PHYSICAL REVIEW C VOLUME 28, NUMBER 3 SEPTEMBER 1983

Quantum tunneling in multidimensional systems

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(Received 16 May 1983)

The effects of coupling to a harmonic oscillator on the quantum tunneling of a macroscopic motion are studied through the influence functional formalism of Feynman's path integral method for the general coupling form factor. As an example, we consider the model in which the potential barrier is parabolic and the coupling Hamiltonian is linear in both coordinates of the macroscopic motion and of the intrinsic harmonic oscillator. The results are then compared with the exact solution obtained through the canonical transformation into normal coordinates in the limiting cases when the normal coordinates reduce to the original coordinates. We found that: (1) In the adiabatic case, i.e., when the recurrence time π/ω of the oscillator is much shorter than the transmission time through the macroscopic potential barrier, the effect of oscillator coupling can be well represented by an effective potential. The coupling enhances the tunneling probability on the whole. (2) There exists a critical energy, above which the tunneling probability is reduced because of the linear oscillator coupling. In the weak coupling limit and when $\omega \rightarrow 0$, the critical energy becomes $-\infty$, so that the coupling to the oscillator always reduces the tunneling probability.

[NUCLEAR REACTIONS Quantum tunneling, coupling of macroscopic motion to intrinsic oscillators, semiclassical method, heavy ion fusion.]

I. INTRODUCTION

The quantum tunneling process in multidimensional systems has become an important and fundamental problem in many areas of physics. The tunneling of the vacuum state in field theories[1] and tunneling of the trapped magnetic flux[2,3] are a few examples. In nuclear physics, it has been discussed in connection with the problem of spontaneous fission.[4] The sub-barrier fusion cross section enhancement discovered recently[5] in heavy ion collisions also draws much attention of nuclear physicists in their attempt to understand the quantum tunneling problem better. The problem is that the observed sub-barrier fusion cross section is greatly enhanced in comparison with the theoretical predictions based on the one dimensional potential model.[6] Esbensen[7] has tried to attribute the enhancement to the zero point motion of the nuclear surface. The success of this idea has, however, been questioned by Landowne and Nix in their dynamical calculation.[8]

Quantum tunneling processes can be tackled by the time dependent Feynman path integral method[9] along the imaginary time axis.[1,10] This method has been applied by Caldeira and Leggett[11] to studying the effect of coupling to harmonic oscillators on the quantum tunneling probability through a potential barrier of a macroscopic motion. They have, however, considered only the dissipation effect on the macroscopic motion due to the coupled oscillators, and left out other important effects such as the potential barrier renormalization and the mass renormalization.

Alternately, Brink and co-workers[12] have used a Wentzel-Kramers-Brillouin (WKB) type of approximation, and have shown that the change of the potential barrier due to the coupling to harmonic oscillators indeed plays a decisive role in determining the tunneling probability. Their prescription to decouple the equation for the vibrators from that for the tunneling degree of freedom is, however, nontrivial. Similarly, Widom and Clark[13] have insisted on the importance of the renormalization of the potential barrier by discussing the effective frequency for the parabolic potential barrier. They have thus reached a different conclusion from that of Ref. 11 concerning the effect of the coupling to harmonic oscillators on the quantum tunneling probability.

The aim of the present paper is to study the multidimensional quantum tunneling process, i.e., the effects of the coupling to a harmonic oscillator on the quantum tunneling process of the macroscopic motion. In Sec. II, Feynman's path integral method with imaginary time shall be reviewed and developed to deal with multidimensional tunneling problems. The influence functional method of Sec. II is then used in Sec. III to discuss the problem that the harmonic oscillator linearly couples to the macroscopic degree of freedom. In Sec. IV, we examine the physics of quantum tunneling in the model that the coupling potential is a linear function of both the intrinsic coordinate and the macroscopic coordinate. Section V discusses the result of the coordinate transformation into the normal modes. The conclusion of the present study and the implication concerning the heavy ion fusion reaction are given in Sec. VI. Appendix A is concerned with the quantum tunneling in the case when the macroscopic motion couples to a harmonic oscillator through a quadratic function of the coordinate of the harmonic oscillator.[4] In Appendix B we discuss the conditions necessary to justify the prescription of representing the effects of coupling in terms of the renormalization of the effective potential in the adiabatic case.

II. PATH INTEGRAL METHOD FOR THE CLASSICALLY FORBIDDEN PROCESS

A. One dimensional problem

Let us consider a particle of mass M moving in a potential $\hat{U}(q)$. The propagator K is then given by

$$K(q_1t_1;q_0t_0)=\langle q_1\,|\,e^{-i\hat{H}(t_1-t_0)/\hbar}\,|\,q_0\rangle\ , \tag{2.1}$$

where

$$\hat{H}=\frac{\hat{p}^2}{2M}+\hat{U}(\hat{q})\ . \tag{2.2}$$

In Eqs. (2.1) and (2.2), and in what follows, quantum operators are marked with a caret. The propagator K gives the transition amplitude needed to find the particle at position q_1 at a later time t_1 when it is at q_0 at the initial time t_0.

We are now interested in a classically forbidden process, i.e., quantum tunneling through a potential barrier. Therefore, we consider the propagation along the imaginary time axis[1,10,11], and set

$$t_1=-i\tau_1,\quad t_0=-i\tau_0\ , \tag{2.3}$$

where both τ and τ_0 are real. τ_0 is the time when the tunneling process is initiated, so that we can choose $\tau_0=0$. The transition amplitude for a classically forbidden process is then given by

$$K(q_1\tau_1;q_0\tau_0)=\langle q_1\,|\,e^{-\hat{H}(\tau_1-\tau_0)/\hbar}\,|\,q_0\rangle\ . \tag{2.4}$$

The path integral representation of the transition amplitude then reads

$$K(q_1\tau_1;q_0\tau_0)=\int\mathscr{D}q(\tau)e^{-\mathscr{S}_0[q(\tau)]/\hbar}\ , \tag{2.5}$$

where the action integral $\mathscr{S}_0$ is defined by

$$\mathscr{S}_0[q(\tau)]=\int_{\tau_0}^{\tau_1}\left[\frac{M}{2}\dot{q}^2+U(q)\right]d\tau\ . \tag{2.6}$$

Notice that the action $\mathscr{S}_0$ for the classically forbidden tunneling process through a potential barrier corresponds to the action for the classically allowed process in the inverted potential well.[1]

In Eq. (2.5) the integral should, in principle, be carried over all paths which connect $q(\tau_1)=q_1$ and $q(\tau_0)=q_0$. The integral would, however, be dominated by the saddle point path $\bar{q}(\tau)$, which satisfies the equation of motion

$$M\frac{d^2\bar{q}}{d\tau^2}=\frac{dU(\bar{q})}{d\bar{q}} \tag{2.7}$$

and the boundary conditions

$$\bar{q}(\tau_0)=q_0 \text{ and } \bar{q}(\tau_1)=q_1\ . \tag{2.8}$$

Equations (2.7) and (2.8) give the classical path in the inverted potential well.

Corresponding to Eq. (2.5), the transition probability is given in terms of double path integrals

$$J_0^{(0)}(q_1\tau_1;q_0\tau_0)=\int\mathscr{D}q(\tau)\mathscr{D}\tilde{q}(\tau)\times\exp(-\{\mathscr{S}_0[q(\tau)]+\mathscr{S}_0[\tilde{q}(\tau)]\}/\hbar)\ . \tag{2.9}$$

Similarly to the classically allowed processes,[9,14] let us change the integration paths into the average and the difference of two paths by introducing $Q(\tau)$ and $\eta(\tau)$ such that

$$Q(\tau)=\tfrac{1}{2}[q(\tau)+\tilde{q}(\tau)] \tag{2.10}$$

and

$$\eta(\tau)=q(\tau)-\tilde{q}(\tau)\ . \tag{2.11}$$

Equation (2.9) can then be rewritten as

$$J_0^{(0)}(q_1\tau_1;q_0\tau_0)=\int\mathscr{D}Q(\tau)\mathscr{D}\eta(\tau)\exp\{-2\mathscr{S}_0[Q(\tau)]/\hbar\}\times\exp\{-\tfrac{1}{4}\eta(\tau)^2\mathscr{S}_0''[Q(\tau)]+\cdots\}\ . \tag{2.12}$$

Here the leading term in $J_0^{(0)}$ is zeroth order with respect to $\eta(\tau)$. This contrasts with the case for classically allowed processes, where the leading term in the argument of the exponent is a linear function of η.[14]

The dominant contribution to the transition probability will be given by the classical path determined by

$$\left.\frac{\delta\mathscr{S}_0}{\delta Q}\right|_{\bar{Q}}=0\ . \tag{2.13}$$

Clearly, $\bar{Q}$ obeys the same equation of motion and the boundary conditions as Eqs. (2.7) and (2.8). The tunneling probability is then given by

$$P_0^{(0)}\sim e^{-2\mathscr{S}_0[\bar{Q}(T_0)]/\hbar}\ , \tag{2.14}$$

where T_0 is the transmission time of the tunneling process.

