho} , \tag{4.11}$$
$$Z'_{\rho}=\left\{\frac{E_{\rho}^{+}}{\Omega\omega_{2}'}[1+(-1)^{\rho}\beta]\right\}^{1/2} .$$

In the case of positive parity, the treatment is somewhat similar. As previously, a number of degenerate noncollective solutions are obtained. In general, there exist also two collective solutions. Here, however, one of the collective roots lies at zero energy and corresponds to the spurious solution which violates the number conservation. In order to obtain this solution at exactly zero energy, use is made of the gap equation (3.7). We would like to stress here that this exact removal of the spurious state is possible only with the help of coherent improved BCS (orbit dependent gaps) and improved RPA ($\mathscr{H}''_{22}$ taken into account) treatment. The other collective solution, called the physical solution, when it exists, has an energy ω given by

$$\omega=\left[\sum_\rho E_\rho^+(E_\rho^- -2\Omega-\vec{k}_\rho^2)+2\Omega(\vec{k}_1+\vec{k}_2)^2\right]^{1/2}, \tag{4.12}$$

where the vector $\vec{k}_\rho$ has the two components $\sqrt{2\Omega}\eta_\rho$ and $(-1)^{\rho+1}\sqrt{2\Omega x}\,\zeta_\rho$.

The corresponding wave function coefficients are

$$X_{\rho\tau\pi}=X_\rho=\tfrac{1}{2}(Z_\rho+Z'_\rho)\,,$$
$$Y_{\rho\tau\pi}=Y_\rho=\tfrac{1}{2}(Z_\rho-Z'_\rho)$$

with

$$Z_1=\left[\frac{E_1^+}{\omega E_2^+}\right]^{1/2},$$
$$Z_2=-\left[\frac{E_2^+}{\omega E_1^+}\right]^{1/2}, \tag{4.13}$$
$$Z'_1=-\frac{E_1^+(E_1^- -\vec{k}_1^2)+E_2^+\vec{k}_1\cdot\vec{k}_2}{\omega E_1^+}Z_1\,,$$
$$Z'_2=-\frac{E_2^+(E_2^- -\vec{k}_2^2)+E_1^+\vec{k}_1\vec{k}_2}{\omega E_1^+}Z_1\,.$$

The RPA is a harmonic approximation, where the B^+ and $B_i'^\dagger$ operators (4.5) are supposed to obey the boson commutation relations:

$$[B,B^\dagger]=[B'_i,B'^\dagger_i]=1\,.$$

It follows that the electromagnetic transitions between states,

$$|n_0n_1n_2\rangle=\frac{B^{\dagger n_0}B_1'^{\dagger n_1}B_2'^{\dagger n_2}}{\sqrt{n_0!n_1!n_2!}}|\mathrm{RPA}\rangle\,, \tag{4.14}$$

where $|\mathrm{RPA}\rangle$ is the correlated RPA ground state, are rather simple to evaluate. In particular, all states (4.14) have the same quadrupole moment Q given by (3.6). Furthermore, only $E2$ cascade transitions take place:

$$\langle m_0m_1m_2|\hat{Q}|n_0n_1n_2\rangle=2\Omega q\left[\sum_\rho(-1)^{\rho+1}\zeta_\rho(X_\rho+Y_\rho)\right]\delta_{m_1,n_1}\delta_{m_2,n_2}(\sqrt{n_0}\delta_{n_0,m_0+1}+\sqrt{m_0}\delta_{m_0,n_0+1})\,.$$

A similar relation holds for the $E3$ cascade transitions,

$$\langle m_0m_1m_2|\hat{R}|n_0n_1n_2\rangle=2\Omega r\delta_{m_0,n_0}\left\{\left[\sum_\rho(-1)^{\rho+1}\zeta_\rho(X_\rho^{(1)}+Y_\rho^{(1)})\right]\delta_{m_2,n_2}(\sqrt{n_1}\delta_{n_1,m_1+1}+\sqrt{m_1}\delta_{m_1,n_1+1})\right.$$
$$\left.+\left[\sum_\rho(-1)^{\rho+1}\zeta_\rho(X_\rho^{(2)}+Y_\rho^{(2)})\right]\delta_{m_1,n_1}(\sqrt{n_2}\delta_{n_2,m_2+1}+\sqrt{m_2}\delta_{m_2,n_2+1})\right\}.$$

V. TAMM-DANCOFF APPROXIMATION

The principle of this approximation is quite analogous to the one developed in the last section. Instead of operators (4.5) we now introduce TDA—type phonons,

$$B_j^\dagger=\sum_{\rho\tau\pi}X^j_{\rho\tau\pi}\gamma^\dagger_{\rho\tau\pi} \tag{5.1}$$

for positive parity and

$$B_j'^\dagger=\sum_{\rho\tau\pi}X'^j_{\rho\tau\pi}\lambda^\dagger_{\rho\tau\pi}$$

for negative parity. We still require equations of motion of the form (4.6), introduce the part $\mathscr{H}''_{22}$ of $\mathscr{H}_{\mathrm{int}}$ but retain in the linearization procedure only the terms of the commutators in $\gamma^\dagger$ and $\lambda^\dagger$. Moreover, the ground state is the unperturbed $|\mathrm{BCS}\rangle$ state. As in the RPA treatment, besides a number of noncollective solutions, two collective roots are obtained for each parity.