As a simple example, if the potential barrier is parabolic and expressed as

$$\hat{U}(\hat{q})=V_0-\tfrac{1}{2}M\Omega_0^2\hat{q}^2\ , \tag{2.15}$$

one easily gets

$$\mathscr{S}_0=(V_0-E)\frac{\pi}{\Omega_0}=(V_0-E)T_0\ , \tag{2.16}$$

which is the well-known result of the WKB approximation.

B. Open system problem

We now consider a system whose macroscopic coordinates are coupled to the microscopic degrees of freedom. The Hilbert space is the product of the space spanned by the macroscopic degrees of freedom and the space spanned by the microscopic degrees of freedom. We call these space A and space B, respectively. For example, in heavy ion collisions the A space corresponds to the relative motion, and the B space to intrinsic excitations of the colliding nuclei, respectively. Our aim is to derive an equation of motion for the macroscopic degrees of freedom alone which contains the influence of the microscopic motions. In Feynman's path integral method, this goal can be accomplished by studying the influence functional.[14]

1. Semiclassical approximation and the influence function

We consider only one degree of freedom for the macroscopic motion, and denote the coordinate, the conjugate momentum, and the corresponding mass by $\hat{q}$, $\hat{p}$, and M, respectively. The coordinates of the space B are denoted by $\hat{\xi}$. We then assume the following Hamiltonian for the total system:

$$\hat{H}=\frac{\hat{p}^2}{2M}+\hat{U}(\hat{q})+\hat{H}_B+\hat{V}_c(\hat{q},\hat{\xi}) , \tag{2.17}$$

where $\hat{H}_B$ is the unperturbed Hamiltonian of the space B, and $\hat{V}_c$ is the coupling Hamiltonian.

Let us introduce a propagator by

$$K_{\beta\alpha}(q_1\tau_1;q_0\tau_0)=\langle\beta q_1\,|\,e^{-\hat{H}(\tau_1-\tau_0)/\hbar}\,|\,\alpha q_0\rangle . \tag{2.18}$$

It represents the amplitude for the system to make a transition from a position q_0 to q_1 concerning the macroscopic motion, and from a quantum state α to β concerning the microscopic degrees of freedom as time elapses from $-i\tau_0$ to $-i\tau_1$. A simple extension of the semiclassical theory of Pechukas[15] to the case of time evolution along the imaginary time axis leads to

$$K_{\beta\alpha}(q_1\tau_1;q_0\tau_0)=\int\mathscr{D}q(\tau)e^{-\mathscr{S}_0[q(\tau)]/\hbar}\times\langle\beta\,|\,\hat{u}[q(\tau),\tau_1,\tau_0]\,|\,\alpha\rangle , \tag{2.19}$$

where the action is given by Eq. (2.6) and the Green's function $\hat{u}$ is defined by

$$\hbar\frac{\partial}{\partial\tau}\hat{u}[q(\tau),\tau,\tau_0]=-\hat{h}_B(q(\tau))\hat{u}[q(\tau),\tau,\tau_0] \tag{2.20}$$

with the boundary condition

$$\hat{u}[q(\tau),\tau_0,\tau_0]=\hat{1} . \tag{2.21}$$

The operator $\hat{h}_B$ in Eq. (2.20) is the effective Hamiltonian for the space B, and is given by

$$\hat{h}_B(q(\tau))=\hat{H}_B+\hat{V}_c(q(\tau),\hat{\xi}) . \tag{2.22}$$

Note that the macroscopic coordinate appears in $\hat{h}_B$ as a time-dependent c number.

In the open system problem, we are not interested in the details of the transition concerning the microscopic degrees of freedom. Accordingly, we define the transition probability concerning the macroscopic motion alone by

$$J(q_1\tau_1;q_0\tau_0)=\sum_\beta|K_{\beta\alpha}(q_1\tau_1;q_0\tau_0)|^2 . \tag{2.23}$$

Equation (2.19) then leads to

$$J(q_1\tau_1;q_0\tau_0)=\int\int\mathscr{D}q(\tau)\mathscr{D}\tilde{q}(\tau)\rho(\tau)\times\exp(-\{\mathscr{S}_0[q(\tau)]+\mathscr{S}_0[\tilde{q}(\tau)]\}/\hbar) , \tag{2.24}$$

where the *influence functional* ρ is given by

$$\rho(\tau)\equiv\rho[q(\tau),\tilde{q}(\tau);\tau_1]=\langle\alpha\,|\,\hat{u}^+[\tilde{q}(\tau),\tau_1,\tau_0]\hat{u}[q(\tau),\tau_1,\tau_0]\,|\,\alpha\rangle . \tag{2.25}$$

Equation (2.24) is the generalization of Eq. (2.9) to the open system problem. The tunneling probability can be estimated by studying J at $\tau_1=T$, where T is the transmission time. Note that T becomes in general different from the transmission time T_0 of a one dimensional problem.

2. Standard form of the influence functional and physical implications

Let us express the influence functional ρ as

$$\rho(\tau_1)=\exp[\phi(\tau_1)/\hbar] . \tag{2.26}$$

The argument ϕ can be expressed in terms of $Q(\tau)$ and $\eta(\tau)$ introduced by Eqs. (2.10) and (2.11). We are interested in the part of ϕ which is independent of η. Denoting this part by ϕ_0, Eqs. (2.24), (2.12), and (2.6) suggest that ϕ_0 has the following integral form:

$$\phi_0(\tau_1)=-2\int_{\tau_0}^{\tau_1}W(\tau)d\tau . \tag{2.27}$$

We call $W(\tau)$ the *influence potential.* As we will see in the examples in Sec. III and in Appendix A, $W(\tau)$ depends on τ either explicitly or through the time dependence of the macroscopic coordinate Q. An important aspect is the memory effect involved in ϕ_0 as a general property of the open system problem. Namely, $W(\tau)$ depends not only on $Q(\tau)$, but also on the values of Q at previous times. Only under certain circumstances can the memory effect be well approximated by introducing velocity dependence and/or an effective mass in $W(\tau)$. The tunneling process then becomes a Markov process. We learn in Sec. IV that the condition to justify the Markov approximation is equivalent to the adiabaticity condition. In this sense, the classically forbidden process differs from the classically allowed process, for which the validity of the Markov approximation is related to the lifetime of the oscillator.

Let us now assume that the influence potential $W(\tau)$ is given by

$$W(\tau)=B(\tau,Q(\tau))+C(\tau,Q(\tau))\dot{Q}(\tau) . \tag{2.28}$$

The physical meaning of the first term is clear, i.e., the potential barrier is renormalized by the amount of $B(\tau,Q(\tau))$ due to the coupling of the macroscopic motion to microscopic degrees of freedom. In order to understand the physical meaning of the second term, we derive the equation of motion to determine the classical path, which dominates the tunneling probability. It reads

$$M\ddot{\bar{Q}}=\frac{\partial}{\partial\bar{Q}}[U(\bar{Q})+B(\tau,\bar{Q}(\tau))]-\frac{\partial C}{\partial\tau}\ . \tag{2.29}$$

This means that the second term in Eq. (2.28) does not change the equation of motion to determine the tunneling path, but only affects the tunneling probability if C does not explicity depend on τ.

III. LINEAR COUPLING TO A HARMONIC OSCILLATOR

A. The influence potential

In the present section, we consider the case when the tunneling degree of freedom couples to a harmonic oscillator. The unperturbed Hamiltonian of the space B is then agiven by

$$\hat{H}_B=\hbar\omega(a^\dagger a+\tfrac{1}{2})\ . \tag{3.1}$$

We further assume a coupling Hamiltonian which linearly depends on the coordinate of the oscillator, i.e.,

$$\begin{aligned}\hat{V}_c&=Cf_0(\hat{q})\hat{\xi}\\&=gf_0(\hat{q})(a^\dagger+a)\ ,\end{aligned} \tag{3.2}$$

where

$$g=C\sqrt{\hbar/2m\omega}\ . \tag{3.3}$$

In Eqs. (3.2) and (3.3), C, $f_0(q)$, and m are the coupling strength, the coupling form factor, and the mass parameter of the harmonic oscillator, respectively.