Explicitly, one gets, for the negative collective states,

$$\omega'_1=\frac{1}{2}\left[\sum_\rho(\tilde{E}_\rho-\Omega y\zeta_\rho^2)-\sqrt{d'}\right], \tag{5.2}$$

where

$$\tilde{E}_\rho=2E_\rho-\tfrac{1}{2}[\zeta_\rho^2+(x+y)\eta_\rho^2]\,,$$
$$d'=\left[\sum_\rho(\tilde{E}_\rho-\Omega y\zeta_\rho^2)\right]^2+4\Omega^2y^2\zeta_1^2\zeta_2^2\,. \tag{5.3}$$

The corresponding wave function coefficients are

$$X'_{\rho\tau\pi}=X_\rho^{(1)}=(-1)^\rho\left[\frac{1+(-1)^\rho\beta}{4\Omega}\right]^{1/2}, \tag{5.4}$$

where

$$\beta=\frac{\sum_\rho(-1)^\rho(\tilde{E}_\rho+\Omega y\zeta_\rho^2)}{\sqrt{d'}}\,.$$

The second negative collective solution has

$$\omega_2' = \frac{1}{2}\left[\sum_\rho (\tilde{E}_\rho - \Omega y \zeta_\rho^2) + \sqrt{d'}\right], \tag{5.5}$$

$$X'_{\rho\tau\pi} = X_\rho^{(2)} = \left[\frac{1+(-1)^{\rho+1}\beta}{4\Omega}\right]^{1/2}. \tag{5.6}$$

Similarly, for the positive parity states, we have

$$\omega_1 = \tfrac{1}{2}(S_1 + S_2 - \sqrt{d}), \tag{5.7}$$

where

$$S_\rho = \tilde{E}_\rho - \Omega(1 + x\zeta_\rho^2 + \eta_\rho^2),$$
$$d = (S_1 - S_2)^2 + 4S^2, \tag{5.8}$$

with

$$S = \Omega(x\zeta_1\zeta_2 - 1 - \eta_1\eta_2),$$
$$X_{1\tau\pi} = X_1^{(1)} = -sg(S)\left[\frac{1-\beta}{4\Omega}\right]^{1/2}, \tag{5.9}$$
$$X_{2\tau\pi} = X_2^{(1)} = \left[\frac{1+\beta}{4\Omega}\right]^{1/2},$$

where

$$\beta = \frac{S_1 - S_2}{\sqrt{d}}.$$

The second positive solution is given by

$$\omega_2 = \tfrac{1}{2}(S_1 + S_2 + \sqrt{d}), \tag{5.10}$$

$$X_{1\tau\pi} = X_1^{(2)} = sg(S)\left[\frac{1+\beta}{4\Omega}\right]^{1/2},$$

$$X_{2\tau\pi} = X_2^{(2)} = \left[\frac{1-\beta}{4\Omega}\right]^{1/2}. \tag{5.11}$$

We can see in Eqs. (5.3) and (5.8) that the two solutions of each parity always exist, contrary to the RPA solutions. Furthermore, neither of the two positive solutions is completely spurious or completely physical as in the RPA. Nevertheless, as in Ref. 5, we shall consider the lowest positive solution ω_1 as almost spurious. Several arguments can be given to justify this choice: First, its energy is close to the zero energy spurious solution of the RPA; second, this lowest solution has the same symmetry of the wave function as the spurious RPA solution; and third, for the peculiar cases of our model $e=0$ and/or $\nu=0$, the root is exactly the spurious state $(\hat{N}-N)\,|\,\mathrm{BCS}\rangle$. As well as the RPA, the TDA is a harmonic approximation where the $B^\dagger$ and $B_i'^\dagger$ operators (5.1) are supposed to be pure bosons. The multiboson states in this approximation are

$$|\,n_0n_1n_2\rangle = \frac{B^{\dagger n_0}B_1'^{\dagger n_1}B_2'^{\dagger n_2}}{\sqrt{n_0!n_1!n_2!}}\,|\,\mathrm{BCS}\rangle, \tag{5.12}$$

where B stands for the physical positive parity phonon defined by Eq. (5.11).

As for the RPA, all states (5.12) have the same quadrupole moment Q given by (3.6). Similarly, only cascade transitions can occur. These are given by

$$\langle m_0m_1m_2\,|\,\hat{Q}\,|\,n_0n_1n_2\rangle = 2\Omega q\left[\sum_\rho (-1)^{\rho+1}\zeta_\rho X_\rho\right]\delta_{m_1,n_1}\delta_{m_2,n_2}(\sqrt{n_0}\,\delta_{n_0,m_0+1} + \sqrt{m_0}\,\delta_{m_0,n_0+1})$$

for $E2$, and

$$\langle m_0m_1m_2\,|\,\hat{R}\,|\,n_0n_1n_2\rangle = 2\Omega r\delta_{m_0,n_0}\left\{\left[\sum_\rho (-1)^{\rho+1}\zeta_\rho X_\rho^{(1)}\right]\delta_{m_2,n_2}(\sqrt{n_1}\,\delta_{n_1,m_1+1} + \sqrt{m_1}\,\delta_{m_1,n_1+1})\right.$$
$$\left. + \left[\sum_\rho (-1)^{\rho+1}\zeta_\rho X_\rho^{(2)}\right]\delta_{m_1,n_1}(\sqrt{n_2}\,\delta_{n_2,m_2+1} + \sqrt{m_2}\,\delta_{m_2,n_2+1})\right\}$$

for $E3$.

VI. MULTIPHONON METHOD

In order to explain the anharmonicities in the nuclear vibrations one needs to treat the Pauli principle properly in piling up phonons to build higher excited states. The multiphonon method[1-7] has been developed to achieve this goal.

In the model of the present study we have three basic TDA phonons: two of negative parity, (5.4) and (5.6), and one of positive parity, (5.11). The use of the general MPM of Ref. 7 is then required. Two possibilities are offered: either the application of a generalized Wick's theorem or the use of recursion formulas. For numerical reasons it appears that the second approach is more convenient. For a more detailed formalation of this, the reader is referred to Sec. III of Ref. 7. Here only a sketch of this approach is given.

We start with the multiphonon states

$$|\,n_0n_1n_2\rangle = \frac{B^{\dagger n_0}B_1^{\dagger n_1}B_2^{\dagger n_2}}{n_0!n_1!n_2!}\,|\,0\rangle, \tag{6.1}$$

where $|\,0\rangle = |\,\mathrm{BCS}\rangle$ is the quasiparticle vacuum, and where, for brevity, we have omitted the primes of the negative phonons. Note the difference of the proportionality factors in the nonorthogonal multiphonon basis (6.1) and in the orthonormal states of the RPA (4.14) and TDA (5.12). To calculate the eigenstates and eigenvalues of the model Hamiltonian (2.1) we use the collective subspace

spanned by the states (6.1). First we remark that these do not form an orthonormalized basis. So we shall have to calculate the overlap matrix elements

$$\mathcal{F}(m_0,m_1,m_2;n_0,n_1,n_2)=\langle m_0m_1m_2\,|\,n_0n_1n_2\rangle \quad (6.2)$$

as well as

$$\langle m_0m_1m_2\,|\,\mathcal{H}\,|\,n_0n_1n_2\rangle . \quad (6.3)$$

To simplify a bit the notation, the total Hamiltonian $\mathcal{H}$ of Eq. (4.4) is rewritten in the form

$$\mathcal{H}=\sum_\rho \tilde{E}_\rho \mathcal{N}_\rho+\sum_{\rho\sigma}[P_{\rho\sigma}(\Gamma^\dagger_\rho\Gamma^\dagger_\sigma+\Gamma_\sigma\Gamma_\rho)+P'_{\rho\sigma}(\Lambda^\dagger_\rho\Lambda^\dagger_\sigma+\Lambda_\sigma\Lambda_\rho)+R_{\rho\sigma}(\Gamma^\dagger_\rho\mathcal{N}_\sigma+\mathcal{N}_\sigma\Gamma_\rho)+R'_{\rho\sigma}(\Lambda^\dagger_\rho\mathcal{L}_\sigma+\mathcal{L}_\sigma\Lambda_\rho)$$
$$+S_{\rho\sigma}\Gamma^\dagger_\rho\Gamma_\sigma+S'_{\rho\sigma}\Lambda^\dagger_\rho\Lambda_\sigma+T_{\rho\sigma}\mathcal{N}_\rho\mathcal{N}_\sigma+T'_{\rho\sigma}\mathcal{L}_\rho\mathcal{L}_\sigma] , \quad (6.4)$$

where the coefficients $\tilde{E}$, P, R, R', S, S', T, and T' can easily be deduced from Eq. (4.4).

The calculation of the quantities (6.3) needs the evaluation of the corresponding matrix elements of the one body operator $\mathcal{N}$ and the two body operators $\Gamma^\dagger\Gamma^\dagger$, $\Lambda^\dagger\Lambda^\dagger$, $\Gamma^\dagger\mathcal{N}$, $\Lambda^\dagger\mathcal{L}$, $\Gamma^\dagger\Gamma$, $\Lambda^\dagger\Lambda$, $\mathcal{N}\mathcal{N}$, and $\mathcal{L}\mathcal{L}$.

It can be shown,[8] that all these can be deduced from the quantities (6.2) and

$$\mathcal{A}_\rho(m_0,m_1m_2;n_0n_1n_2)=\langle m_0m_1m_2\,|\,\Gamma^\dagger_\rho\,|\,n_0n_1n_2\rangle ,$$
$$\mathcal{B}_\rho(m_0m_1m_2;n_0n_1n_2)=\langle m_0m_1m_2\,|\,\Lambda^\dagger_\rho\,|\,n_0n_1n_2\rangle ,$$
$$\mathcal{C}_{\rho\rho'}(m_0m_1m_2;n_0n_1n_2)=\langle m_0m_1m_2\,|\,\Gamma^\dagger_\rho\Gamma^\dagger_{\rho'}\,|\,n_0n_1n_2\rangle ,$$
$$\mathcal{D}_{\rho\rho'}(m_0m_1m_2;n_0n_1n_2)=\langle m_0m_1m_2\,|\,\Gamma^\dagger_\rho\Lambda^\dagger_{\rho'}\,|\,n_0n_1n_2\rangle ,$$
$$\mathcal{E}_{\rho\rho'}(m_0m_1m_2;n_0n_1n_2)=\langle m_0m_1m_2\,|\,\Lambda^\dagger_\rho\Lambda^\dagger_{\rho'}\,|\,n_0n_1n_2\rangle , \quad (6.5)$$

which in turn are calculated using recursion formulas[8] deduced from the general procedure.[7]

The eigenvalues and eigenstates of $\mathcal{H}$ are then obtained using the canonical orthogonalization procedure of Lödwin,[9] and standard diagonalization codes.

Once the eigenstates $|\Psi_K\rangle$ are obtained, we calculate the matrix elements $\langle\psi_{K'}|\hat{Q}|\psi_K\rangle$ and $\langle\psi_{K'}|\hat{R}|\psi_K\rangle$ of the quadrupole and octupole operators (2.9) and (2.10). To have a more physical picture, it will also be useful to express the eigenfunctions $|\psi_K\rangle$ in terms of the vectors (6.1),

$$|\Psi_K\rangle=\sum c(n_0n_1n_2)\,|\,n_0n_1n_2\rangle . \quad (6.6)$$

In our applications, we shall call the multiphonon method where all states (6.1) are incorporated MPM3. We will also solve the eigensolution problem with a space reduced to states built on two kinds of phonons, MPM2, [i.e., $n_2=0$ in (6.1)], and that built with one phonon, MPM1 [i.e., $n_0=n_2=0$ in (6.1), or the case studied in Ref. 4].

A comparison of the results of the three calculations will allow us to analyze the effects of switching in successively the different phonons. MPM1 will simulate the experimental situation observed[4] in nuclei of the mass region $222\leq A\leq 226$ while MPM2 is expected to be used for heavier nuclei.