The corresponding Green's function $\hat{u}$ for the oscillator can be easily obtained by making use of Glauber's coherent state representation for quantum operators.[16] The result reads

$$\hat{u}[q(\tau),\tau,\tau_0]=\int\frac{d^2z}{\pi}e^{A+Bz+Cz^*+Dz^*z}\,|z\rangle\langle z|\ , \tag{3.4}$$

where $|z\rangle$ is the coherent state and the coefficients A through D are given by

$$\begin{aligned}A=\tfrac{1}{2}\omega(\tau-\tau_0)-\int_{\tau_0}^{\tau}d\tau_1 f(q(\tau_1))e^{\omega\tau_1}&\\\times\int_{\tau_0}^{\tau_1}d\tau_2 f(q(\tau_2))e^{-\omega\tau_2}&\ ,\end{aligned} \tag{3.5a}$$

$$B=-e^{\omega\tau}\int_{\tau_0}^{\tau}d\tau_1 f(q(\tau_1))e^{-\omega\tau_1}\ , \tag{3.5b}$$

$$C=-e^{-\omega\tau_0}\int_{\tau_0}^{\tau}d\tau_1 f(q(\tau_1))e^{\omega\tau_1}\ , \tag{3.5c}$$

and

$$D=1-e^{\omega(\tau-\tau_0)}\ . \tag{3.5d}$$

The form factor f in these equations is related to the form factor $f_0(q)$ in Eq. (3.2) as

$$f(q)=gf_0(q)/\hbar\ . \tag{3.6}$$

We now assume that the harmonic oscillator is in the ground state when the tunneling process is initiated at one end of the potential barrier. Equations (3.4) and (3.5) then lead to the following expression for the influence potential $W(\tau)$:

$$W(\tau)=\tfrac{1}{2}\hbar\omega+W_0(\tau)\ , \tag{3.7}$$

with

$$\begin{aligned}W_0(\tau)=-\hbar[1+e^{-2\omega(T-\tau)}]f(Q(\tau))e^{-\omega\tau}&\\\times\int_{\tau_0}^{\tau}d\tau_1 f(Q(\tau_1))e^{\omega\tau_1}&\ ,\end{aligned} \tag{3.8}$$

where T is the transmission time of the tunneling process. The first term of Eq. (3.7) is nothing but the zero point energy of the harmonic oscillator. The second term $W_0(\tau)$ represents the physical effect of the coupling to a harmonic oscillator on the quantum tunneling process. It vanishes as the frequency of the harmonic oscillator approaches infinity. This is physically sensible, because the harmonic oscillator should not be excited at all, and hence should cause no effect on the tunneling process if the energy quanta of the oscillator becomes too large.

Equations (2.24) and (3.8) now lead to the following equation of motion for the dominant tunneling path:

$$M\ddot{\bar{Q}}(\tau)=\frac{dU}{d\bar{Q}}+F(\tau)\ , \tag{3.9a}$$

where

$$\begin{aligned}F(\tau)=-\hbar\frac{df}{d\bar{Q}}\Bigg\{\Bigg[&e^{-\omega\tau}\int_{\tau_0}^{\tau}d\tau_1 f(\bar{Q}(\tau_1))e^{\omega\tau_1}\\&+e^{\omega\tau}\int_{\tau}^{T}d\tau_1 f(\bar{Q}(\tau_1))e^{-\omega\tau_1}\Bigg]\\&+e^{-2\omega T+\omega\tau}\int_{\tau_0}^{T}d\tau_1 f(\bar{Q}(\tau_1))e^{\omega\tau_1}\Bigg\}\ .\end{aligned} \tag{3.9b}$$

Equation (3.9) has been derived by assuming that the transmission time and the terminal positions of the tunneling process are given. These quantities should, however, be determined self-consistently by solving Eq. (3.9).

We remark that Eqs. (3.8) and (3.9) have been obtained without making any assumption concerning the strength of the coupling Hamiltonian. They are exact in the sense of perturbation expansion with respect to the powers of f. This is a special property of the linear coupling. In fact, if the coupling Hamiltonian is a quadratic function of $\hat{\xi}$, then the third or higher order powers of f appear in W_0, as well as in the equation of motion to determine the dominant tunneling path (see Appendix A).

B. Effective potential and the effective mass in the adiabatic case

Equation (3.9) is in general a non-Markovian equation of motion. In the adiabatic case, however, it can be well approximated by a Markovian equation of motion, in which the main effect to the macroscopic equation of motion is the renormalization of the potential and of the mass. This is because the response time of the oscillator

π/ω is much shorter than the transmission time in the adiabatic case.

In this case, the third term on the rhs of Eq. (3.9b) can be ignored except at times $\tau=0$ and $\tau=T$. On the other hand, the dominant contribution to the integral in the first and second terms comes from the vicinity of $\tau_1=\tau$. We thus expand $f(\bar{Q}(\tau_1))$ in these terms around $\bar{Q}(\tau)$. This prescription leads, for example for the first term, to

$$
\begin{aligned}
e^{-\omega\tau}\int_{\tau_0}^{\tau} d\tau_1 f(\bar{Q}(\tau_1))e^{\omega\tau_1}
&= f(\bar{Q}(\tau))\frac{1}{\omega}(1-e^{-\omega(\tau-\tau_0)})-\dot{\bar{Q}}(\tau)\frac{df}{d\bar{Q}}\left[-\frac{(\tau-\tau_0)}{\omega}e^{-\omega(\tau-\tau_0)}+\frac{1}{\omega^2}(1-e^{-\omega(\tau-\tau_0)})\right] \\
&\quad +\tfrac{1}{2}\left[\ddot{\bar{Q}}\frac{df}{d\bar{Q}}+\dot{\bar{Q}}^2\frac{d^2f}{d\bar{Q}^2}\right]\left[-\frac{(\tau-\tau_0)^2}{\omega}e^{-\omega(\tau-\tau_0)}-\frac{2(\tau-\tau_0)}{\omega^2}e^{-\omega(\tau-\tau_0)}+\frac{2}{\omega^3}(1-e^{-\omega(\tau-\tau_0)})\right]+\cdots .
\end{aligned}
\tag{3.10}
$$

This is the moment expansion in terms of the moments defined by

$$
\chi_n=\int_0^{\tau-\tau_0}\tau_1^n e^{-\omega\tau_1}d\tau_1 . \tag{3.11}
$$

When $\omega(\tau-\tau_0)\gg 0$, Eq. (3.10) becomes

$$
e^{-\omega\tau}\int_{\tau_0}^{\tau} d\tau_1 f(\bar{Q}(\tau_1))e^{\omega\tau_1}\simeq\frac{1}{\omega}f(\bar{Q}(\tau))-\frac{1}{\omega^2}\frac{df}{d\bar{Q}}\dot{\bar{Q}}(\tau)+\frac{1}{\omega^3}\left[\ddot{\bar{Q}}(\tau)\frac{df}{d\bar{Q}}+\dot{\bar{Q}}^2\frac{d^2f}{d\bar{Q}^2}\right]+\cdots . \tag{3.12}
$$

Similarly, if $\omega(T-\tau)\gg 0$, then the second term on the rhs of Eq. (3.9b) can be approximated as

$$
e^{\omega\tau}\int_{\tau}^{T} d\tau_1 f(\bar{Q}(\tau_1))e^{-\omega\tau_1}\simeq\frac{1}{\omega}f(\bar{Q}(\tau))+\frac{1}{\omega^2}\dot{\bar{Q}}(\tau)\frac{df}{d\bar{Q}}+\frac{1}{\omega^3}\left[\ddot{\bar{Q}}\frac{df}{d\bar{Q}}+\dot{\bar{Q}}^2\frac{d^2f}{d\bar{Q}^2}\right]+\cdots . \tag{3.13}
$$

Note that Eqs. (3.12) and (3.13) are the expansions in powers of Ω_0/ω (see Appendix B for more details). We thus obtain the following equation of motion for the dominant tunneling path if Ω_0/ω is sufficiently small and if $\omega^2(T-\tau)(\tau-\tau_0)\gg 0$:

$$
M\ddot{\bar{Q}}(\tau)=\frac{dU}{d\bar{Q}}-2\hbar\frac{df}{d\bar{Q}}\left\{\frac{1}{\omega}f(\bar{Q}(\tau))+\frac{1}{\omega^3}\left[\ddot{\bar{Q}}(\tau)\frac{df}{d\bar{Q}}+\dot{\bar{Q}}^2\frac{d^2f}{d\bar{Q}^2}\right]\right\}. \tag{3.14}
$$

Equation (3.14) leads to the following expression for the additional effective potential $\delta U(Q)$ and the additional effective mass δM due to the linear coupling to a harmonic oscillator:

$$
\delta U(Q)=-\frac{\hbar}{\omega}[f(Q)]^2 \tag{3.15a}
$$

$$
=-\frac{C^2}{2m\omega^2}[f_0(Q)]^2 \tag{3.15b}
$$

and

$$
\delta M=\frac{2\hbar}{\omega^3}\left[\frac{df}{d\bar{Q}}\right]^2 \tag{3.16a}
$$

$$
=\frac{C^2}{m\omega^4}\left[\frac{df_0}{d\bar{Q}}\right]^2 . \tag{3.16b}
$$

These formulae recover the result in Ref. 12 when the form factor $f_0(Q)$ equals Q. Note also here that the renormalization of mass is of higher order than that of the effective potential.

C. Simple interpretation of the results in the adiabatic limit

The result given in Eq. (3.15) can be simply understood in the following way. The total Hamiltonian of the present problem reads

$$
\hat{H}=\frac{\hat{P}^2}{2M}+\hat{U}(\hat{Q})+\frac{\hat{p}_\xi^2}{2m}+\tfrac{1}{2}m\omega^2\hat{\xi}^2+C\hat{f}_0(\hat{Q})\hat{\xi} . \tag{3.17a}
$$

This can be rewritten as

$$
\hat{H}=\frac{\hat{P}^2}{2M}+\hat{U}(\hat{Q})-\tfrac{1}{2}\frac{C^2}{m\omega^2}(f_0(\hat{Q}))^2+\frac{\hat{p}_\xi^2}{2m}+\tfrac{1}{2}m\omega^2\left[\hat{\xi}+\frac{Cf_0(\hat{Q})}{m\omega^2}\right]^2 . \tag{3.17b}
$$

Therefore, if one can treat $\hat{Q}$ as a c number during the period when the microscopic system evolves with time, then the effective potential for the macroscopic degree of freedom can be considered to be given by

$$\hat{U}_{\rm eff}=\hat{U}(\hat{Q})-\frac{1}{2}\frac{C^2}{m\omega^2}(f_0(\hat{Q}))^2 \ . \tag{3.18}$$

This is nothing but the result given in Eq. (3.15).

An alternative interpretation of Eq. (3.15) can be obtained as follows. In the adiabatic case, the intrinsic motion will make a zero point oscillation around the minimum value $\xi_0(Q)$ determined by

$$\left.\frac{\partial H}{\partial \xi}\right|_{\xi=\xi_0(Q)}=0 \tag{3.19}$$

for each given value of Q. Equations (3.17) and (3.19) lead to

$$\xi_0(Q)=-\frac{C}{m\omega^2}f_0(Q) \ . \tag{3.20}$$

The classical motion of the oscillator is then expected to be given by

$$\xi_{\rm cl}(t)=\xi_0(Q)+A_0\cos\omega t \ , \tag{3.21}$$

where A_0 is the amplitude of the zero point motion of the oscillator, and is given by

$$A_0=\left[\frac{\hbar}{m\omega}\right]^{1/2} . \tag{3.22}$$

Treating the oscillator motion classically, we then obtain the following effective Hamiltonian for the macroscopic motion:

$$\hat{H}=\frac{\hat{P}^2}{2M}+\hat{U}(\hat{Q})+\frac{1}{2}\hbar\omega-\frac{1}{2}\frac{C^2}{m\omega^2}(f_0(\hat{Q}))^2 \ . \tag{3.23}$$

The last term on the rhs of Eq. (3.23) is again nothing but the effective potential obtained in Eq. (3.15).

IV. PARABOLIC POTENTIAL BARRIER WITH A LINEAR COUPLING FORM FACTOR

Let us consider a simple solvable model that the A space potential is parabolic [see Eq. (2.15)] and the coupling form factor is linear, i.e.,

$$f_0(q)=q \ . \tag{4.1}$$

In this simple model, we shall encounter the difficulty of defining a physical state in the tunneling problem. That is, the initial state cannot in general be easily prepared such that the intrinsic harmonic oscillator is in its ground state, because the coupling form factor is of infinite range. Here the physical tunneling probability shall correspond to the penetrability of the normal mode. Although the tunneling process of the original macroscopic coordinate is an interesting problem in its own right, it does not correspond to a realistic physical process except in the weak coupling limit or in the adiabatic limit, when the normal coordinates reduce to the original coordinates. Still this model provides us with some concrete conclusion concerning the oscillator coupling to the macroscopic coordinate.

A. Influence potential $W(\tau)$ and a characteristic equation for the eigenfrequencies of the normal modes

Equation (3.8) gives the influence potential $W(\tau)$ as

$$W_0(\tau)=-g^2(1+e^{-2\omega(T-\tau)})q(\tau)e^{-\omega\tau}\times\int_{\tau_0}^{\tau}d\tau_1 q(\tau_1)e^{\omega\tau_1} \ . \tag{4.2}$$

The equation of motion for the dominant tunneling path Eq. (3.9) in the macroscopic coordinate becomes

$$\begin{aligned}\ddot{q}+\Omega_0^2 q&=-\frac{g^2}{M}\left[e^{-\omega\tau}\int_{\tau_0}^{\tau}d\tau_1 q(\tau_1)e^{\omega\tau_1}+e^{\omega\tau}\int_{\tau}^{T}d\tau_1 q(\tau_1)e^{-\omega\tau_1}+e^{-2\omega T+\omega\tau}\int_{\tau_0}^{T}d\tau_1 q(\tau_1)e^{\omega\tau_1}\right]\\&=-\frac{g^2}{M}\left\{-2\int_{\tau_0}^{\tau}d\tau_1 q(\tau_1)\sinh\omega(\tau-\tau_1)+e^{\omega(\tau-T)}[q^{(+)}+q^{(-)}]\right\} ,\end{aligned} \tag{4.3}$$

where

$$q^{(\pm)}\equiv\int_{\tau_0}^{T}d\tau_1 q(\tau_1)e^{\pm\omega(T-\tau_1)} \ . \tag{4.4}$$

Here we use $q(\tau)$ to denote the dominant tunneling path $\bar{Q}(\tau)$ of Sec. III. The integrodifferential equation (4.3) can now be cast into the fourth order differential equation as

$$\ddddot{q}+(\Omega_0^2-\omega^2)\ddot{q}-\left[\omega^2\Omega_0^2+2\frac{\omega g^2}{M}\right]q=0 \ . \tag{4.5}$$

Equation (4.5) is the natural result of two linearly coupled oscillators.

Setting $q\propto e^{i\Omega t}$, we observe that the frequency Ω obeys the dispersionlike characteristic equation which has been obtained by Widom and Clark,[13] i.e.,

$$\Omega^2=\Omega_0^2+\frac{2g^2\omega}{M(\Omega^2+\omega^2)} \ . \tag{4.6}$$

Denoting the real and imaginary solutions of Eq. (4.6) by $\pm\Omega$ and $\pm i\tilde{\omega}$, respectively, we get

$$\Omega^2=\Omega_0^2+\tfrac{1}{2}(\Omega_0^2+\omega^2)[(1+x)^{1/2}-1] \ , \tag{4.7a}$$

$$\tilde{\omega}^2=\omega^2+\tfrac{1}{2}(\Omega_0^2+\omega^2)[(1+x)^{1/2}-1] \ , \tag{4.7b}$$

where

$$x\equiv\frac{8g^2\omega}{M(\omega^2+\Omega_0^2)^2} \tag{4.7c}$$

is a dimensionless constant, which plays the role of the *effective coupling strength.* One notes that x is small in the weak coupling limit ($g \to 0$) or in the adiabatic limit ($\omega \to \infty$) (see Sec. V for precise conditions). In these two cases, Ω and $\tilde{\omega}$ become

$$\Omega \simeq \Omega_0 + \frac{g^2 \omega}{M\Omega_0(\Omega_0^2+\omega^2)} , \tag{4.8a}$$

$$\tilde{\omega} \simeq \omega + \frac{g^2}{M(\Omega_0^2+\omega^2)} . \tag{4.8b}$$

Equation (4.6) is indeed the equation to determine the normal mode frequencies of the present model (see Sec. V for details).

Because of the difficulty of preparing the initial physical state in the present model, the tunneling probability obtained from the Feynman path integral method, which prepares the initial intrinsic harmonic oscillator in the ground state, may not be the same as that of the normal coordinate, which prepares the initial state of the normal mode intrinsic harmonic oscillator in its ground state. This point has been overlooked in Ref. 13. Accordingly, in Ref. 13 the tunneling probability of the original macroscopic motion has been assigned to be

$$P = \{1+\exp[2\pi(E-V_0)/\Omega\}^{-1} . \tag{4.9}$$

This is not correct, because the effective potential for the original coordinate is not necessarily parabolic (see Sec. IV C) and also because the renormalization of the frequency of the intrinsic harmonic oscillator has to be considered if one wants to discuss the tunneling process in the normal coordinate (see Sec. V).