In principle, within our model, it can be shown that the space $\mathcal{E}_{C\pi}$ spanned by all the multiphonon states (6.1) compatible with the Pauli principle, including the removed spurious 0^+ phonon, is the direct sum of the exact collective spaces $\mathcal{E}_{NC\pi}$ for all different number of particles $\mathcal{E}_{C\pi}=\sum_N\oplus\mathcal{E}_{NC\pi}$. Thus, the diagonalization of $\mathcal{H}$ in the whole MPM collective space would give rise to all collective states of all nuclei. However, this is not the aim of the MPM, which we want to apply later to more realistic situations; the procedure used is to choose the best phonons (through a TDA treatment, for example) for a given nucleus and to diagonalize the Hamiltonian in a truncated space [i.e., restriction in the number of states (6.1)], which is expected to overlap rather well with $\mathcal{E}_{NC\pi}$.

Thus, we start with a trial basis including all states $|n_0n_1n_2\rangle$ up to $n=n_0+n_1+n_2=4$, and we enlarge progressively n by one unit and stop this procedure at n_C when a numerical stability is obtained for the 10 lowest states of each parity. This number 10 corresponds to the number of basis states (6.1) up to $n=3$. In practical situations $n_C\simeq 8$.

VII. RESULTS AND DISCUSSION

The different approaches discussed in the previous sections are now compared. For this purpose several observables have been calculated. We paid special attention to the ground state energy, to the energy spectrum, to the quadrupole moments, and to the off-diagonal matrix elements of the quadrupole and octupole operators (2.9) and (2.10). (Strictly speaking, only the squares of these last quantities are observables, but for numerical convenience we shall give their absolute values and consider these as observables.) In order to reduce the presentation of the tremendously large number of results somewhat, we only considered the ten lowest lying states of each parity. Despite this restriction we still have to compare for each set of parameters: 20 energies, 20 quadrupole moments, 90 off-diagonal quadrupole matrix elements, and 100 octupole transitions, leading to 230 data calculated in RPA, TDA, the three versions MPM1, MPM2, and MPM3 of the multiphonon method, and the exact solution.

As mentioned in the previous sections our problem has five parameters: two of them, Ω and $\nu=N/4\Omega-1$, define the system under consideration and the three others are of dynamical nature: e, x, and y.

We first study the 230 observables in the different approaches for a rather typical set of parameters:

$$\Omega=8,\quad \nu=-\tfrac{1}{8},$$
$$e=2.5,\quad x=y=0.25. \quad (7.1)$$

(These values correspond to a natural extension of those used as a standard parameter set in Refs. 5 and 6.) We are immediately faced with the delicate problem of the correspondence of the states obtained in each approach.

TABLE II. Absolute energy of the ground state, relative energies of the lowest positive parity 0^+ excited state, and diagonal matrix elements of the deviation of the particle number (last column). All energies are given in G units. The standard parameter set (7.1) is used.

	TDA	RPA	MPM1	MPM2	MPM3	Exact	$\langle \mid \hat{N}-N \mid \rangle$
$\mid 000\rangle$	−264.1		−264.3	−264.5	−264.5	−272.4	0.003
$\mid 100\rangle$	27.70	27.36		26.98	27.33	27.79	0.257
$\mid 200\rangle$	55.40	54.73		52.38	52.72	53.64	0.497
$\mid 020\rangle$	55.62	55.01	53.21	53.01	53.35	54.33	0.393
$\mid 011\rangle$	59.63	59.32			57.50	58.62	0.443
$\mid 002\rangle$	63.64	63.64			61.66	62.75	0.582
$\mid 300\rangle$	83.11	82.09		76.04	76.40	77.85	0.769
$\mid 120\rangle$	83.22	82.37		77.42	77.74	79.38	0.576
$\mid 111\rangle$	87.33	86.68			81.37	83.42	0.685
$\mid 102\rangle$	91.34	91.00			85.39	87.22	0.809

However, a careful analysis of all energy and transition properties removes practically all ambiguities. Furthermore, it is possible to label the different states with the quantum numbers (n_0,n_1,n_2) which characterize the state in the harmonic approaches TDA or RPA. We note that in the MPM, these labels correspond generally to the largest component in the expansion (6.6). This labeling is somewhat similar to the use of asymptotic quantum numbers used to identify Nilsson states. We remind the reader that, because of the nonorthogonality of the MPM basis (6.1), this main component can be greater than 1, and it is not simply related to the occupation probability of the state.

In Tables II and III we give, respectively, the energies of the 0^+ and 0^- states obtained in the different approaches, TDA, RPA, MPM1, MPM2, and MPM3 compared to the exact values in the seventh column. The first line of Table II refers to the absolute ground state energy while the other values deal with the relative energy spectrum. The BCS or TDA ground state energy reproduces the exact value within 4%. The multiphonon treatment slightly improves the description. One knows from Ref. 6 that the remaining difference can be removed by a better treatment of the nonconservation of the particle number. It is also evident from Tables II and III that the MPM3 approach leads to an overall better agreement than the harmonic approximations. The MPM systematically underestimates the energies but stays within a precision of less than 2.5% for the 20 states considered. The TDA (or RPA) adjusts very nicely the "one phonon" states but all other energies are overestimated: Some "three phonon" states show a deviation of the order of 7%. We also note that for the states where the three MPM can be compared, no large difference concerning the energies are seen.