B. Adiabatic limit

In the adiabatic limit, Eq. (3.14) becomes

$$M_e \ddot{q} = M\Omega_e^2 q , \tag{4.10}$$

with

$$\Omega_e^2 = \Omega_0^2 + 2\frac{g^2}{M\omega} \equiv \Omega_{\rm ad}^2 , \tag{4.11a}$$

$$M_e = M + 2\frac{g^2}{\omega^3} . \tag{4.11b}$$

Equation (4.10) is manifestly parabolic; this is also clear from Eq. (3.18). One can let $\tau_0 = 0$ and write

$$q(\tau) = Q_e \cos(\Omega_e \tau) , \tag{4.12a}$$

with

$$Q_e = -[2(V_0 - E)/(M\Omega_e^2)]^{1/2} . \tag{4.12b}$$

Here we have ignored the mass renormalization, because it is of higher order compared with the frequency renormalization in the adiabatic limit [cf. Eq. (4.11a) with Eq. (4.11b)]. Combining Eqs. (3.8) and (4.12), one obtains

$$W_0(\tau) = -\frac{g^2}{\hbar}[(1-e^{-\omega\tau})(1+e^{-2\omega(T-\tau)})] \times q(\tau)\left[\frac{\omega}{\omega^2+\Omega_e^2}q(\tau) - \frac{\dot{q}(\tau)}{\omega^2+\Omega_e^2}\right] . \tag{4.13}$$

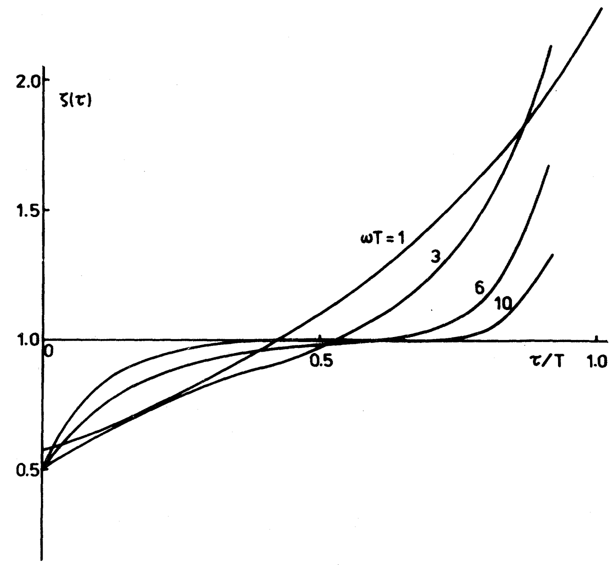

FIG. 1. $\zeta(\tau)$ in Eq. (4.16) is shown for a number of ωT parameters; when $\zeta(\tau)$ equals 1, the extra force exerted on the tunneling degree of freedom is also linear [see Eq. (4.15)].

Equation (4.13) is exactly the form of Eq. (2.28), where the quantities B and C are given by

$$B(\tau, q(\tau)) = -\frac{g^2}{\hbar}\frac{\omega}{\omega^2+\Omega_e^2}C_0(\tau)[q(\tau)]^2 , \tag{4.14a}$$

$$C(\tau, q(\tau)) = \frac{g^2}{\hbar}\frac{1}{\omega^2+\Omega_e^2}C_0(\tau)q(\tau) , \tag{4.14b}$$

where

$$C_0(\tau) = (1-e^{-\omega\tau})(1+e^{-2\omega(T-\tau)}) . \tag{4.14c}$$

Accordingly, the extra force exerted on the tunneling degree of freedom due to the linear coupling to a harmonic oscillator is given by

$$F(\tau) = -2\frac{g^2}{\hbar}\frac{\omega}{\omega^2+\Omega_e^2}\zeta(\tau)q(\tau) , \tag{4.15}$$

where

$$\zeta(\tau) = 1 - \tfrac{1}{2}e^{-\omega(\tau-\tau_0)} + 2e^{-2\omega(T-\tau)} - \tfrac{3}{2}e^{-\omega(2T-\tau-\tau_0)} . \tag{4.16}$$

Figure 1 shows $\zeta(\tau)$ as a function of τ/T for a few typical values of the adiabaticity parameter ωT. It is nearly equal to unity except for the vicinity of $\tau/T = 0$ and $\tau/T = 1$ for large values of ωT.

The tunneling probability is now given as a product of three factors,

$$P \simeq P_0 P_R P_D , \tag{4.17a}$$

where

$$P_0 = \exp\left\{-2\int_{\tau_0}^{T}\left[\frac{M}{2}\dot{q}^2 + U(q)\right]d\tau/\hbar\right\}, \quad (4.17b)$$

$$P_R = \exp\left[-2\int_{\tau_0}^{T} B(\tau, q(\tau))d\tau/\hbar\right], \quad (4.17c)$$

and

$$P_D = \exp\left[-2\int_{\tau_0}^{T} C(\tau, q(\tau))\dot{q}\, d\tau/\hbar\right]. \quad (4.17d)$$

The integral in each factor can be easily evaluated by using Eqs. (4.12) and (4.14) as

$$\int_0^T \left[\frac{M}{2}\dot{q}^2 + U(q)\right]d\tau = (V_0 - E)\frac{T}{2}\left\{2 + \frac{M}{M_e}\left[1 - \left[\frac{\Omega_0}{\Omega_e}\right]^2\right]\right\}, \quad (4.18a)$$

$$\int_0^T B(\tau, q(\tau))d\tau = -\frac{g^2}{\hbar}\frac{\omega}{\omega^2 + \Omega_e^2}Q_e^2\left[\tfrac{1}{2}T - \tfrac{1}{4}\left[\frac{1}{\omega} + \frac{2\omega}{\omega^2 + 4\Omega_e^2} - \frac{\omega}{\omega^2 + \Omega_e^2}\right](1 - e^{-2\omega T})\right], \quad (4.18b)$$

and

$$\int_0^T C(\tau, q(\tau))\dot{q}\, d\tau = \frac{g^2}{\hbar}\frac{\Omega_e}{\omega^2 + \Omega_e^2}Q_e^2(1 - e^{-\omega T})\left[\frac{\Omega_e}{\omega^2 + 4\Omega_e^2}(1 - e^{-\omega T}) + \frac{\Omega_e}{4(\omega^2 + \Omega_e^2)}(1 + e^{-\omega T})\right]. \quad (4.18c)$$

Caldeira and Leggett[11] have paid attention only to the dissipation factor P_D. Since $P_D < 1$, in Ref. 11 it is concluded that the coupling to a harmonic oscillator reduces the tunneling probability. Comparison of Eq. (4.18c) with Eq. (4.18b), however, clearly shows that the enhancement effect of the potential renormalization factor P_R is more than to cancel the reduction effect of the factor P_D in the adiabatic case. Moreover, it is also expected that the factor P_0 becomes larger than the tunneling probability for a one-dimensional problem $P_0^{(0)}$.

In fact, Eqs. (4.17) and (4.18) lead to

$$P \approx P_{\rm ad} = (1 + e^{2\pi(V_0 - E)/\hbar\Omega_{\rm ad}})^{-1} \quad (4.19)$$

in the adiabatic limit. This is actually what is expected from the effective equation of motion Eq. (4.10) if one ignores the mass renormalization. Since $\Omega_{\rm ad} > \Omega_0$, one reaches the same conclusion as that of Ref. 13 in the adiabatic limit. Namely, the linear coupling to the intrinsic oscillator enhances the quantum tunneling probability of a macroscopic motion at sub-barrier energy.

C. Weak coupling limit ($g \to 0$)

In the weak coupling limit, we shall use a perturbation method to solve Eq. (4.3). The zeroth order solution of Eq. (4.3) is clearly

$$q^{(0)} = -Q_0 \cos\Omega_0 \tau \quad (4.20)$$

with

$$Q_0 = [2(V_0 - E)/M\Omega_0^2]^{1/2}$$

and

$$T_0 = \frac{\pi}{\Omega_0}.$$

The first order solution of Eq. (4.3) becomes

$$q(\tau) = \alpha\cos(\Omega_0\tau) + \beta\sin(\Omega_0\tau) + \gamma\tau\sin\Omega_0\tau + \tilde{\delta}e^{-\omega\tau} + \epsilon e^{\omega\tau}, \quad (4.21)$$

where

$$\alpha = A + \frac{x}{8}Q_0(1 + e^{-2\omega T_0}), \quad (4.22a)$$

$$\beta = \frac{x}{8}\frac{\omega}{\Omega_0}Q_0(-1 + e^{-2\omega T_0}), \quad (4.22b)$$

$$\gamma = \frac{x}{8}\frac{\omega^2 + \Omega_0^2}{\Omega_0}Q_0, \quad (4.22c)$$

$$\tilde{\delta} = -\frac{x}{8}Q_0, \quad (4.22d)$$

and

$$\epsilon = -\frac{x}{8}Q_0 e^{-2\omega T_0}. \quad (4.22e)$$

The parameter A in Eq. (4.22a) is $-Q_0$ in the standard perturbation treatment. One would, however, be allowed to assume other values for A. For example, in the adiabatic case, another reasonable choice is $A = -Q_e$. Equation (4.21) then provides us with a refined formula of the dominant tunneling path for the problem considered in the preceding subsection.

In Fig. 2 we show $q(\tau)$ (the thick solid line), as well as each term on the rhs of Eq. (4.21) (the thin solid lines). The numbers attached to the thin solid lines correspond to the order of each term on the rhs of Eq. (4.21). Figure 2 and Eq. (4.21) clearly show that in general the macroscopic motion in the open system problem need not be a simple harmonic motion. We remark that the fourth and the fifth terms on the rhs of Eq. (4.21) can be ignored in comparison with the first three terms in the adiabatic case, i.e., if $\Omega_0/\omega \ll 1$. Equation (4.21) is then identical to a resonance phenomenon in the problem of forced oscillations.[17]

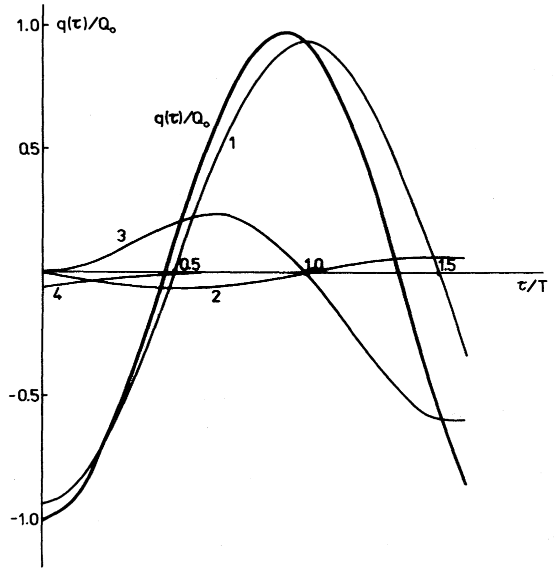

FIG. 2. The thick solid line shows $q(\tau)/Q_0$ of Eq. (4.21) for $x=0.5$ and $\omega/\Omega_0=1$. The thin lines marked 1, 2, 3, and 4 correspond to the first, second, third, and fourth terms, respectively, on the rhs of Eq. (4.21). The fifth term of Eq. (4.21) is too small to be shown on the figure.

It also corresponds to the first order perturbation expansion of the adiabatic limit with respect to the strength of the coupling Hamiltonian.

The transmission time T will correspond to the time when $Q(\tau)$ is bounced back. On the other hand, if we denote the value of $Q(\tau)$ at that time as Q_T, then $(-A+Q_T)$ will give the transmission distance of the tunneling process. If we write

$$T=T_0+\delta T \tag{4.23a}$$

and

$$Q_T=A+\delta Q_0 , \tag{4.23b}$$

then δT and δQ_0 are given as follows up to the first order with respect to g^2:

$$\delta T \approx -\frac{g^2}{M\hbar}\frac{\omega}{(\omega^2+\Omega_0^2)\Omega_0^2}\left[\frac{\pi}{\Omega_0}-\frac{\omega}{\omega^2+\Omega_0^2}(1-e^{-2\omega T_0})\right] , \tag{4.24}$$

thus

$$\delta T \approx \begin{cases} -\dfrac{g^2}{M\hbar}\dfrac{\pi}{\omega\Omega_0^3} & \text{if } \omega \gg \Omega_0 \\ -\dfrac{g^2}{M\hbar}\dfrac{\pi\omega}{\Omega_0^5} & \text{if } \omega \ll \Omega_0 , \end{cases} \tag{4.25}$$

and

$$\delta Q_0 \approx -Q_0\frac{g^2}{M\hbar}\frac{\omega}{(\omega^2+\Omega_0^2)^2}(1+e^{-\omega T_0})^2 . \tag{4.26}$$

These equations imply that both the transmission time and the transmission distance are reduced due to the linear coupling to a harmonic oscillator. This can be clearly seen in Fig. 2.

V. QUANTUM TUNNELING OF A NORMAL MODE

We consider the linear coupling model of Sec. IV. For this simple model, there exists a canonical coordinate transformation which decouples the total Hamiltonian into a sum of Hamiltonians for the normal modes. In this section, we thus study the exact solution of the quantum tunneling process in normal mode. We shall compare the quantum tunneling probability of the coupled system with that of an uncoupled system. We shall discuss also the relation between the quantum tunneling of the normal mode with that of the original macroscopic motion viewed as an open system problem.

Let us make a canonical coordinate transformation by

$$\tilde{q}=a(\hat{q}+b\hat{\xi}) , \tag{5.1a}$$

$$\tilde{\xi}=d(c\hat{q}+\hat{\xi}) , \tag{5.1b}$$

with

$$a=d=(1+Z^2)^{-1/2} , \tag{5.2a}$$

$$b=-\sqrt{m/M}Z , \tag{5.2b}$$

$$C=\sqrt{M/m}Z , \tag{5.2c}$$

$$Z=\frac{\sqrt{1+x}-1}{\sqrt{x}} , \tag{5.2d}$$

where x is the dimensionless effective coupling strength defined by Eq. (4.7c). The total Hamiltonian becomes

$$\hat{H}=\frac{1}{2M}\tilde{p}^2-\tfrac{1}{2}M\Omega^2\tilde{q}^2+\frac{1}{2m}\tilde{p}_\xi^2+\tfrac{1}{2}m\tilde{\omega}^2\tilde{\xi}^2+V_0 . \tag{5.3}$$

The eigenfrequencies corresponding to these normal modes, Ω and $\tilde{\omega}$, are identical to those introduced by Eq. (4.7).

Equation (5.3) shows that the normal mode $\tilde{q}$ is governed by a simple parabolic potential barrier. This contrasts with the motion concerning the original macroscopic coordinate q, which may not necessarily be described in terms of an effective parabolic potential barrier. Therefore, if we set the physical question as the quantum tunneling of the normal coordinate $\tilde{q}$, then the tunneling probability is exactly given by

$$\tilde{P}(E_t)=\frac{1}{1+e^{2\pi\{V_0-[E_t-(1/2)\hbar\tilde{\omega}]\}/\hbar\Omega}} , \tag{5.4}$$

where E_t is the energy of the total system. In Eq. (5.4) the notation $\tilde{P}$, instead of P, has been used in order to distinguish the tunneling probability of the normal mode $\tilde{q}$ from that concerning the original macroscopic coordinate q.

The effect of the coupling on the quantum tunneling could be discussed by comparing $\tilde{P}(E_t)$ with the tunneling probability of the macroscopic motion q in the case when it is not coupled with ξ. We denote this tunneling probability by $P_0(E_t)$. Obviously, it is given by Eq. (5.4) with $\tilde{\omega}$ and Ω replaced by the bare ω and Ω_0. One easily observes that

$$\tilde{P}(E_t) \geq P_0(E_t) \text{ if } E_t \leq E_c \tag{5.5a}$$

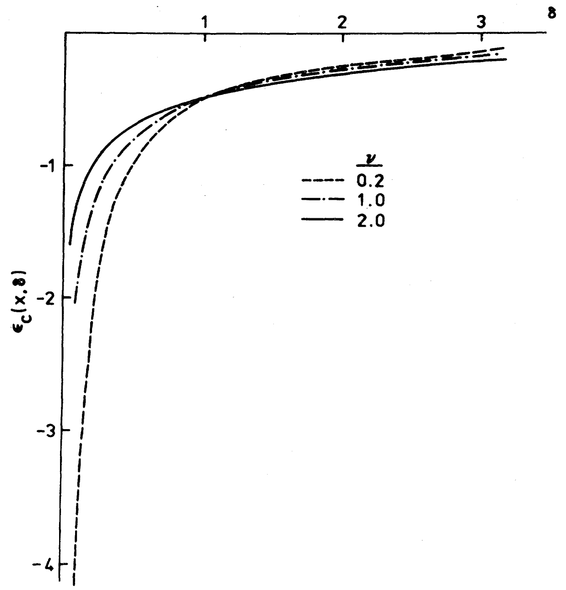

FIG. 3. The critical energy parameter ϵ_c of Eq. (5.6) as a function of $\delta=\omega/\Omega_0$ is shown for several coupling strengths ν [Eq. (5.8)].

and

$$\tilde{P}(E_t)\le P_0(E_t) \text{ if } E_t \ge E_c , \tag{5.5b}$$

where the critical energy E_c is given by

$$\epsilon_c \equiv \frac{E_c-\frac{1}{2}\hbar\omega - V_0}{\hbar\Omega_0} = -\frac{1}{2}\delta \frac{\left[1+\frac{1}{2\delta^2}(1+\delta^2)\sqrt{x}Z\right]^{1/2}-1}{[1+\frac{1}{2}(1+\delta^2)\sqrt{x}Z]^{1/2}-1} \tag{5.6}$$

with

$$\delta=\omega/\Omega_0 . \tag{5.7}$$

Figure 3 shows ϵ_c as a function of the adiabaticity parameter δ for several different values of the coupling strength ν defined by

$$\nu=\frac{g}{(M\Omega_0^3)^{1/2}} . \tag{5.8}$$

The normal coordinates reduce to the original coordinates if Z of Eq. (5.2) approaches zero. This can be reached with a small dimensionless effective coupling strength x, i.e., in the limit of $x\rightarrow 0$; we note that

$$x=\frac{8\nu^2\delta}{(1+\delta^2)^2} . \tag{5.9}$$

The condition of small effective coupling strength can therefore be realized by two situations, i.e., the adiabatic limit ($\delta\rightarrow\infty$) and the weak coupling limit ($\nu\rightarrow 0$).