In Tables IV and V we present the values, in q units, of the quadrupole moments, respectively, for the 0^+ and 0^- states for the MPM's and exact solution. The results are also compared with 3.33, the unique value given by the TDA or RPA [see Eq. (3.6)]. The MPM3 approach reproduces very nicely the variations from state to state of the quadrupole moments, though a slight overestimation can be noticed: The quantitative description is fine for "zero, one, and two phonon" states; for the three phonon states, however, some deviations are observed but they stay within 10%. When it is possible to compare the different MPM calculations, it appears that MPM2 and MPM3 give similar results whereas MPM1 leads to very poor agreement. This is due to the fact that the parameter set (7.1) corresponds to a case where the second 0^- phonon is far away from the two other phonons and consequently has only a small effect on the electromagnetic properties of the states built on 0^+ and 0_1^- phonons.

Among the 90 pieces of information contained in the off-diagonal $E2$ transitions we have selected some noteworthy data. In Table VI we give the ten $E2$ transitions allowed in the harmonic approximations. These are

TABLE III. Relative energies (in G units) of the ten lowest negative parity excited states 0^- and diagonal matrix elements of the deviation of the particle number (last column) for the standard set (7.1).

	TDA	RPA	MPM1	MPM2	MPM3	Exact	$\langle \mid \hat{N}-N \mid \rangle$
$\mid 010\rangle$	27.81	27.50	27.48	27.50	27.49	27.95	0.197
$\mid 001\rangle$	31.82	31.82			31.82	32.36	0.286
$\mid 110\rangle$	55.51	54.87		53.36	53.35	54.34	0.432
$\mid 101\rangle$	59.52	59.18			57.37	58.47	0.534
$\mid 210\rangle$	83.21	82.23		77.75	77.72	79.34	0.577
$\mid 201\rangle$	87.22	86.54			81.06	82.63	0.766
$\mid 030\rangle$	83.43	82.51	77.17	76.85	76.85	78.26	0.627
$\mid 021\rangle$	87.44	86.82			81.44	83.38	0.597
$\mid 012\rangle$	91.45	91.14			85.50	87.34	0.695
$\mid 003\rangle$	95.46	95.45			89.51	91.20	0.896

TABLE IV. Quadrupole moments in q units for the ten lowest positive parity states 0^+, obtained for the standard parameter set (7.1). The given values are also to be compared to 3.33, the result of the harmonic approximations TDA and RPA.

	MPM1	MPM2	MPM3	Exact
$\|000\rangle$	3.33	3.34	3.34	3.34
$\|100\rangle$		3.62	3.62	3.58
$\|200\rangle$		3.84	3.84	3.73
$\|020\rangle$	3.03	3.49	3.49	3.46
$\|011\rangle$			3.31	3.17
$\|002\rangle$			3.15	3.01
$\|300\rangle$		4.33	4.29	4.14
$\|120\rangle$		3.77	3.83	3.74
$\|111\rangle$			3.57	3.45
$\|102\rangle$			3.41	3.12

TABLE VI. Absolute values in q units, of the ten allowed $E2$ matrix elements, between states labeled by quantum numbers n_0 which differ by one unit, for the standard parameter set (7.1).

	TDA	RPA	MPM2	MPM3	Exact
$\langle 000 \| E2 \| 100\rangle$	5.58	6.03	6.04	6.04	6.09
$\langle 100 \| E2 \| 200\rangle$	7.89	8.53	7.57	7.57	7.46
$\langle 010 \| E2 \| 110\rangle$	5.58	6.03	5.80	5.79	5.83
$\langle 001 \| E2 \| 101\rangle$	5.58	6.03		5.85	5.87
$\langle 200 \| E2 \| 300\rangle$	9.66	10.4	9.43	9.43	9.36
$\langle 110 \| E2 \| 210\rangle$	7.89	8.53	6.16	6.18	6.35
$\langle 101 \| E2 \| 201\rangle$	7.89	8.53		7.78	7.04
$\langle 020 \| E2 \| 120\rangle$	5.58	6.03	6.12	6.09	6.20
$\langle 011 \| E2 \| 111\rangle$	5.58	6.03		5.59	5.56
$\langle 002 \| E2 \| 102\rangle$	5.58	6.03		5.65	5.63

the "cascade transitions" which take place between states $|n_0,n_1,n_2\rangle$ and $|n_0\pm1,n_1,n_2\rangle$. In the TDA or RPA the matrix elements of these transitions are proportional to $\sqrt{n_0}$ leading to only three different values in the second and third columns of Table VI. Here, too, the MPM3 demonstrates an overall improvement when compared with the harmonic approximations. The agreement is obtained within less than 2%. We notice again that MPM2 is, for this set of parameters, equivalent to the MPM3 approach.

Apart from these ten allowed transitions one observes four other matrix elements having a value greater than q. These transitions, strictly forbidden in TDA and RPA, are given in Table VII. They are mainly due to the strong mixture of the states labeled by $|020\rangle$ and $|200\rangle$ for the two first ones, $|030\rangle$ and $|210\rangle$ for the third one, and $|201\rangle$ and $|021\rangle$ for the last one.

The values of these four elements are very sensitive to the choice of the parameters of our problem. Their importance is more or less "accidental," but it remains, however, that they are reproduced rather correctly by MPM2 or MPM3.

Another interesting fact, which appears systematically (i.e., whatever the parameters may be) is the existence of ten other $E2$ transitions, an order of magnitude smaller, completely forbidden in harmonic approximation, and which connect states $|n_0n_1n_2\rangle$ with states $|n_0,n_1\pm1,n_2\mp1\rangle$.

These transitions, somewhat induced by an $E2$ transition between the two different octupole phonons, are reported in Table VIII. The MPM3 approach reproduces these data very nicely.

In a similar way we will not give the 100 $E3$ matrix elements. We restrict ourselves to the most noteworthy data. In Table IX (Table X) we give the "allowed" $E3$ cascade transitions, which take place between states labeled by quantum numbers n_1 (n_2) differing by one unit. For reasons similar to that given for the $E2$ case, only three different values appear in the TDA or RPA results. These results confirm the evident superiority of the MPM over the harmonic approximations. This is noticeable particularly for the cases where the state $|020\rangle$ is involved; see Table IX. We also note that the values in Tables X are an order of magnitude smaller than those of Table IX, demonstrating that the 0_2^- phonon is of a less collective nature than the 0_1^- phonon. In Table IX the MPM1 is seen to be insufficient, whereas the MPM2 and MPM3 may be considered again as equally good. The agreement obtained with MPM3 is satisfactory. The MPM3 is very good for the large values of Table IX and a little less good performing for the small values of Table X. In Table XI, we give the values of five strong $E3$ forbidden transitions which were obtained with the use of the standard parameter set (7.1). Conclusions similar to those given for Table VII and $E2$ transitions can be drawn here. Other rather important $E3$ transitions may be observed. They imply states where one of the negative phonon 0_1^- or 0_2^- is transformed in the positive phonon 0^+. As for the equivalent $E2$ transition shown in Table VIII, the MPM gives a satisfactory description[8] of these data.

In order to estimate the effects of the nonconservation of the number of particles induced by the BCS treatment and present in the MPM we have also calculated the matrix elements of the "deviation" operator $\hat{N}-N$. The absolute values of its diagonal matrix elements are given in

TABLE V. Same as in Table IV but for the ten lowest negative parity states 0^+.

	MPM1	MPM2	MPM3	Exact
$\|010\rangle$	3.18	3.36	3.36	3.33
$\|001\rangle$			3.26	3.21
$\|110\rangle$		3.66	3.67	3.62
$\|101\rangle$			3.53	3.41
$\|210\rangle$		3.78	3.83	3.77
$\|201\rangle$			3.82	3.45
$\|030\rangle$	2.87	3.62	3.62	3.53
$\|021\rangle$			3.39	3.26
$\|012\rangle$			3.22	2.96
$\|003\rangle$			2.98	2.72

TABLE VII. Absolute values, in q units, of four strong forbidden $E2$ matrix elements obtained with the standard parameter set (7.1).

	MPM2	MPM3	Exact
$\langle 100 \| E2 \| 020\rangle$	3.69	3.68	4.02
$\langle 020 \| E2 \| 300\rangle$	2.70	2.71	3.01
$\langle 110 \| E2 \| 030\rangle$	5.22	5.17	5.03
$\langle 101 \| E2 \| 021\rangle$		2.40	4.11

TABLE VIII. Absolute values, in q units, of the ten forbidden $E2$ matrix elements occurring between states labeled by $|n_0,n_1,n_2\rangle$ and $|n_0,n_1\mp1,n_2\pm1\rangle$ with the use of the standard parameter set (7.1).

	MPM3	Exact
$\langle 010\|E2\|001\rangle$	0.416	0.437
$\langle 110\|E2\|101\rangle$	0.442	0.521
$\langle 020\|E2\|011\rangle$	0.527	0.542
$\langle 011\|E2\|002\rangle$	0.630	0.682
$\langle 210\|E2\|201\rangle$	0.219	0.199
$\langle 120\|E2\|111\rangle$	0.589	0.680
$\langle 111\|E2\|102\rangle$	0.667	0.818
$\langle 030\|E2\|021\rangle$	0.546	0.360
$\langle 021\|E2\|012\rangle$	0.870	0.807
$\langle 012\|E2\|003\rangle$	0.824	0.928

TABLE X. Absolute values, in r units, of the ten allowed $E3$ matrix elements occurring between states labeled by $|n_0,n_1,n_2\rangle$ and $|n_0,n_1,n_2\pm1\rangle$ obtained with the use of the standard parameter set (7.1).

	TDA	RPA	MPM3	Exact
$\langle 000\|E3\|001\rangle$	0.608	0.607	0.618	0.637
$\langle 100\|E3\|101\rangle$	0.608	0.607	0.747	0.783
$\langle 010\|E3\|011\rangle$	0.608	0.607	0.706	0.681
$\langle 001\|E3\|002\rangle$	0.859	0.858	0.955	1.01
$\langle 200\|E3\|201\rangle$	0.608	0.607	0.802	0.927
$\langle 110\|E3\|111\rangle$	0.608	0.607	1.01	0.784
$\langle 101\|E3\|102\rangle$	0.859	0.858	1.05	1.28
$\langle 020\|E3\|021\rangle$	0.608	0.607	0.878	0.779
$\langle 011\|E3\|012\rangle$	0.859	0.858	1.10	1.06
$\langle 002\|E3\|003\rangle$	1.05	1.05	1.27	1.40

the last column of Tables II and III. For the ground state it is very small; for the other states it remains smaller than $\simeq 0.3n$ where $n=n_0+n_1+n_2$. Since we are dealing with an even number of particles, we see from these values that the contribution from the neighboring nuclei may not be too large, at least for the lower part of the energy spectrum.

It is now natural to ask the question: How is the agreement obtained with the multiphonon method preserved when one changes the different parameters of the problem? We have studied[8] the variation of the evaluated quantities by changing one by one each of the five involved parameters. The effects of the variation of v, Ω, e, and x are the same as those obtained in our previous study[5] where only $K=0^+$ phonons were considered and are not exhibited in the present paper. We again find that the quality of the results of the MPM improves with decreasing quadrupole force, decreasing splitting of the shells, decreasing $|v|$, and higher degeneracy. We just want to report here the effects of the variation of y, the parameter which measures the octupole force. To illustrate this study we restrict ourselves to some selected observables concerning states where