A. The adiabatic limit

In the adiabatic limit, Ω and $\tilde{\omega}$ approach $\Omega_{\rm ad}$ and ω, respectively, up to the first order of Ω_0/ω. We then obtain

$$\tilde{P}(E_t)\simeq 1/(1+\exp\{2\pi[V_0-(E_t-\tfrac{1}{2}\hbar\omega)]/\hbar\Omega_{\rm ad}\}) . \tag{5.10}$$

The result agrees with that in Sec. IV B. Note that $E_t-\frac{1}{2}\hbar\omega$ in Eq. (5.10) corresponds to E in Eq. (4.19).

B. Weak coupling limit

In this case, the critical parameter ϵ_c defined by Eq. (5.6) is approximately given by

$$\epsilon_c \simeq -\frac{1}{2\delta} . \tag{5.11}$$

Therefore one expects that there exists a sub-barrier energy range where the coupling to the harmonic oscillator reduces the quantum tunneling probability of the open system problem in comparison with that of an uncoupled system. Equation (5.11) indicates that such an energy range increases as δ becomes smaller. This is natural, because the energy of the macroscopic motion would be easily transferred to the coupled oscillator if the energy quanta of the oscillator are small, so that the dissipation effect would dominate.

VI. SUMMARY AND SOME COMMENTS ON THE SUB-BARRIER FUSION CROSS SECTION

We have studied the effects of the coupling to a harmonic oscillator on the quantum tunneling of a macroscopic motion. To this end, we have used the influence functional formalism of the Feynman path integral method of quantum mechanics along the imaginary time axis.

Our result shows that the coupling to other degrees of freedom introduces the *influence potential* into the effective Lagrangian for the macroscopic tunneling degree of freedom. We have derived the expression of the influence potential as a functional of the coupling form factor for the case of coupling to a harmonic oscillator by assuming that the coupling Hamiltonian is a linear function of the coordinate of the harmonic oscillator. The influence potential [Eq. (3.8)] involves memory effects. In the adiabatic case, in other words, if the recurrence time π/ω of the harmonic oscillator is much shorter than the transmission time, the influence potential can be well approximated by renormalizing the potential barrier and the mass and by introducing a linearly velocity dependent term in the Lagrangian. The explicit forms of the effective potential and the effective mass are given in Eqs. (3.15) and (3.16). They are the generalizations of the corresponding formulae given in Ref. 12 to the case of the general coupling form factor concerning the macroscopic tunneling coordinate.

We have studied in detail the model that the potential barrier is parabolic and the coupling Hamiltonian is linear with respect to both the macroscopic coordinate and the intrinsic coordinate. Our result shows that the coupling to the harmonic oscillator enhances the tunneling probability as a net effect if the adiabatic condition is satisfied [Eq. (4.19)]. On the other hand, in the weak coupling limit, we have used a perturbation treatment to write the behavior of the dominant tunneling path as a function of time [Eq. (4.21)]. It indicates that the quantum tunneling of the macroscopic coordinate is not a simple harmonic motion. In this linear coupling model, the equation of motion for the dominant tunneling path leads to the dispersionlike equation to determine the eigenfrequencies of the normal modes.

In this simple model, a canonical coordinate transformation decouples the original Hamiltonian into the sum of Hamiltonians for the normal harmonic modes. The normal coordinates reduce to the original coordinates of the macroscopic motion and of the intrinsic harmonic oscillator if the dimensionless *effective coupling strength* x defined by Eq. (4.7) is small. In this case, the exact solution of the quantum tunneling probability through the canonical transformation confirms and/or supplements the results of the influence functional method. That is to say: (1) The linear coupling to a harmonic oscillator enhances the sub-barrier fusion cross section in the adiabatic case. (2) There exists a critical energy E_c, above which the tunneling probability is reduced due to the coupling to a harmonic oscillator. In the weak coupling limit defined as $v \rightarrow 0$, and if ω is small, the critical energy becomes $-\infty$, so that the linear coupling to a harmonic oscillator always reduces the tunneling probability.

Finally, we wish to use the results of the present study to draw some comments concerning the enhancement of the sub-barrier fusion cross section in heavy ion collisions. The external potential barrier of the heavy ion fusion reaction is of the order of $\hbar\Omega_0 \approx 4$ MeV.[5] Let us then consider the effects of intrinsic degrees of freedom on the macroscopic motion. There are two types of collective vibrational excitations in nuclei. These are the low frequency collective vibrational states and the high lying giant resonances.

The intrinsic motion regarding the giant resonances with $\hbar\omega \simeq 10-20$ MeV may be considered to be in the adiabatic limit. Their contribution to the macroscopic motion is then to renormalize the effective potential. Since the deformation parameters for the giant resonances are about equal to those of low lying collective states[18] and the coupling strength between the macroscopic coordinate and the intrinsic motion is proportional to the deformation parameter, the effective renormalization potential due to these high lying giant resonances may not be so important, because the effective renormalization potential is proportional to $1/\omega^2$. On the other hand, our result indicates that the coupling to the intrinsic degree of freedom concerning the low-lying collective vibrational states with $\hbar\omega = 1-2$ MeV does not necessarily enhance the tunneling probability, even at the sub-barrier energy. One therefore doubts the validity of the zero point motion prescription[7] in explaining the enhancement of the sub-barrier fusion cross section.

In the present work, we have not treated (1) the reorientation of deformed nuclei and (2) the induced static polarization of nuclear density. Recent phenomenological study indicates that the polarization degree of freedom may be the origin of the enhancement of the sub-barrier fusion cross section.[19] More work is needed in understanding the role of the deformation degree of freedom during the tunneling process.[20]

ACKNOWLEDGMENTS

This work was supported in part by the National Science Council, Republic of China, and by the Deutsche Forschungs Gemeinschaft, Federal Republic of Germany. The authors acknowledge the warm hospitality extended to them at the Academia Sinica, Taipei. One of the authors (N.T.) also acknowledges the hospitality extended to him during his stay at Technische Universität, München. The authors are grateful for useful discussions with Prof. H. Morinaga, Prof. E. K. Lin, Prof. J. A. Alexander, Dr. S. Landowne, and Dr. M. C. Nemes.

APPENDIX A: QUADRATIC COUPLING TO A HARMONIC OSCILLATOR

In this appendix we consider the quantum tunneling problem, where the macroscopic motion couples to a harmonic oscillator through a quadratic function of the coordinate of the harmonic oscillator as

$$\hat{H} = \frac{\hat{p}^2}{2M} + \hat{U}(\hat{q}) + \frac{\hat{p}_\xi^2}{2m} + \tfrac{1}{2}k\left[1+\lambda\exp\left[-\frac{\hat{q}^2}{b^2}\right]\right]\hat{\xi}^2 . \tag{A1}$$

This is equivalent to assuming that the coupling Hamiltonian is given by

$$V_c = \tfrac{1}{2}k\lambda\exp\left[-\frac{\hat{q}^2}{b^2}\right]\hat{\xi}^2$$

$$= qf_0(\hat{q})(a^\dagger a^\dagger + aa + 2a^\dagger a + 1) , \tag{A2}$$

$$g = \hbar k\lambda/(4m\omega) , \tag{A3a}$$

$$\omega = \sqrt{k/m} , \tag{A3b}$$

$$f_0(\hat{q}) = \exp(-\hat{q}^2/b^2) . \tag{A3c}$$

The quantum tunneling problem for the Hamiltonian in Eq. (A1) has been previously discussed for the problem of nuclear fission.[4]

For each value of q, let us define the q-dependent oscillator frequency as

$$\hbar\omega_a(q) = \hbar\omega\sqrt{1+\lambda f_0(q)} . \tag{A4}$$

The sum of the third and fourth terms on the rhs of Eq. (A1) is then

$$\hbar\omega_a(q)(a^\dagger a + \tfrac{1}{2}) . \tag{A5}$$

This implies that the tunneling process given by Eq. (A1) can be well described as a one dimensional tunneling process through an effective potential barrier given by

$$\hat{U}_{\rm eff}(\hat{q})=\hat{U}(\hat{q})+\tfrac{1}{2}\hbar[\omega_a(\hat{q})-\omega]\ , \tag{A6}$$

if the adiabatic condition is satisfied, i.e., if ω is much larger than the inverse of the transmission time. This is actually the prescription suggested in Ref. 4.