$$0\le n_0+n_1+n_2\le 2\ .$$

Tables XII and XIII, devoted, respectively, to positive and negative parity states, show a mesh. The abscissa of each point of the mesh is given by y. Five values, 0.01, 0.25, 0.5, 0.75, and 1 are retained for this parameter. The ordinate of each point of the mesh is given by the state. At each point we give the values obtained in MPM3 (upper line) and in the exact solution (lower line) for at most four observables. In the lower left-hand corner we give the relative energy for the state in G units, and in the lower right-hand corner the quadrupole moment in q units. In the upper left-hand corner we report the allowed $E2$ matrix element (in q units) decreasing from the state, and in the upper right-hand corner one finds the allowed $E3$ matrix element (in r units) coming from this state. There is one exception for this presentation for the two phonon state $|011\rangle$ where the two upper divisions are concerned with allowed $E3$ transitions: the left-hand one deals with that due to the change of n_1, and the right-hand with that involving a variation of n_2. Several remarks can be made: First, as could be expected, the variation of the strength of the octupole force does not practically affect the spectroscopic properties of the state $|100\rangle$. Second, with increasing y, the energy of the lowest octupole state $|010\rangle$ decreases while that of the second negative state $|001\rangle$ remains rather stable. The collectivity of this last state is large for small values of y (i.e., one observes a large $E3$ matrix element) and decreases with y, for the benefit of the lowest octupole state. Third, we note that the labeling of some states (mainly of positive parity) is, in some cases, rather difficult. When some ambiguity remains we use an asterisk. This in particular is the case for $y=0.50$ where the states $|011\rangle$ and $|200\rangle$ are practically degenerated. Apart from this, we see that the MPM3 gives an overall

TABLE IX. Absolute values, in r units, of the ten allowed $E3$ matrix elements which take place between states labeled by $|n_0,n_1,n_2\rangle$ and $|n_0,n_1,\pm1,n_2\rangle$ in the case of the standard parameter set (7.1).

	TDA	RPA	MPM1	MPM2	MPM3	Exact
$\langle 000\|E3\|010\rangle$	5.55	5.98	5.99	5.98	5.99	6.04
$\langle 100\|E3\|110\rangle$	5.55	5.98		5.74	5.72	5.75
$\langle 010\|E3\|020\rangle$	7.85	8.46	8.36	7.35	7.36	7.23
$\langle 001\|E3\|011\rangle$	5.55	5.98			5.79	5.80
$\langle 200\|E3\|210\rangle$	5.55	5.98		1.08	1.10	0.954
$\langle 110\|E3\|120\rangle$	7.85	8.46		7.40	7.36	7.35
$\langle 101\|E3\|111\rangle$	5.55	5.98			5.53	5.44
$\langle 020\|E3\|030\rangle$	9.61	10.4	10.1	5.62	5.68	5.64
$\langle 011\|E3\|021\rangle$	7.85	8.46			7.59	6.74
$\langle 002\|E3\|012\rangle$	5.55	5.98			5.57	5.53

TABLE XI. Absolute values, in r units, of five strong forbidden $E3$ matrix elements obtained with the use of the standard parameter set (7.1).

	MPM2	MPM3	Exact
$\langle 200 \mid E3 \mid 010 \rangle$	3.95	3.94	4.28
$\langle 200 \mid E3 \mid 030 \rangle$	6.69	6.65	6.88
$\langle 020 \mid E3 \mid 210 \rangle$	7.34	7.27	7.20
$\langle 011 \mid E3 \mid 201 \rangle$		2.66	4.41
$\langle 300 \mid E3 \mid 110 \rangle$	3.01	2.96	3.12

good agreement for values of y up to 0.75. The quadrupole moments are overestimated, and all other quoted observables slightly underestimated. Finally, in conclusion we may add that the quality of the MPM increases with decreasing octupole force, i.e., with an increasing pairing.

Another interesting point to be discussed is the possibility that MPM1 or MPM2 is sufficient to obtain a good description of typical states for a given situation. Intuitively we felt that a case where the 0_1^- state is much lower than the 0^+ and 0_2^- states would be correctly described by MPM1, while MPM2 would be more suited for the case where the 0_1^- and 0^+ states are much lower than the 0_2^- state. To distinguish between the various situations we introduce the quantity

$$d=\frac{E(0_2^-)-E(0^+)}{E(0^+)-E(0_1^-)}\ ,$$

where the energies have been calculated in the exact case; for the major part of the parameters used in our numerous analyses, the 0^+ state lies in between the 0_1^- and 0_2^- states and d is a positive number. The case for which $d \ll 1$ is called type I, and it corresponds to a 0_1^- state well separated from a "doublet" of 0^+ and 0_2^- states. The other extreme case $d \gg 1$ corresponds to a "doublet" of 0_1^- and 0^+ states much lower than the remaining 0_2^- state, and is called type II. Finally, the case $d \simeq 1$ corresponds to a 0^+ state equidistant from the 0_1^- and 0_2^- states and is referred to as type III. If we restrict ourselves to states with $n_0+n_1+n_2 \leq 3$ the number of levels which can be described in MPM1, MPM2, and MPM3 is, respectively, 2, 6, and 10 for positive parity and 2, 4, and 10 for negative parity. Thus, when we compare the various approximations MPMi, we mean that we compare the results for the states which can be described in these given approximations.

The standard case extensively studied in this paper is of type II ($d=-28$). The MPM1 version is good for the energies but does not reproduce the transition probabilities very well. On the contrary, the MPM2 version gives results which are practically as good as MPM3 for both energies as well as transition probabilities. This is the con-

TABLE XII. Values of the energy (in G units) in the lower left-hand corner, of the quadrupole moment (in q units) in the lower right-hand corner, and of the allowed $E2$, upper left-hand corner, and $E3$, upper right-hand corner transitions or upper part for state $\mid 011 \rangle$ decreasing from the one and two phonon positive parity states. The lower line corresponds to the exact solution, the upper to the MPM3. Variation with y is shown, the other parameters being fixed to $\nu=-\frac{1}{8}$, $\Omega=8$, $e=2.5$, $x=0.25$. Asterisk denotes ambiguity. (See text.)