Glauber's coherent state representation for quantum operators, which was powerful to determine the Green's function $\hat{u}$ in the case of linear coupling, can be used for the present problem as well. It becomes, however, rather complicated to solve the equations for A, B, etc., which enter the coherent state representation of the Green's function [see Eq. (3.4)]. We, therefore, use the perturbation theory to determine the influence potential up to the third order with respect to the coupling constant g. After somewhat lengthy but standard calculations we obtain

$$W\cong W^{(1)}+W^{(2)}+W^{(3)}+\cdots\ , \tag{A7}$$

where

$$W^{(1)}(\tau)=f(q(\tau))\ , \tag{A8a}$$

$$\begin{aligned}W^{(2)}(\tau)&=-2f(q(\tau))\int_{\tau_0}^{\tau}d\tau_1 f(q(\tau_1))e^{-2\omega(\tau-\tau_1)}\\&=-\tfrac{1}{2}e^{-4\omega T}f(q(\tau))e^{2\omega\tau}\int_{\tau_0}^{\tau}d\tau_1 f(q(\tau_1))e^{2\omega\tau_1}\ ,\end{aligned} \tag{A8b}$$

and

$$W^{(3)}(\tau)=8f(q(\tau))\int_{\tau_0}^{\tau}d\tau_1\int_{\tau_0}^{\tau_1}d\tau_2 f(q(\tau_1))f(q(\tau_2))[e^{-2\omega(\tau-\tau_2)}+e^{-4\omega T}(e^{2\omega(\tau+\tau_2)}+2e^{2\omega(\tau_1+\tau_2)})] \tag{A8c}$$

with

$$f(q)=gf_0(q)/\hbar\ . \tag{A9}$$

In Eq. (A7), we have written only those important terms which are not related to $\eta(\tau)$. The constant zero point energy of the harmonic oscillator is also left out.

In contrast to the exact result of Eq. (3.8) for the linearly coupled oscillator, Eq. (A7) is correct only up to the third order with respect to f. Note also that the influence potential for the case of quadratic coupling has a linear term with respect to f. This term orignates from the c-number term on the rhs of Eq. (A2), and dominates the effect of coupling to harmonic oscillators on a quantum tunneling problem in the weak coupling limit. Expanding the influence potential $W(\tau)$ in the power series of $1/\omega$, we obtain

$$W(\tau)=\hbar f(q)-\hbar f(q)\left[\frac{f}{\omega}-\frac{1}{2\omega^2}\dot{q}\frac{df}{dq}+\frac{1}{4\omega^3}\left[\ddot{q}\frac{df}{dq}+\dot{q}^2\frac{d^2f}{dq^2}\right]+\cdots\right]+\hbar f(q)\left[2\left[\frac{f}{\omega}\right]^2+\cdots\right]+\cdots\ , \tag{A10}$$

if $\omega^2(T-\tau)(\tau-\tau_0)\gg 0$ (see Sec. III B).

There are two parameters related to the expansion given by Eq. (A10). They are Ω_0/ω, Ω_0 being the inverse of the transmission time, and f/ω. The former and the latter are the measures of the adiabaticity and of the strength of the coupling Hamiltonian, respectively. As the parameter Ω_0/ω becomes small, $W(\tau)$ approaches the limiting value given by

$$W(\tau)=\hbar f(q)\left[1-\frac{f}{\omega}+2\left[\frac{f}{\omega}\right]^2+\cdots\right]\ . \tag{A11}$$

This is nothing but the first three terms in the expansion of the adiabatic potential

$$W_{\rm ad}(\tau)=\tfrac{1}{2}\hbar[\omega_a(q)-\omega]$$

in powers of $f(q)/\omega$.

Note that the quadratic coupling to a harmonic oscillator either enhances or reduces the sub-barrier tunneling probability, depending on the sign of the coupling Hamiltonian even in the adiabatic case. This can be easily seen in Eq. (A6) together with Eq. (A4). In this respect, the quadratic coupling essentially differs from the linear coupling, by which the sub-barrier tunneling probability is always enhanced irrespective of the sign of the coupling Hamiltonian, at least in the adiabatic case [see Eq. (3.15)].

APPENDIX B: CONDITION TO JUSTIFY THE PRESCRIPTION IN TERMS OF AN EFFECTIVE POTENTIAL

Equation (3.10) suggests that whether the effect of the coupling to harmonic oscillators on the tunneling probability can be well prescribed in terms of an effective potential or not depends not only on the adiabaticity parameter, but also on the property of the coupling Hamiltonian. The aim of the present appendix is to clarify this point.

We first recall that the integral part in the influence potential $W(\tau)$ can be expressed as

$$e^{-\omega\tau}\int_{\tau_0}^{\tau}d\tau_1 f(Q(\tau_1))e^{\omega\tau_1}=\sum_{n=0}^{\infty}V_n(\tau)\ , \tag{B1}$$

with

$$V_0(\tau)=\chi_0 f(Q(\tau))\ , \tag{B2a}$$

$$V_1(\tau)=-\chi_1 Q(\tau)\frac{df}{dQ}\ , \tag{B2b}$$

$$V_2(\tau)=\chi_2\frac{1}{2!}\left\{\dot{Q}(\tau)\frac{df}{dQ}+[\dot{Q}(\tau)]^2\frac{d^2f}{dQ^2}\right\}\ , \tag{B2c}$$

$$V_3(\tau)=-\chi_3\left\{\frac{1}{3!}\ddot{Q}(\tau)\frac{df}{dQ}+\frac{1}{2!}\dot{Q}(\tau)\ddot{Q}(\tau)\frac{d^2f}{dQ^2}-\frac{1}{3!}[\dot{Q}(\tau)]^3\frac{d^3f}{dQ^3}\right\}, \tag{B2d}$$

where χ_n are defined by Eq. (3.11). As can be seen in Eq. (3.10), χ_n $(n\geq 1)$ consists of a constant term and the terms which are proportional to $e^{-\omega(\tau-\tau_0)}$. Let us consider only the constant term among them, and furthermore, only the first term on the rhs of each of Eq. (B2). We denote the resultant part of each $V_n(\tau)$ as $V_n^{(0)}(\tau)$. For $n\geq 1$, they are given by

$$V_n^{(0)}(\tau)=(-1)^n\frac{1}{\omega^{n+1}}\left[\frac{d^n}{d\tau^n}Q(\tau)\right]\frac{df}{dQ}\ . \tag{B3}$$

Let us now assume a Gaussian function for the coupling form factor,

$$f=\alpha\exp(-Q^2/b^2)\ . \tag{B4}$$

This enables us to express each $V_n^{(0)}(\tau)$ as a product of $V_0(\tau)$ and a multiplication factor $\epsilon_n^{(0)}$, which is given by

$$\epsilon_n^{(0)}=2(-1)^{n+1}\frac{Q}{b^2}\frac{1}{\omega^n}\frac{d^n}{d\tau^n}Q(\tau)\ . \tag{B5}$$

If we further approximate $Q(\tau)$ by a single harmonic motion as

$$Q(\tau)=Q_0\cos(\Omega\tau) \tag{B6}$$

then

$$\epsilon_n^{(0)}=\left[\frac{Q_0}{b}\right]^2\left[\frac{\Omega}{\omega}\right]^n\kappa_n\ ,$$

where

$$\begin{aligned}\kappa_n&=(-1)^{m+1}2\cos^2(\Omega\tau)\ \text{ if } n=2m,\ m \text{ an integer}\\&=(-1)^{m+1}\sin(2\Omega\tau)\ \text{ if } n=2m+1\ .\end{aligned} \tag{B7}$$

The condition to justify the prescription to calculate the tunneling probability as a one dimensional process through an effective potential barrier is then given by

$$\left[\frac{Q_0}{b}\right]^2\left[\frac{Q}{\omega}\right]^k\ll 1\ , \tag{B8}$$

where $k=1,2,3,\ldots$.

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