	$y=0.01$		$y=0.25$		$y=0.50$		$y=0.75$		$y=1$	
$\mid 002 \rangle$		4.86		0.955		0.480		0.305		0.020
		4.95		1.01		0.501		0.327*		0.019*
	62.1	2.64	61.7	3.15	61.9	3.20	62.1	3.16	62.3	3.01
	63.2	2.47	62.7	3.01	63.0	3.05	63.4*	2.87*	65.9*	3.58*
$\mid 011 \rangle$	4.32	3.69	5.79	0.706	6.43	0.330	7.55	0.207	8.71	0.247
	4.26	3.61	5.80	0.681	4.03*	0.436*	7.54	0.233	12.1*	0.267*
	61.2	3.40	57.5	3.31	53.1	3.13	47.5	2.88	40.8	2.55
	62.1	3.37	58.6	3.17	54.2*	3.50*	48.5	2.60	38.2*	1.41*
$\mid 020 \rangle$		5.89		7.36		9.27		10.7		12.7
		6.01		7.23		9.32		10.7		14.3
	60.1	4.18	53.4	3.49	43.8	2.95	32.3	2.42	19.6	2.45
	61.2	4.08	54.3	3.46	44.6	2.90	32.5	2.29	15.1	2.00
$\mid 200 \rangle$	8.45		7.57		8.35		8.20		7.69	
	8.51		7.46		5.27*		8.22		7.16*	
	52.5	4.01	52.7	3.84	53.3	3.91	53.6	4.01	55.3	3.03
	53.5	3.93	53.6	3.73	54.2*	3.27*	54.6	3.94	56.3*	2.39*
$\mid 100 \rangle$	6.06		6.03		6.01		5.90		5.64	
	6.11		6.09		6.05		5.93		5.58	
	27.2	3.66	27.3	3.62	27.5	3.56	27.7	3.56	28.4	2.91
	27.6	3.63	27.8	3.58	28.0	3.53	28.2	3.55	28.9	2.79

TABLE XIII. Same as in Table XII, but for one and two phonon negative parity states.

	$y=0.01$		$y=0.25$		$y=0.50$		$y=0.75$		$y=1$	
$\|101\rangle$	5.87	3.48	5.85	0.747	5.83	0.364	5.75	0.214	4.68	0.127
	5.88	3.51	5.87	0.783	5.85	0.380	5.71	0.186	5.46*	0.016*
	57.5	3.29	57.4	3.53	57.6	3.53	57.8	3.56	58.3	2.98
	58.6	3.14	58.5	3.41	38.8	3.43	59.0	3.54	58.8*	1.90*
$\|110\rangle$	5.85	4.12	5.79	5.72	5.69	6.34	5.27	7.47	4.57	9.25
	5.91	4.14	5.83	5.75	5.72	6.36	5.14	7.52	3.72	8.60
	56.4	4.08	53.4	3.67	49.1	3.45	43.6	3.58	38.3	1.89
	57.4	4.08	54.3	3.62	50.1	3.40	43.3	3.50	37.5	2.09
$\|001\rangle$		3.56		0.618		0.304		0.186		0.340
		3.59		0.637		0.321		0.206		0.100
	32.0	3.03	31.8	3.26	31.9	3.25	32.0	3.17	32.3	2.47
	32.6	2.98	32.4	3.21	32.5	3.18	32.7	3.08	32.0	2.27
$\|010\rangle$		4.35		5.99		6.64		7.78		10.8
		4.39		6.04		6.68		7.83		11.8
	31.1	3.75	27.5	3.36	22.7	3.14	16.4	2.83	7.33	2.30
	31.6	3.73	28.0	3.33	23.1	3.11	16.7	2.70	6.93	2.08

clusion we expect for a type II case.

We have also studied a number of other situations, but to avoid the presentation of too many results, we report here only the qualitative features which emerge from this analysis. Concerning a type I case (a typical example is obtained for $\Omega=8$, $\nu=-\frac{1}{8}$, $e=2.5$, $x=0.25$, $y=0.75$, for which $d=0.39$), the MPM1 describes correctly the energies and the allowed transitions, but describes poorly the forbidden transitions. The MPM2 approximation is generally as good as MPM3, and often even better for all types of observables. This conclusion is rather consistent with our first idea, but the improvement by passing from MPM1 to MPM2 is even more impressive than expected. Finally, a typical type III case ($\Omega=8$, $\nu=-\frac{1}{8}$, $e=2.5$, $x=0.12$, $y=0.25$, for which $d=1$) was examined. The conclusion is quite similar to the type II case. Here, too, we find that MPM2 gives results completely equivalent to MPM3 (and often better); this was not expected *a priori*.

VIII. CONCLUSIONS

In the frame of our simple exactly solvable model, we have been able to show that the multiphonon method provides a fairly good approach for the spectroscopic properties of the coupled vibrational low lying levels of the system having definite physical situations.

The anharmonicities in the energy spectra are well reproduced even with the simplest versions of the multiphonon method. On the other hand, the allowed quantities (quadrupole moments or cascade $E2$ and $E3$ transitions) are systematically improved by making a proper MPM treatment in comparison with harmonic TDA or RPA results. But one of the most important points of our study is that the MPM is able to explain quite nicely the forbidden transitions (in particular, transitions between one phonon states), some of them being of the same order of magnitude as the allowed ones. In that case, the coupling of different types of phonon is of crucial importance. Finally, it was shown that in practically all situations, MPM2, that is, the coupling of only the most collective phonon of each parity, is sufficient tc obtain a very good overall agreement with the exact solution. This conclusion is a very encouraging and exciting result prompting the attempt to apply the MPM2 method in a realistic case—for example, in the actinide region (see Ref. 4).

We are very much indebted to Dr. Z. Szymanski for fruitful discussions and Dr. A. Jain for a careful reading of the manuscript.

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